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# Contents

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# Basic Concepts 1

The study of transfer phenomena, which include transfer of momentum, energy, mass, electricity, etc. has been recognized as a unified discipline of fundamental importance on the basis of generalized fluxes and forces. A flux like heat transfer, momentum transfer, mass transfer, electricity and chemical reaction rate is linearly proportional to the respective *conjugate force* of temperature gradient, velocity gradient, concentration gradient, electric potential gradient and chemical affinity, the constant of proportionality being a property of the medium, like thermal conductivity, viscosity, diffusion coefficient and electrical conductivity. It is a law of nature (phenomenological law) which states that a driving force causes the respective flux from a higher to a lower potential. The reverse never happens spontaneously. The transfer process indicates the tendency of a system to proceed towards equilibrium. For example, in a solid body with a nonuniform temperature distribution, energy is transferred so as to establish a uniform temperature distribution in the body.

Heat is defined as energy transferred by virtue of a temperature difference or gradient. Heat transfer is a vector quantity, flowing in the direction of decreasing temperature, with a negative temperature gradient. In the science of thermodynamics, the important parameter is the quantity of heat transferred during a process. In the subject of heat transfer, attention is directed to the rate at which heat is transferred. Thermodynamics is concerned with the transition of a system from one equilibrium state to another, and is based principally on the two laws of nature, the first law and the second law of thermodynamics. It is the science of heat transfer which is concerned with the estimation of the rate at which heat is transferred, the duration of heating and cooling for a certain heat duty and the surface area required to accomplish that heat duty.

When a small amount of perfume vapour is sprayed into a room of air, the mass transfer process causes the perfume vapour to diffuse throughout the room until its concentration is uniform, indicating an equilibrium condition. In an electrically conducting material with a nonuniform electrical potential (voltage) distribution, electric charge will flow until a uniform potential distribution is set up. In all transfer processes we are concerned with rates at which changes in properties of a system occur. In the flow of a viscous fluid, the viscous (frictional) stresses may be related to the rate of change of momentum of a system. Heat conduction may be related to the rate of change of internal energy of system. Mass diffusion may be related to the rate of change of composition of a mixture due to transfer of one or more of the component species.

There are three distinct modes in which heat transmission can take place: conduction, radiation and convection. Strictly speaking, only conduction and radiation should be classified as heat transfer processes, because only these two modes depend on the existence of a temperature difference. Convection refers to the mass motion of a fluid, and the convective heat transfer between a solid wall and a fluid depends not only on the temperature difference, but also on the mass transport of the fluid. However, since convection, like conduction and radiation, also accomplishes energy transfer from regions of higher temperature to regions of lower temperature, the term 'heat transfer by convection' has become generally accepted.

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## 1.1 THERMODYNAMICS AND HEAT TRANSFER

The science of heat transfer is concerned with the calculation of the rate at which heat flows within a medium, across an interface, or from one surface to another, and the associated temperature distribution. Thermodynamics deals with systems in equilibrium and calculates the energy transferred to change a system from one equilibrium state to another. However, it cannot tell the duration for which heat has to flow to change that state of equilibrium. For example, if 1 kg ingot of iron is quenched from 1000°C to 100°C in an oil bath, thermodynamics tells us that the loss in internal energy of the ingot is

$$mc\Delta T = 1 \text{ kg} \times 0.45 \frac{\text{kJ}}{\text{kg K}} \times 900 \text{ K} = 405 \text{ kJ}$$

But thermodynamics cannot tell us about the time required for the temperature to drop to 100°C. The time depends on various factors such as the temperature of the oil bath, physical properties of the oil, motion of the oil etc. An appropriate heat transfer analysis considers all these factors.

Analysis of heat transfer processes requires some concepts of thermodynamics.

The first law of thermodynamics states the principle of conservation of energy and it is expressed in the form of an energy balance for a system.

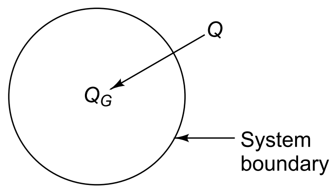


**Fig. 1.1** *Application of first law*

A *closed* system containing a fixed mass of a solid (Fig. 1.1) has a volume V(m$^3$) and density $\rho$(kg/m$^3$). There is heat transfer into the system at a rate $Q$(W), and heat may be generated internally within the solid, say, by nuclear fission or electrical current at a rate $Q_G$(W).The principle of energy conservation requires that over a time interval $\Delta t$(s).

Change in internal energy within the system

= Heat transferred into the system + Heat generated within the system

$$\Delta U = Q \cdot \Delta t + Q_G \, \Delta t \tag{1.1}$$

Dividing by $\Delta t$ and equating it to zero in the limit.

$$\frac{dU}{dt} = Q + Q_G$$

Now, $dU = \rho V \, du = \rho V c dT$, where $du = c_v \, dT$ and $c_v = c_p = c$ for an incompressible fluid or a rigid solid,

$$\rho V c \frac{dT}{dt} = Q + Q_G \tag{1.2}$$

This is the energy equation on a *rate basis*.

Figure 1.2 shows an open system for which a useful form of the first law is the *steady flow energy equation* (SFEE), given below:

$$\dot{m}\left(h_1 + \frac{V_1^2}{2} + gz_1\right) + \dot{Q} = \dot{m}\left(h_2 + \frac{V_2^2}{2} + gz_2\right) + \dot{W} \tag{1.3}$$

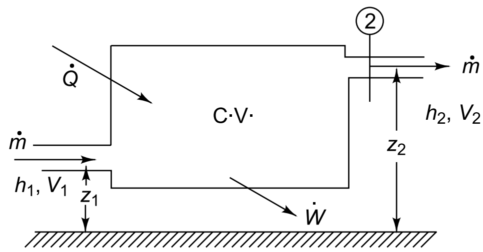


**Fig. 1.2** *Energy conservation for a steady flow open system*

For most heat transfer equipment, changes in kinetic and potential energy are negligible and no external work is done, thus, SFEE reduces to

$$\dot{m}h_1 + \dot{Q} = \dot{m}h_2$$

or,
$$\dot{Q} = \dot{m}(h_2 - h_1) \tag{1.4}$$

For an ideal gas or an incompressible liquid,

$$h_2 - h_1 = \int_{T_1}^{T_2} c_p \, \mathrm{d}T$$

The *second law of thermodynamics* states that if two bodies at temperatures $T_1$ and $T_2$ are connected, and if $T_1 > T_2$, then heat will flow *spontaneously and irreversibly* from body 1 to body 2, causing entropy increase of the universe or entropy generation. Since, all heat transfer processes occur through finite temperature differences overcoming thermal irreversibility, the heat transfer area or operating variables can be optimized in regard to two or more irreversibilities following the principle of minimization of entropy generation or exergy destruction [3].

The roles of thermodynamics, cost or economics, and heat transfer, simultaneously act upon to yield an energy-efficient equipment, which is now a concern of the engineers.

## 1.2 HEAT CONDUCTION

Conduction refers to the transfer of heat between two bodies or two parts of the same body through molecules which are, more or less, stationary, as in the case of solids.

Fourier's law (after the French scientist J.B.J. Fourier who proposed it in 1822) of heat conduction states that the rate of heat transfer is linearly proportional to the temperature gradient. For one-dimensional or uni-directional heat conduction

$$q_k \propto \cdot \frac{\mathrm{d}T}{\mathrm{d}x}$$

or

$$q_k = -k \frac{\mathrm{d}T}{\mathrm{d}x} \tag{1.5}$$

where $q_k$ is the rate of heat flux (a vector) in W/m$^2$, d$T$/d$x$ is the temperature gradient in the direction of heat flow $x$ and $k$ is the constant of proportionality, which is a property of the material through which heat propagates. This property of the material is called *thermal conductivity* (W/m K). The negative sign is used because heat flows from a high to a low temperature and the slope d$T$/d$x$ is negative (Fig. 1.3). It may be noted that temperature can be given in kelvin or degree Celsius in Eq. (1.5) and the temperature gradient which does not depend on these units is used since one kelvin is equal to one degree Celsius (1 K = 1°C). Thus, the unit of thermal conductivity could also be written as W/m °C, but this is not the recommended practice when using the SI system of units. The magnitude of the thermal conductivity $k$ for a given substance very much depends on its microscopic structure and also tends to vary somewhat with temperature. Table 1.1 gives some selected values of $k$.

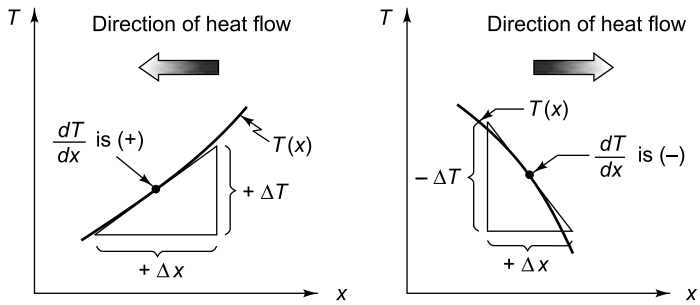


**Fig. 1.3** *Sign convention for conduction heat flow*

For the simple case of steady-state one-dimensional heat flow through a plane wall (Fig. 1.4), the temperature gradient and the heat flow do not vary with time, so that from Eq. (1.5)

$$\bar{q}_k \int_0^L \mathrm{d}x = -\int_{T_1}^{T_2} k \, \mathrm{d}T$$

where the temperature at the left face ($x = 0$) is uniform at $T_1$ and the temperature at the right face ($x = L$) is uniform at $T_2$.

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**Table 1.1** *Thermal conductivity of various materials at 0°C*

| *Material* | *Thermal conductivity* $k$ $W/m \cdot K$ |
|---|---|
| Metals: | |
| Silver (pure) | 410 |
| Copper (pure) | 385 |
| Aluminum (pure) | 202 |
| Nickel (pure) | 93 |
| Iron (pure) | 73 |
| Carbon steel, 1% C | 43 |
| Lead (pure) | 35 |
| Chrome-nickel steel (18% Cr, 8% Ni) | 16.3 |
| Nonmetallic solids: | |
| Diamond | 2300 |
| Quartz, parallel to axis | 41.6 |
| Magnesite | 4.15 |
| Marble | 2.08–2.94 |
| Sandstone | 1.83 |
| Glass, window | 0.78 |
| Maple or oak | 0.17 |
| Sawdust | 0.059 |
| Glass wool | 0.038 |
| Ice | 2.22 |
| Liquids: | |
| Mercury | 8.21 |
| Water | 0.556 |
| Ammonia | 0.540 |
| Lubricating oil, SAE 50 | 0.147 |
| Freon 12, $CCl_2F_2$ | 0.073 |
| Gases: | |
| Hydrogen | 0.175 |
| Helium | 0.141 |
| Air | 0.024 |
| Water vapor (saturated) | 0.0206 |
| Carbon dioxide | 0.0146 |

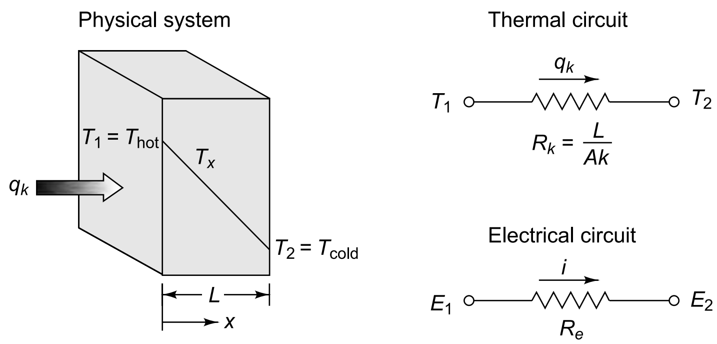


**Fig. 1.4** *Temperature distribution for steady-state conduction through a plane wall and analogy between thermal and electrical circuits*

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If $k$ is independent of $T$, we obtain after integration

$$\bar{q}_k = k\,\frac{T_1 - T_2}{L}$$

If $A$ is the surface area normal to heat flow, then the rate of heat transfer in watts is

$$Q_k = \bar{q}_k A = k\,A\,\frac{T_1 - T_2}{L} \tag{1.6}$$

Since $dt/dx = -\,q_k/k$, for the same $\bar{q}_k$, if $k$ is low (i.e. for an insulator), $dT/dx$ will be large i.e., there will be a large temperature difference across the wall, and if $k$ is high (i.e. for a conductor), $dT/dx$ will be small, or there will be a small temperature difference across the wall (Fig. 1.5).

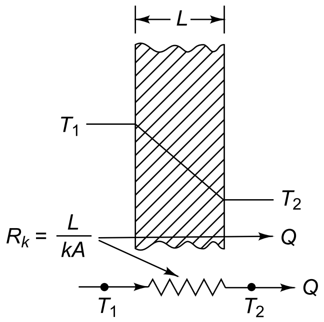


**Fig. 1.5** *Thermal resistance offered by a wall*

### 1.2.1 Resistance Concept

Heat flow has an analogy in the flow of electricity. Ohm's law states that the current $I$ (Ampere) flowing through a wire (Fig. 1.4) is equal to the voltage potential $E_1 - E_2$ (V), divided by the electrical resistance $R_e$ ($\Omega$) or

$$I = \frac{E_1 - E_2}{R_e} \tag{1.7}$$

Since the temperature difference and heat flux in conduction are similar to the potential difference and electric current respectively, the rate of heat conduction through the wall [(Eq. (1.6)] can be written as

$$Q = \frac{T_1 - T_2}{L/kA} = \frac{T_1 - T_2}{R_k} \tag{1.8}$$

where $R_k = L/kA$ is the conductive thermal resistance to heat flow offered by the wall (Fig. 1.5).

Again, the electrical resistance $R_e = \rho\,\frac{l}{A}$, where $\rho$ is the specific resistance ($\Omega$ m), $l$ is the length of the conductor (m) and $A$ is the cross-sectional area of the conductor (m$^2$), Eq. (1.7) can now be written as

$$I = \frac{E_1 - E_2}{\rho\,\frac{l}{A}} = \sigma A\,\frac{E_1 - E_2}{l} = \sigma A\,\frac{dE}{dl}$$

or

$$i = \frac{I}{A} = \text{current density (A/m}^2\text{)}$$

$$= -\,\sigma\,\frac{dE}{dl} \tag{1.9}$$

where $\sigma$ (= $1/\rho$) is the electrical conductivity [($\Omega$ m)$^{-1}$ or mho] and $dE/dl$ is the potential gradient (V/m). The similarity of Eqs (1.5) and (1.9) can be noticed. The reciprocal of the thermal resistance is referred to as the *thermal conductance* $K_k$, defined by

$$K_k = \frac{kA}{L} \tag{1.10}$$

The ratio $k/L$ is the thermal conductance per unit area.

For many materials, thermal conductivity can be approximated as a linear function of temperature over limited temperature ranges:

$$k(T) = k_0(1 + \beta_k T) \tag{1.11}$$

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where $\beta_k$ is an empirical constant and $k_0$ is the value of thermal conductivity at a reference temperature.

The rate of heat conduction is then

$$Q_k = -k_0(1+\beta_k T)A\frac{\mathrm{d}T}{\mathrm{d}x}$$

On integration, $\int_0^L Q_k dx = \int_{T_1}^{T_2} -k_0 A(1+\beta_k T)\mathrm{d}T$

$$Q_k = \frac{k_0 A}{L}\left[(T_1 - T_2) + \frac{\beta_k}{2}(T_1^2 - T_2^2)\right] \quad (1.12)$$

This can be rewritten more simply as

$$Q_k = \frac{k_{av} A}{L}(T_1 - T_2) \quad (1.13)$$

where $k_{av}$ is the value of $k$ at the arithmetic average temperature $(T_1 + T_2)/2$.

The temperature distributions for a constant thermal conductivity ($\beta_k = 0$) and for thermal conductivities that increase ($\beta_k > 0$) and decrease ($\beta_k < 0$) with temperature are shown in Fig. 1.6.

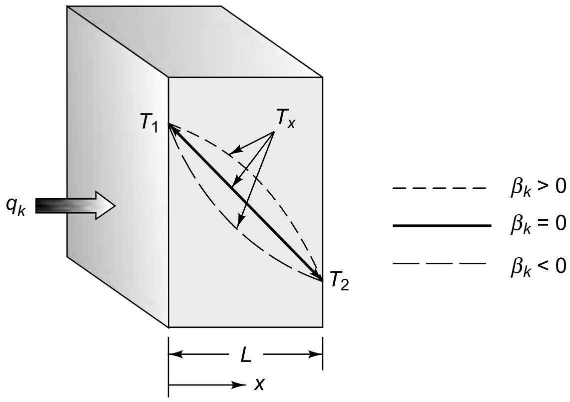


**Fig. 1.6** *Temperature distribution in conduction through a plane wall with constant and variable thermal conductivity*

### 1.2.2 Composite Walls

For a composite wall, as shown in Fig. 1.7, there are three resistances in series. The rate of heat conduction is the same through all sections. The slope of the temperature profile in each depends on the thermal conductivity $k$ of the material of that section. The lower the $k$, the more will be the slope and the higher is the temperature difference. The higher the $k$, the less will be the slope and the lower is the temperature difference. The total thermal resistance.

$$R = R_1 + R_2 + R_3 = \frac{L_1}{k_1 A} + \frac{L_2}{k_2 A} + \frac{L_3}{k_3 A}$$

and the rate of heat conduction

$$Q_k = \frac{T_1 - T_4}{R} = \frac{T_1 - T_4}{\dfrac{L_1}{k_1 A} + \dfrac{L_2}{k_2 A} + \dfrac{L_3}{k_3 A}} \quad (1.14)$$

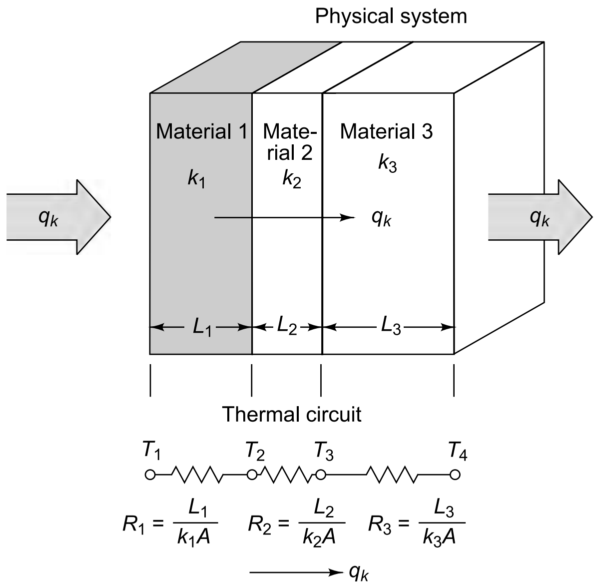


**Fig. 1.7** *Conduction through three resistances in series*

Also,

$$Q_k = \left(\frac{kA}{L}\right)_1 (T_1 - T_2) = \left(\frac{kA}{L}\right)_2 (T_2 - T_3)$$

$$= \left(\frac{kA}{L}\right)_3 (T_3 - T_4) \quad (1.15)$$

where $T_2$ and $T_3$ are the interface temperatures. The walls are assumed to be in good thermal contact, with no contact resistance.

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Conduction can occur in a wall with two different materials in parallel (Fig. 1.8). If the temperatures over the left and right faces are uniform at $T_1$ and $T_2$, the equivalent thermal circuit is shown to the right of the physical system. The total resistance $R$ is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

or

$$R = \frac{R_1 R_2}{R_1 + R_2}$$

where $R_1 = \dfrac{L}{k_1 A_1}$ and $R_2 = \dfrac{L}{k_2 A_2}$

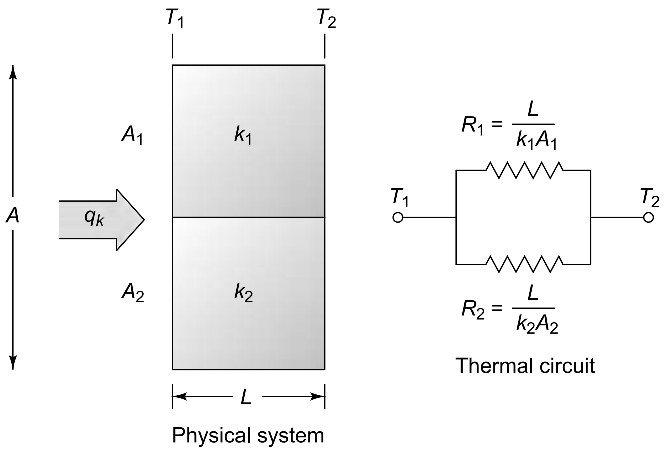


**Fig. 1.8** *Conduction through two resistances in parallel*

The rate of heat conduction

$$Q_k = \frac{T_1 - T_2}{R} = \frac{(T_1 - T_2)}{L}(k_1 A_1 + k_2 A_2) \tag{1.16}$$

Since heat is conducted through two separate parallel paths between the same temperature difference,

$$Q_k = Q_1 + Q_2 = \frac{T_1 - T_2}{\left(\frac{L}{kA}\right)_1} + \frac{T_1 - T_2}{\left(\frac{L}{kA}\right)_2}$$

In Fig. 1.9 heat is transferred through a more complex composite structure involving thermal resistances in series and in parallel. The total resistance is

$$R_T = R_1 + R_2 + R_3 = R_1 + \frac{R_2 R_3}{R_2 + R_3} + R_4$$

$$= \frac{L_1}{k_1 A_1} + \left(\frac{L_2}{k_2 A_2}\frac{L_3}{k_3 A_3}\right)\left(\frac{1}{\frac{L_2}{k_2 A_2} + \frac{L_3}{k_3 A_3}}\right) + \frac{L_4}{k_4 A_4}$$

$$= \frac{L_1}{k_1 A_1} + \frac{L_2}{k_2 A_2 + k_3 A_3} + \frac{L_4}{k_4 A_4} \tag{1.17}$$

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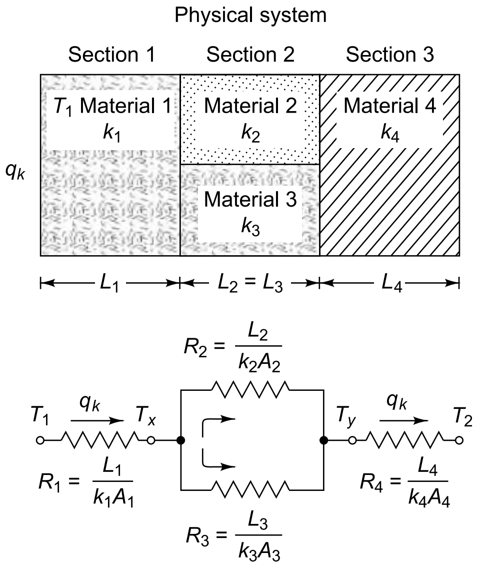


**Fig. 1.9** *Heat conduction through resistances in series and in parallel*

The rate of heat flow is

$$Q_k = \frac{(\Delta T)_{\text{overall}}}{R_T} = \frac{T_1 - T_2}{R_T} \tag{1.18}$$

### 1.2.3 Contact-Resistance

Thermal contact resistance develops when two conducting surfaces do not fit tightly together and a thin layer of fluid is trapped between them (Fig. 1.10). This resistance is primarily a function of surface

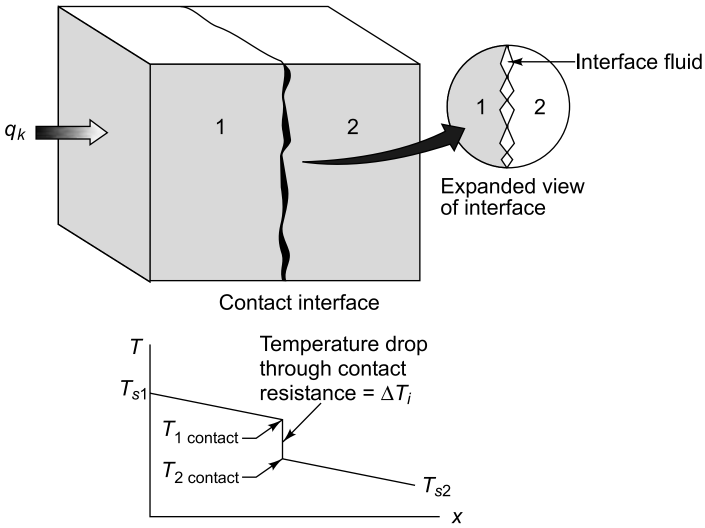


**Fig. 1.10** *Contact resistance between two solid bodies*

roughness, the pressure holding the two surfaces in contact, the interface fluid and the interface temperature.

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The direct contact between the solid surfaces, as shown in the expanded view, takes place at a limited number of spots, and the voids between them usually are filled with air or the surrounding fluid. Heat transfer through the fluid filling the voids is mainly by conduction, since there is no convection in such a thin layer of fluid and the radiation effects are negligible at normal temperatures.

If the heat flux through the two solid surfaces in contact is $\bar{q}$ and the temperature difference across the fluid gap is $\Delta T_i$ (= $T_{s1} - T_{s2}$), the interface resistance $R_i$ is defined by

$$R_i = \frac{\Delta T_i}{q}$$

The effect of contact pressure on the thermal contact resistance between metal surfaces under vacuum conditions is presented in Table 1.2. An increase in contact pressure can reduce the contact resistance significantly. The interfacial fluid also affects the thermal resistance, as shown in Table 1.3. Putting a viscous liquid like glycerin on the interface reduces the contact resistance 10 times with respect to air at a given pressure. A thermally conducting liquid called a *thermal grease* such as silicone oil is applied between the contact surfaces before they are pressed against each other. It is commonly done when attaching electronic components such as power transistors to heat sinks.

**Table 1.2** *Thermal contact resistance at different contact pressures under vacuum conditions [1]*

| *Interface material* | *Resistance $R_i$($m^2$ K/W × $10^4$)* | |
|---|---|---|
| | *Contact pressure (1 bar)* | *Contact pressure (100 bar)* |
| Stainless steel | 6–25 | 0.7–4.0 |
| Copper | 1–10 | 0.1–0.5 |
| Magnesium | 1.5–3.5 | 0.2–0.4 |
| Aluminium | 1.5–5.0 | 0.2–0.4 |

**Table 1.3** *Thermal contact resistance for aluminium–aluminium interface with different interfacial fluids, having 1 μm surface roughness under 1 bar contact pressure [1]*

| *Interfacial fluid* | *Resistance $R_i$ ($m^2$ K/W)* |
|---|---|
| Air | $2.75 \times 10^{-4}$ |
| Helium | $1.05 \times 10^{-4}$ |
| Hydrogen | $0.72 \times 10^{-4}$ |
| Silicone oil | $0.525 \times 10^{-4}$ |
| Glycerine | $0.265 \times 10^{-4}$ |

Numerous experimental measurements have been made of the contact resistance at the interface between dissimilar metallic surfaces, but no satisfactory correlations have been found [2].

### 1.2.4 Thermal Conductivity

As defined by Fourier's law, Eq. (1.5), the thermal conductivity is

$$\mathbf{k} = -\frac{\bar{\mathbf{q}}_\mathbf{k}}{\mathbf{dT/dx}}$$

This equation can be used to determine the thermal conductivity of a material. A layer of solid material of known thickness and area can be heated from one side by an electric resistance heater of known output. If the other surface of the heater is perfectly insulated, all the heat generated by the resistance heater will be transferred through the material whose conductivity is to be determined. Then measuring the two surface temperatures $T_1$ and $T_2$ of the layers of material when steady state has been reached, the thermal

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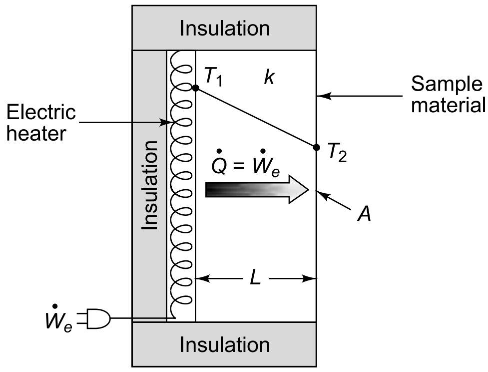
conductivity can be estimated, as shown in Fig. 1.11.

For engineering purposes, the experimentally measured values of thermal conductivity are generally used. These values can be predicted fairly well for gases with the help of kinetic theory of gases. But in the case of liquids and solids, theories are not adequate to predict thermal conductivity with sufficient accuracy. Table 1.1 gives values of thermal conductivity for several materials. It may be noted that pure metals are the best conductors and gases are the poorest ones.




$$k = \frac{L}{A(T_1 - T_2)} \dot{Q}$$

**Fig. 1.11** *A simple experimental set up to determine the thermal conductivity of a material*

The mechanism of thermal conduction in a gas can be explained on a molecular level from basic concepts of the kinetic theory of gases [3]. The kinetic energy (KE) of a molecule is a function of temperature. Molecules in a high-temperature region have higher KE and hence higher velocities than those in a lower-temperature region. Since molecules are in continuous random motion, as they collide with one another they exchange energy as well as momentum. When a molecule moves from a higher-temperature region to a lower-temperature region, it transports KE from the higher- to the lower-temperature part of the system. Upon collision with slower molecules, the faster molecule gives up some of its energy. In this manner thermal energy is transferred from higher to lower-temperature regions in gas by molecular motion.

The faster the molecules move, the faster they will transport energy. Thus, the transport property called thermal conductivity depends on the temperature of the gas. At moderate pressures the space between molecules is large compared to the size of a molecule. Thermal conductivity of gases is therefore essentially independent of pressure (and density). Figure 1.12(a) shows how the thermal conductivities of some typical gases vary with temperature.

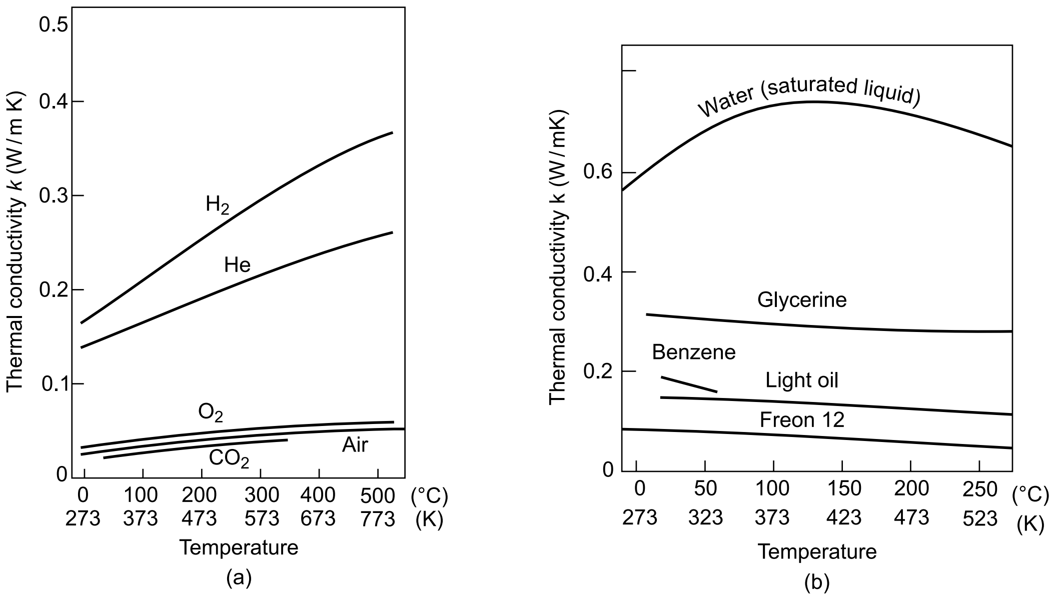



**Fig. 1.12** *Variation of thermal conductivity with temperature for (a) gases and (b) liquids*

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The basic mechanism of heat conduction in liquids is qualitatively similar to that in gases. However, molecular conditions in liquids are more difficult to describe and the details of the conduction mechanisms in liquids are not well understood. The curves in Fig. 1.12(b) show the thermal conductivity of some non-metallic liquids as a function of temperature. For most liquids, the thermal conductivity decreases with temperature, but water is a notable exception. Generally, the thermal conductivity of liquids decreases with increasing molecular weight.

Solid materials consist of free electrons and atoms in a periodic lattice arrangement. Heat can be conducted in a solid by two mechanisms:

(a) migration of free electrons ($k_e$)

(b) lattice vibration ($k_l$)

These two effects are additive, i.e. $k = k_e + k_l$. But in general, the transport due to electrons is more effective than the transport due to vibrational energy in the lattice structure ($k_e > k_l$). Since electrons transport electric charge in a manner similar to the way in which they carry thermal energy from a higher to a lower-temperature region, good electrical conductors are also good heat conductors, whereas good electrical insulators are poor heat conductors. In non-metallic solids there is little or no electronic transport and the conductivity is therefore determined primarily by lattice vibration. Thus, these materials have lower thermal conductivities than metals. Thermal conductivities of some typical metals and alloys are shown in Fig. 1.13.

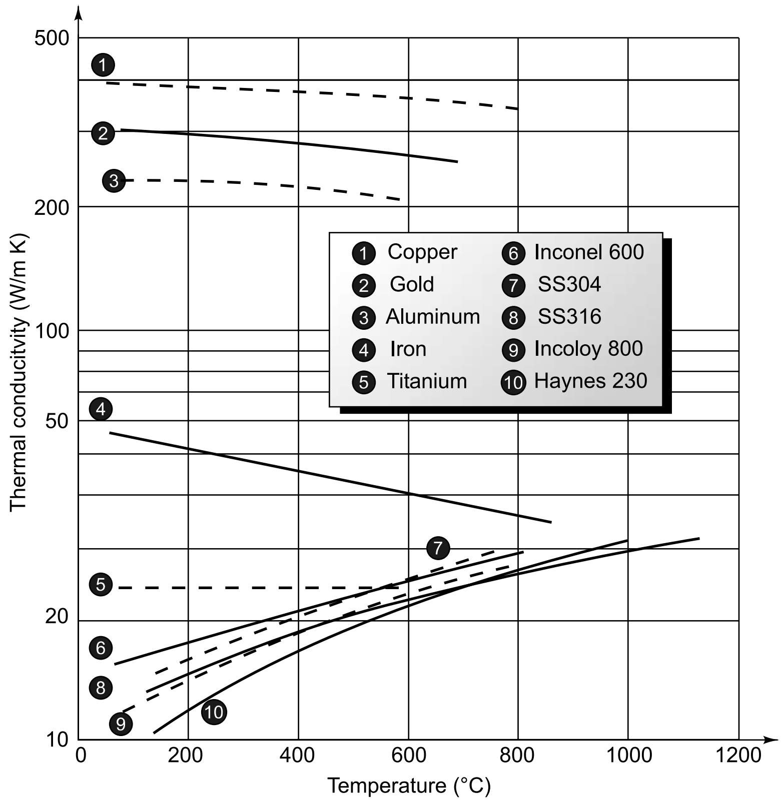


**Fig. 1.13** *Variation of thermal conductivity with temperature for metals and alloys*

The regularity of lattice arrangement has an important effect on $k_l$, with crystalline (well-ordered) materials such as quartz having a higher thermal conductivity than amorphous materials such as glass. That's why diamond, a crystalline substance, has a high thermal conductivity.

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Thermal conductivity of a substance depends on temperature

$$k = k_0 (1 + \beta_k T)$$

where $\beta_k$ is small and negative for most solids and liquids, and positive for gases.

### 1.2.5 Heat Conduction Through a Cylinder

Let us assume that the inside and outside surfaces of the cylinder (Fig. 1.14) are maintained at temperatures $T_1$ and $T_2$ respectively, and $T_1$ is greater than $T_2$. Heat will be assumed to be flowing under steady state only in the radial direction, and there is no heat conduction along the length or the periphery of the cylinder. The rate of heat transfer through the thin cylinder of thickness $dr$ is given by

$$Q_k = -kA \frac{\mathrm{d}T}{\mathrm{d}r} = -k2\pi rL \frac{\mathrm{d}T}{\mathrm{d}r} \tag{1.19}$$

where $L$ is the length of the cylinder.

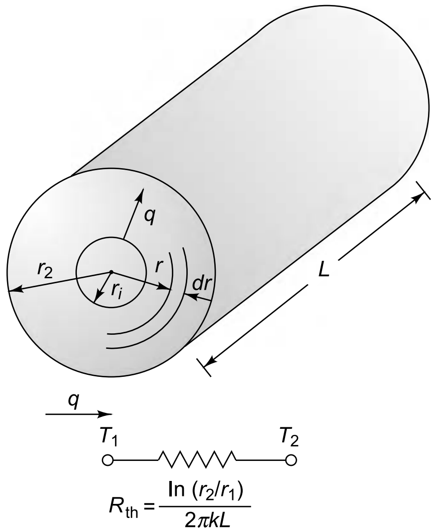


**Fig. 1.14** *One-dimensional heat flow through a hollow cylinder and electrical analog*

$$\int_{T_1}^{T_2} \mathrm{d}T = \int_{r_1}^{r_2} -\frac{Q_k}{2\pi k L} \frac{dr}{r}$$

$$T_2 - T_1 = -\frac{Q_k}{2\pi k L} \ln \frac{r_2}{r_1}$$

$$Q_k = \frac{2\pi k L (T_1 - T_2)}{\ln \left(\dfrac{r_2}{r_1}\right)} \tag{1.20}$$

The above equation can also be written in the following form,

$$Q_k = \frac{2\pi (r_2 - r_1) Lk (T_1 - T_2)}{(r_2 - r_1) \ln \dfrac{2\pi r_2 L}{2\pi r_1 L}} = k \frac{(A_2 - A_1)}{\ln \dfrac{A_2}{A_1}} \frac{(T_1 - T_2)}{r_2 - r_1}$$

$$= -\mathbf{kA_{lm}} \frac{\mathbf{T_2 - T_1}}{\mathbf{x_\omega}} \tag{1.21}$$

where $A_{1m}$ = log-mean area = $\dfrac{A_2 - A_1}{\ln \dfrac{A_2}{A_1}}$, $A_2$ = outside surface area = $2\pi r_2 L$, $A_1$ = inside surface area = $2\pi r_1 L$

and $x_\omega$ = wall thickness of the cylinder = $r_2 - r_1$.

The thermal resistance offered by the cylinder wall to radial heat conduction is then

$$R_k = \frac{T_2 - T_1}{Q_k} = \frac{x}{k \cdot A_{lm}} \tag{1.22}$$

From Eq. (1.19),

$$\mathrm{d}T = -\frac{Q_k}{2\pi k L} \frac{\mathrm{d}r}{r} = C_1 \frac{\mathrm{d}r}{r}$$

$$T = C_1 \ln r + C_2 \tag{1.23}$$

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where $C_1$ and $C_2$ are constants to be evaluated from the conditions:

when $\quad r = r_1, T = T_1$

$\quad r = r_2, T = T_2$

The temperature across the wall of the cylinder varies logarithmically with radius.

For two cylindrical resistances in series (Fig. 1.15),

$$R = R_1 + R_2 = \frac{x_{\omega_1}}{k_1\ A_{lm1}} + \frac{x_{\omega_2}}{k_2\ A_{lm2}}$$

where $x_{\omega_1} = r_2 - r_1, x_{\omega_2} = r_3 - r_2$

$$A_{lm1} = \frac{A_2 - A_1}{\ln \dfrac{A_2}{A_1}} = \frac{2\pi (r_2 - r_1) L}{\ln \dfrac{r_2}{r_1}}$$

and

$$A_{lm2} = \frac{A_3 - A_2}{\ln \dfrac{A_3}{A_2}} = \frac{2\pi (r_3 - r_2) L}{\ln \dfrac{r_3}{r_2}}$$

The rate of heat transfer will be

$$\mathbf{Q_k} = \frac{\mathbf{T_1 - T_3}}{\mathbf{R}} = \frac{\mathbf{T_1 - T_2}}{\mathbf{R_1}} = \frac{\mathbf{T_2 - T_3}}{\mathbf{R_2}}$$

from which the interface temperature $T_2$ can be evaluated.

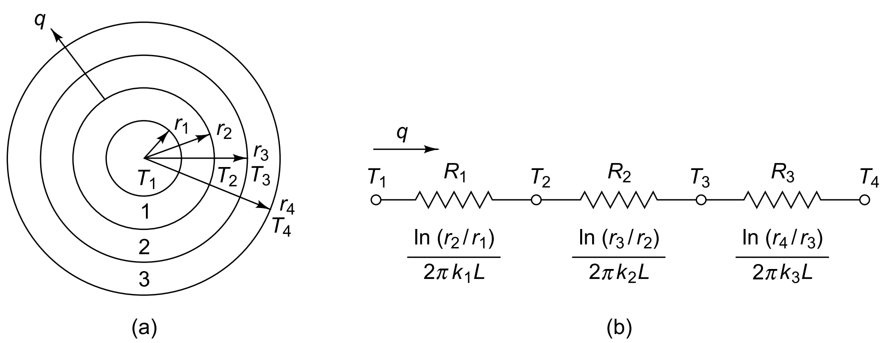


**Fig. 1.15** *One-dimensional heat flow through multiple cylindrical sections and electrical analog*

### 1.2.6 Heat Conduction Through a Sphere

Heat flowing through a thin spherical strip (Fig. 1.16) at radius $r$ of thickness d$r$ is

$$Q_k = -kA\frac{\mathrm{d}T}{\mathrm{d}r}$$

where $A$ is the spherical surface at radius $r$ normal to heat flow

$$Q_k = -k4\pi r^2\frac{\mathrm{d}T}{\mathrm{d}r}$$

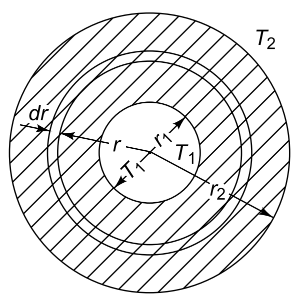


**Fig. 1.16** *Heat conduction through a sphere*

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$$\int_{T_1}^{T_2} \mathrm{d}T = \int_{r_1}^{r_2} -\frac{Q_k}{4\pi k}\frac{\mathrm{d}r}{r^2}$$

$$T_2 - T_1 = -\frac{Q_k}{4\pi k}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

$$Q_k = \frac{4\pi k (T_1 - T_2)\, r_1 r_2}{r_2 - r_1} \tag{1.24}$$

or
$$\mathbf{Q_k = -kA_{gm}\frac{T_2 - T_1}{x_\omega}} \tag{1.24a}$$

where $A_{gm}$ = geometric mean area

$= (A_1A_2)^{1/2} = (4\pi r_1^2\, 4\pi r_2^2)^{1/2}$

$= 4\pi r_1 r_2$

and $x_\omega$ = wall thickness of the sphere = $r_2 - r_1$

Here, the thermal resistance offered by the wall to heat conduction is

$$R_k = \frac{x}{k\,A_{gm}} \tag{1.24b}$$

Thus, similar expressions of thermal resistance hold good for flat plate, cylinder and sphere, which are

$$R_{\text{plate}} = \frac{x}{kA} \qquad R_{\text{cylinder}} = \frac{x}{kA_{lm}}$$

$$R_{\text{sphere}} = \frac{x}{k\,A_{gm}}$$

where $k$ is the thermal conductivity of the wall material.

## 1.3 CONVECTION HEAT TRANSFER

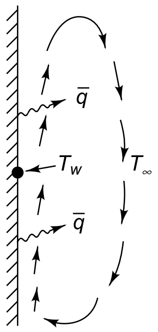


**Fig. 1.17** *Natural or free convection of air ($T_w > T_\infty$)*

Convection is a process involving mass movement of fluids. When a temperature difference produces a density difference which results in mass movement (Fig. 1.17), the process is called *free* or *natural convection*. Here the plate is maintained isothermal at temperature $T_w$, which is higher than the surrounding fluid temperature $T_\infty$. The fluid near the wall, on getting heated, moves up due to the effect of buoyancy, and is replaced by the cold fluid moving towards the wall. Thus a circulation current is set up due to the density difference.

When the mass motion of the fluid is caused by an external device like a pump, compressor, blower or fan, the process is called *forced convection* (Fig. 1.18). Here the fluid is made to flow along the hot surface due to the pressure difference generated by the device and heat is transferred from the wall to the fluid.

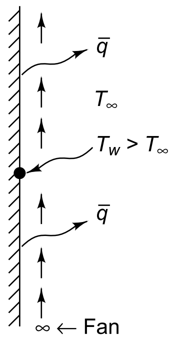


**Fig. 1.18** *Forced convection of air ($T_w > T_\infty$).*

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Whether the convection process is natural or forced, there is always a boundary layer adjacent to the wall where the velocity and temperature vary from the wall to the free stream. Figure 1.19 shows the velocity and temperature boundary layers for forced flow over a hot horizontal surface. Ludwig Prandtl suggested that the field of flow can be divided into two regions: a thin layer next to the wall, which he called the *boundary layer* where the shear stress is confined, and the region outside this layer, where the fluid is

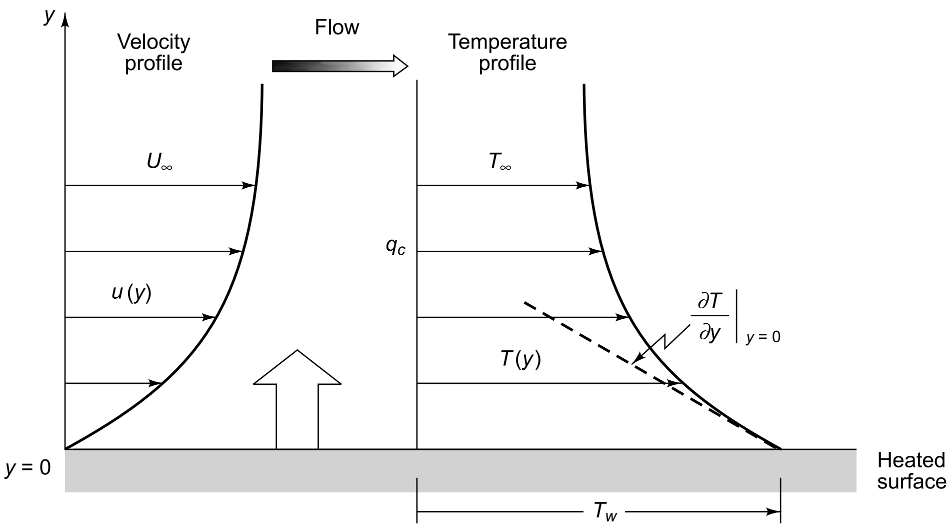


**Fig. 1.19** *Velocity and temperature profiles for forced convection heat transfer*

"ideal" i.e., nonviscous and incompressible. The *boundary layer thickness* $\delta$ is defined as the distance from the wall where $u = 0.99u_\infty$ i.e., the fluid velocity is 99% of the free stream velocity. As a result of viscous forces the velocity of the fluid is zero at the wall and increases to $u_\infty$, as shown.

A *thermal boundary layer thickness* $\delta_t$ is defined in an exactly analogous manner to the velocity boundary layer thickness $\delta$. Within this layer, the temperature varies from $T_w$ at the wall to $T_\infty$ in the undisturbed flow. Since temperature within the boundary layer approaches $T_\infty$ asymptotically, $\delta_t$ is defined as the thickness at which $T_w - T = 0.99 (T_w - T_\infty)$. In general, $\delta_t$ is not equal to $\delta$.

Figure 1.20 shows the velocity and temperature profiles under conditions of natural convection. The velocity at first increases with increasing distance from the surface, reaches a maximum and then decreases to approach zero value. The reason for this behaviour is that the action of viscosity diminishes rapidly with distance from the surface, while the density difference decreases more slowly. The buoyancy force decreases as the fluid density approaches that of unheated surrounding fluid. The temperature field is, however, the same as in forced convection.

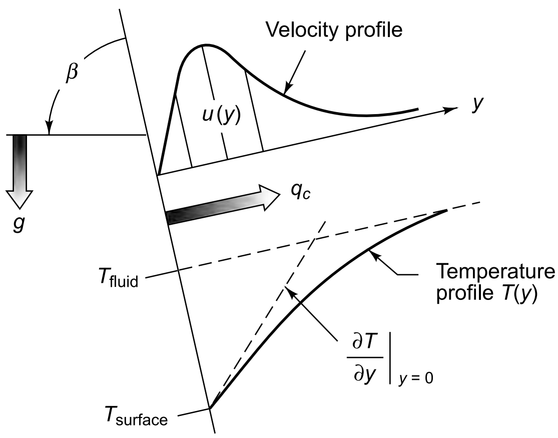


**Fig. 1.20** *Velocity and temperature profiles for natural convection heat transfer*

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The thermal boundary layer is regarded as consisting of a stationary fluid film (Fig. 1.21) through which heat is conducted and then it is transported by fluid motion [4]. The rate of convection heat transfer from the wall to the fluid

$$\mathbf{Q_c = -k_f A \frac{T_\infty - T_w}{\delta_t}} \tag{1.25}$$

where $k_f$ is the thermal conductivity of the fluid film.

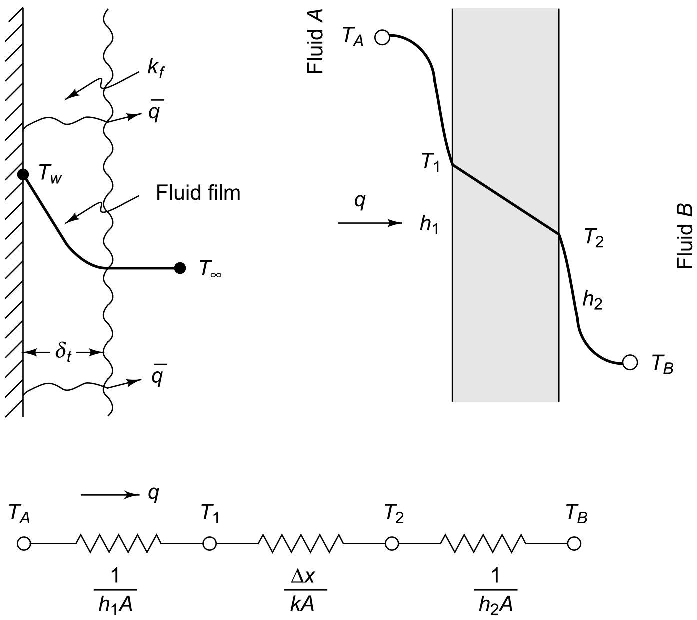


**Fig. 1.21** *Heat transfer through a stationary fluid film*

The film or surface coefficient of heat transfer $h_c$ may be defined as

$$\boldsymbol{h_c = \frac{k_f}{\delta_t}} \tag{1.26}$$

$$\mathbf{Q_c = h_c A (T_w - T_\infty)} \tag{1.27}$$

The rate of heat transfer $Q_c$ increases with the increase in the value of heat transfer coefficient $h_c$. The higher the value of $k_f$ and the lower the value of $\delta_t$, the higher will be the value of $h_c$, and hence, $Q_c$. As the velocity of the fluid increases, film thickness $\delta_t$ decreases and $h_c$ increases. For gases $k_f$ is low, and so $Q_c$ will also be low, compared to a liquid.

The above equation is known as *Newton's law of cooling*. Strictly speaking, convection applies to fluid motion. The mechanism of heat transfer is by conduction:

$$Q_c = -k_f A \left(\frac{\partial T}{\partial y}\right)_{y=0} = h_c A (T_w - T_\infty)$$

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and

$$h_c = \frac{Q_c / A}{T_w - T_\infty} = \frac{-k_f\,(\partial T/\partial y)_w}{T_w - T_\infty}$$

The thermal resistance offered by the fluid film

$$\mathbf{R_c = \frac{T_w - T_\infty}{Q_c} = \frac{1}{h_c A}\ (K/W)} \tag{1.28}$$

## 1.4 COMBINED CONDUCTION AND CONVECTION

Both forced and natural convection flows can be either *laminar* or *turbulent*, with laminar flows being predominant at lower velocities, for smaller sizes, and for more viscous fluids. Flow in a pipe becomes turbulent when the dimensionless group called the *Reynolds number*, $\mathrm{Re}_d = VD/v$, exceeds about 2300, where $V$ is the velocity (m/s), $D$ is the pipe diameter (m), and $v$ is the kinematic viscosity of the fluid (m$^2$/s). Heat transfer rates tend to be much higher in turbulent flows than in laminar flows owing to the vigorous mixing of the fluid. Figure 1.22 shows some commonly encountered flows.

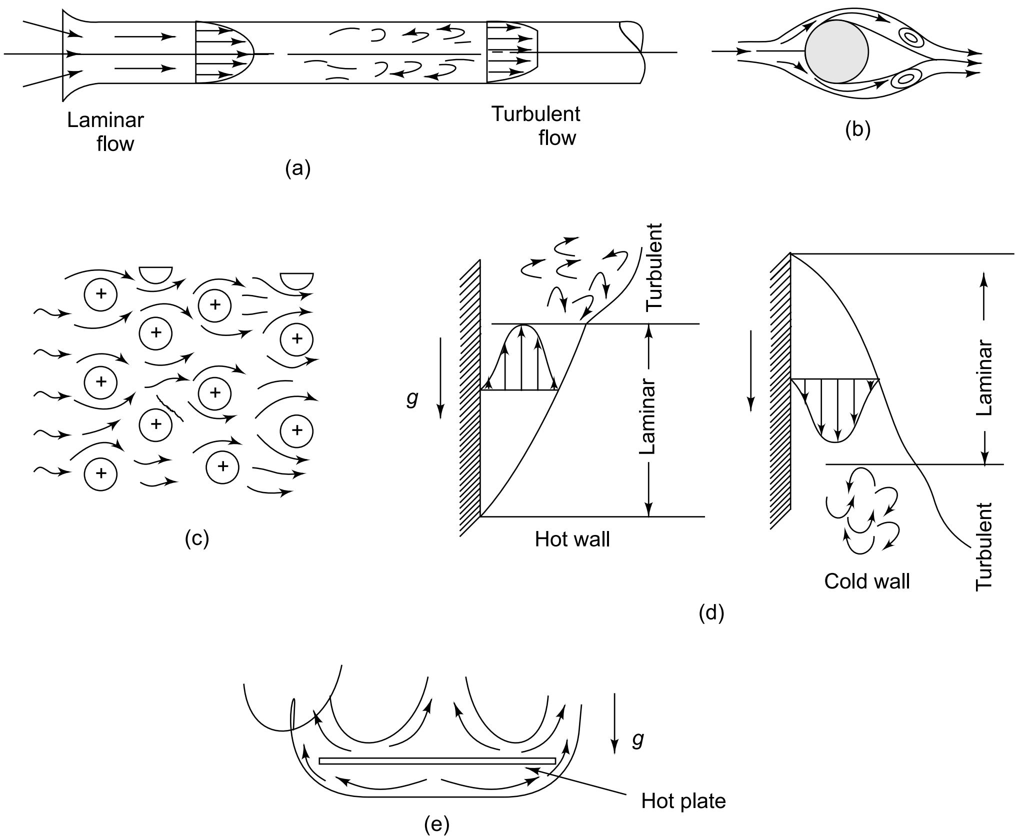


**Fig. 1.22** *Some common flows: (a) Forced flow in a pipe, $\mathrm{Re}_d \cong 50{,}000$. The flow is initially laminar because of 'bell-mouth' entrance, but becomes turbulent downstream. (b) Laminar forced flow over a cylinder, $\mathrm{Re}_d = 25$. (c) Forced flow through a tube bank, (d) Laminar and turbulent natural convection boundary layers on vertical walls. (e) Laminar natural convection about a heated horizontal plate.*

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In all flows involving heat transfer and, therefore, temperature changes, the buoyancy forces arising from the gravitational field will exist. The term forced convection is only applied to flows in which the effects of these buoyancy forces are negligible. In some flows in which a forced velocity exists, the effects of these buoyancy forces may not be negligible. Such flows are termed combined or mixed, free and forced convection flows (Fig. 1.23).

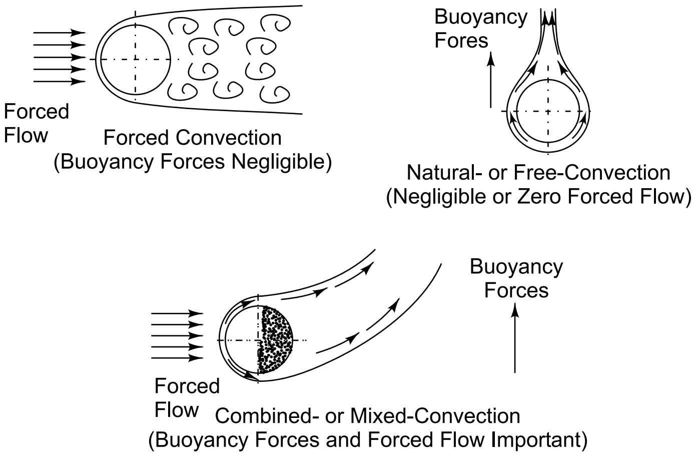


**Fig. 1.23** *Forced, free and mixed convection*

Convective flows can be classified as external and internal flows (Fig. 1.24). External flows involve a flow over the outer surface of a body and internal flows involve the flow through a duct or channel. The velocity and temperature profiles of a flow through a pipe are given in Fig. 1.25. The rate of heat transfer from the wall to the mean or bulk temperature of the fluid is given by

$$q_c = \frac{Q_c}{A} = h_c\,(T_w - T_b) \tag{1.29}$$

where $h_c$ is the heat transfer coefficient (W/m$^2$ K) and $T_b$ is the bulk temperature of the fluid given by

$$\mathbf{T_b} = \frac{\int_A \rho\, u\, c_p A\, T\, dA}{c_p \int_A \rho\, u\, dA} = \frac{\int_A u\, T\, dA}{\int_A u\, dA} \tag{1.30}$$

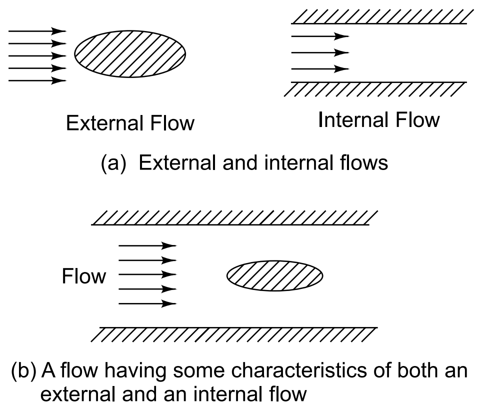


**Fig. 1.24** *External and internal fluid flows*

Since the velocity and temperature of the fluid are varying with temperature, the bulk temperature is equal to the temperature that would be attained if the fluid at a particular section of the duct being considered was discharged into a container and without any heat transfer occurring,

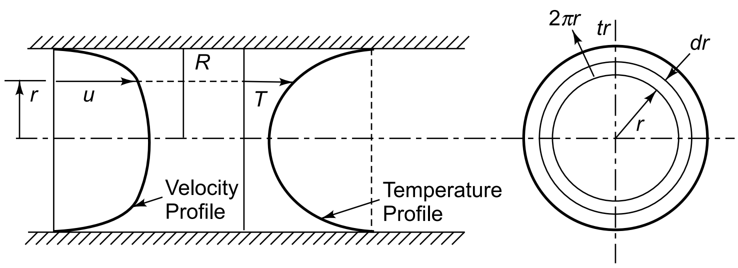


**Fig. 1.25** *Velocity and temperature profiles of flow in a pipe*

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was mixed until a uniform temperature was obtained. Thus, it is sometimes referred to as the '*mixing cup temperature*'.

Equation (1.29) gives the local rate of heat transfer per unit area, the local heat transfer coefficient, the local wall temperature and the local fluid (bulk) temperature. In general, all of these quantities, i.e., $q_w$, $h_c$, $T_w$ and $T_b$ vary with position on the wall or surface. It is, therefore, convenient to define a mean or average heat transfer coefficient, $\bar{h}_c$ such that if $Q$ is the total heat transfer rate from the entire surface of area $A$, then

$$Q = \bar{h}_c\, A(\bar{T}_w - \bar{T}_b)$$

Normally, the bars on $\bar{h}_c$, $\bar{T}_w$ and $\bar{T}_b$ indicating the mean values are omitted, thus

$$Q = h_c\, A\,(T_w - T_b) \tag{1.31}$$

If the pipe has a uniform wall temperature $T_w$ along its length and the flow is laminar ($Re_d \le 2300$), then sufficiently far from the pipe entrance, the heat transfer coefficient is given by the exact relation.

$$h_c = 3.66\,\frac{k}{D} \tag{1.32}$$

where $k$ is the fluid thermal conductivity and $D$ is the pipe diameter. It may be noted that $h_c$ is directly proportional to $k$, inversely proportional to $D$, and surprisingly independent of flow velocity.

On the other hand, for turbulent flow ($Re_d \ge 10{,}000$), $h_c$ is given by the following equation correlated from experimental data

$$\mathbf{h_c = 0.023\,\frac{V^{0.8}\,(\rho\, c_p)^{0.4}\,k^{0.6}}{D^{0.2}\;\nu^{0.4}}} \tag{1.33}$$

In contrast to laminar flow, $h_c$ is now strongly dependent on velocity $V$, but weekly dependent on diameter, in addition to fluid properties $k$, $\rho$, $c_p$ and $\nu$.

Figure 1.26 shows a natural convection flow on a heated vertical surface, as well as associated variation of $h_c$ along the surface. Transition from a laminar to a turbulent boundary layer is shown. In gases, the location of the transition is determined by a critical value of a dimensionless group called the Grashof number, defined as

$$\mathbf{Gr_x = \frac{g\beta\,\Delta T\,x^3}{\nu^2}}\ ,$$

where $\Delta T = T_w - T_\infty$, $x$ = distance from the bottom of the surface where the boundary layer starts, and $\beta$ = volumetric coefficient of expansion, which for an ideal gas is simply $1/T$ (K$^{-1}$). On a vertical plate, transition occurs at $Gr_x = 10^9$. For air, experiments show that for

Laminar flow: $h_c = 1.07\,(\Delta T/x)^{1/4}$ W/m$^2$ K (1.34)
for $10^4 < Gr_x < 10^9$

Turbulent flow: $h_c = 1.3\,(\Delta T)^{1/3}$ W/m$^2$ K (1.35)
for $10^9 < Gr_x < 10^{12}$

Since these are dimensional equations, it is necessary to specify the units: $h_c$ in W/m$^2$ K, $\Delta T$ in K and $x$ in m. It is noted that $h_c$ varies with $x^{-1/4}$ in laminar region but is independent of $x$ in the turbulent region.

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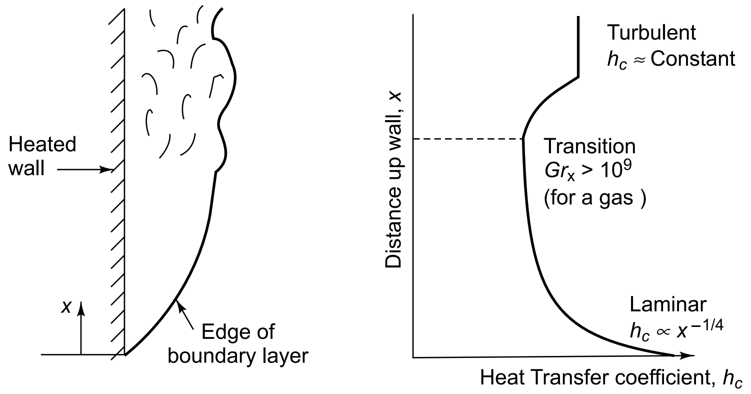


**Fig. 1.26** *A natural convection heat transfer coefficient varying with distance x along with the development of boundary layer*

Table 1.4 gives some order-of-magnitude values of average heat transfer coefficients for various situations.

**Table 1.4** *Typical values of the convective heat transfer coefficient h*

| *Type of flow* | *h, W/(m²K)* |
|---|---|
| *Free comection, ΔT = 25°C* | |
| • 0.25-m vertical plate in: | |
| Atmospheric air | 5 |
| Engine oil | 37 |
| Water | 440 |
| • 0.02-m-OD* horizontal cylinder in: | |
| Atmospheric air | 8 |
| Engine oil | 62 |
| Water | 741 |
| • 0.02-m-diameter sphere in: | |
| Atmospheric air | 9 |
| Engine oil | 60 |
| Water | 606 |
| *Forced convection* | |
| • Atmospheric air at 25°C with $u_\alpha$ = 10 m/s over a flat plate: | |
| $L$ = 0.1 m | 39 |
| $L$ = 0.5 m | 17 |
| • Flow at 5 m/s across 1-cm-OD cylinder of: | |
| Atmospheric air | 85 |
| Engine oil | 1,800 |
| • Water at 1 kg/s inside 2.5-cm-ID† tube | 10,500 |
| *Boiling of water at 1 atm* | |
| • Pool boiling in a container | 3,000 |
| • Pool boiling at peak heat flux | 35,000 |
| • Film boiling | 300 |
| *Condensation of steam at 1 atm* | |
| • Film condenstation on horizontal tubes | 9,000-25,000 |
| • Film condensation on vertical surfaces | 4,000-11,000 |
| • Dropwise condensation | 60,000-120,000 |

* OD = Outer diameter
† ID = inner diameter

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## 1.5 OVERALL HEAT TRANSFER COEFFICIENT

The problem largely encountered in engineering practice is heat being transferred between two fluids of specified temperatures separated by a wall (Fig. 1.27). In such a situation the surface temperatures are not known, but they can be calculated if the convection heat transfer coefficients on both sides of the wall are known.

There are three resistances in series:

$$R = R_1 + R_2 + R_3 = \frac{1}{h_{c,1}A} + \frac{x}{kA} + \frac{1}{h_{c,2}A}$$

Now,

$$Q_c = \frac{T_h - T_c}{R} = \frac{T_h - T_c}{(1/h_{c,1}A) + (x/kA) + (1/h_{c,2}A)}$$

$$= \mathbf{UA\ (T_h - T_c)} \qquad (1.36)$$

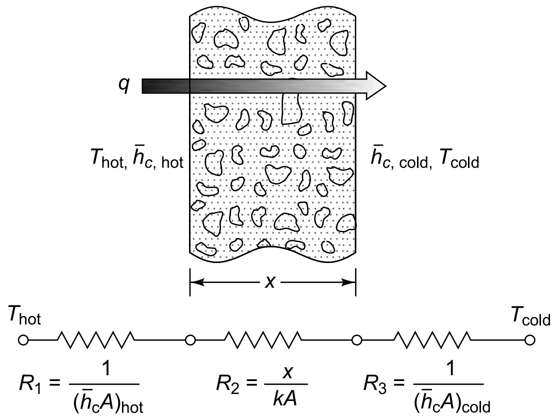


**Fig. 1.27** *Thermal circuit with conduction and convection in series*

where $U$ is known as the *overall heat transfer coefficient* (W/m$^2$K) and is given by

$$\frac{1}{UA} = \frac{1}{h_{c,1}A} + \frac{x}{k\,A} + \frac{1}{h_{c,2}A} \qquad (1.37)$$

or

$$\frac{1}{U} = \frac{1}{h_{c,1}} + \frac{x}{K} + \frac{1}{h_{c,2}}$$

For a composite wall with three different layers in series (Fig. 1.28)

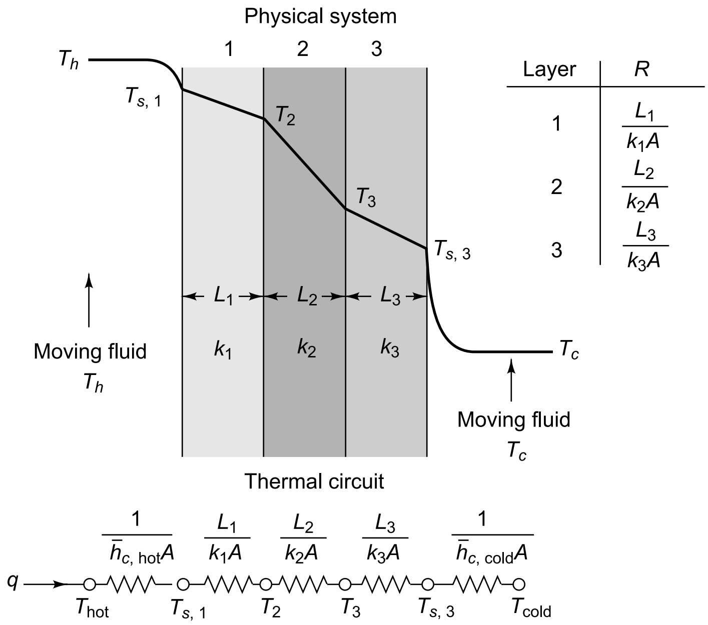


**Fig. 1.28** *Heat transfer through a three-layer composite wall with convection over both exterior surfaces*

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$$\frac{1}{UA} = \frac{1}{h_{c,\,\mathrm{hot}}\,A} + \frac{L_1}{k_1 A} + \frac{L_2}{k_2 A} + \frac{L_3}{k_3 A} + \frac{1}{h_{c,\,\mathrm{cold}}\,A}$$

and

$$Q = UA\,(T_h - T_c) \tag{1.38}$$

Similarly, for heat transfer from a hot fluid inside a cylinder to the cold fluid outside (Fig. 1.29)

$$Q_c = \frac{T_h - T_c}{R_1 + R_2 + R_3} = \frac{T_h - T_c}{\dfrac{1}{h_i\,A_i} + \dfrac{x_w}{k_w\,A_{lm}} + \dfrac{1}{h_o\,A_o}}$$

$$= \mathbf{U_o\,A_o\,(T_h - T_c)}$$

where
$$\frac{1}{U_o\,A_o} = \frac{1}{h_i\,A_i} + \frac{x_w}{k_w\,A_{lm}} + \frac{1}{h_o\,A_o} \tag{1.39}$$

$U_o$ being the overall heat transfer coefficient based on the outside surface area $A_o$, $h_i$ the inside heat transfer coefficient and $h_o$ the outside heat transfer coefficient.

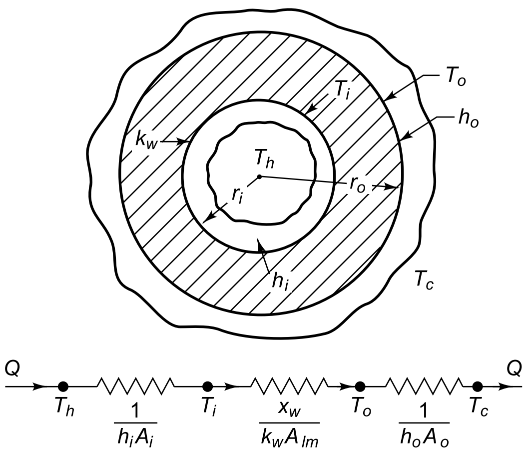


**Fig. 1.29** *Radial heat transfer from a hot to a cold fluid through a cylindrical wall*

Now,

$$T_h - T_1 = Q_c R_1 = Q_c \frac{1}{h_i\,A_i}$$

$$T_1 - T_2 = Q_c R_2 = Q_c \frac{x_w}{k_w\,A_{lm}}$$

$$T_2 - T_c = Q_c R_3 = Q_c \frac{1}{h_o\,A_o}$$

from which the interface temperatures $T_1$ and $T_2$ can be estimated.

When the wall thickness $x_w$ is small,

$$A_o = A_{1m} = A_i$$

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Then

$$\frac{1}{U_o} = \frac{1}{h_i} + \frac{x_w}{k_w} + \frac{1}{h_o} = \frac{1}{U_i} \tag{1.40}$$

where $U_i$ is the overall heat transfer coefficient based on the inside surface area $A_i$. It may be noted that $U_o A_o = U_i A_i$.

If more resistances are put in series, these are to be added up and the same procedure will follow.

## 1.6 RADIATION HEAT TRANSFER

All bodies radiate heat. The phenomenon is identical to the emission of light. Two similar bodies isolated together in a vacuum radiate heat to each other, but the colder body will receive more heat than the hot body and thus become heated.

If $Q$ is the total radiant energy incident upon the surface of a body, some part of it ($Q_a$) will be absorbed, some ($Q_r$) will be reflected and some ($Q_c$) will be transmitted through the body.

$$Q = Q_a + Q_r + Q_t$$

or

$$\frac{Q_a}{Q} + \frac{Q_r}{Q} + \frac{Q_t}{Q} = 1$$

$$\boldsymbol{\alpha + \rho + \tau = 1} \tag{1.41}$$

where $\alpha$ is known as absorptivity, $\rho$ as reflectivity and $\tau$ as transmissivity.

For an opaque body, $\tau = 0$ and so $\alpha + \rho = 1$. Most solids are opaque. By increasing $\rho$ with high surface polishing, $\alpha$ can be decreased.

A body which absorbs all the incident radiation is called a *black body*. A *black body* is also the best emitter. Most radiating surfaces are gray and have an emissivity factor $\varepsilon$ less than unity, where

$$\varepsilon = \frac{\text{Actual radiation of a gray body at } T(\text{K})}{\text{Radiation of a black body at } T(\text{K})}$$

The rate at which energy is radiated by a black body at the absolute temperature $T$ is given by the *Stefan-Boltzmann law*

$$\mathbf{Q_r = \sigma A\, T^4} \tag{1.42}$$

where $\sigma$ is the *Stefan-Boltzmann constant*, $5.67 \times 10^{-8}$ W/m$^2$ K$^4$. The constant was named after two Austrian scientists, J. Stefan, who in 1879 proved Eq. (1.42) experimentally, and L. Boltzmann, who in 1884 derived it theoretically.

The radiant heat exchange between two gray bodies at temperatures $T_1$ and $T_2$ depends on how the two bodies view each other, and their emissivities, and it is given by

$$Q_{1-2} = \sigma A_1\, \mathcal{F}_{1-2}\,(T_1^4 - T_2^4) \tag{1.43}$$

where $\mathcal{F}_{1-2}$ is the view factor or configuration factor for gray bodies or the fraction of total radiant energy leaving, gray surface 1 and reaching gray surface 2.

It can be shown that

$$\mathcal{F}_{1-2} = \frac{\mathbf{1}}{\left[\dfrac{\mathbf{1}}{\boldsymbol{\varepsilon}_1} - \mathbf{1}\right] + \dfrac{\mathbf{1}}{\mathbf{F}_{12}} + \dfrac{\mathbf{A}_1}{\mathbf{A}_2}\left[\dfrac{\mathbf{1}}{\boldsymbol{\varepsilon}_2} - \mathbf{1}\right]} \tag{1.44}$$

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where $\varepsilon_1$ and $\varepsilon_2$ are the emissivities of the two bodies of surface areas $A_1$ and $A_2$, and $F_{12}$ is the view factor of two similar black bodies, or the fraction of energy that leaves the black surface 1 and is incident on the black surface 2. It can be shown that

$$A_1F_{12} = A_2F_{21} \tag{1.45}$$

which is known as the *reciprocity theorem*.

The $T^4$ dependence of radiant heat transfer, Eq. (1.43) often complicates engineering calculations. For the special case of surface 1 surrounded by surface 2, where either area $A_1$ is small as compared to area $A_2$, or surface 2 is nearly black, $F_{12} \simeq \varepsilon_1$ and Eq. (1.43) becomes

$$Q_{12} = \sigma A_1\, \varepsilon_1\, (T_1^4 - T_2^4) \tag{1.46}$$

When $T_1$ and $T_2$ are not too different, it is convenient to linearize Eq. (1.46) by factoring the term $(T_1^4 - T_2^4)$ to obtain

$$\begin{aligned} Q_{12} &= \sigma A_1\, \varepsilon_1\, (T_1^2 + T_2^2)\,(T_1 + T_2)\,(T_1 - T_2) \\ &= \sigma A_1\, \varepsilon_1\, (4\mathrm{T}_m^3)\,(T_1 - T_2) \end{aligned}$$

for $T_1 \simeq T_2$, where $T_\mathrm{m}$ is the mean of $T_1$ and $T_2$. This result can be written concisely as

$$Q_{12} = A_1\, h_r\, (T_1 - T_2) \tag{1.47}$$

where $\quad h_r = 4\varepsilon_1\, \sigma\, T_m^3$ (1.47a)

is called the *radiation heat transfer coefficient* (W/m$^2$ K)

At 25°C or 298 K,

$$\begin{aligned} h_r &= 4\varepsilon_1 \times 5.67 \times 10^{-8}\ (\mathrm{W/m^2\,K^4}) \times (298)^3\ (\mathrm{K^3}) \\ &\simeq \mathbf{6\varepsilon_1\ W/m^2\ K} \end{aligned} \tag{1.47b}$$

This result can be easily remembered. The radiation heat transfer coefficient at *room temperature* is about six times the surface emissivity. For $T_1$ = 320 K and $T_2$ = 300 K, the error incurred in using the approximation of Eq. (1.47) is only 0.1%; for $T_1$ = 400 K and $T_2$ = 300 K, the error is 2%.

### 1.6.1 Combined Convection and Radiation

Heat is transferred from a hot body both by natural convection and radiation.

Rate of heat transfer by natural convection,

$$Q_c = h_c\, A\, (T_w - T_\infty)$$

and that by radiation

$$Q_r = \sigma A_1\, \mathcal{F}_{1-2}\, (T_w^4 - T_\infty^4) = h_\mathrm{r}\, A_1\, (T_w - T_\infty) \tag{1.48}$$

where $h_r$ is known as the *radiation heat transfer coefficient* (W/m$^2$K).

$$h_r = \sigma\, \mathcal{F}_{1-2}\, (T_w + T_\infty)\,(T_w^2 + T_\infty^2) \tag{1.49}$$

The equivalent thermal resistance to radiation heat transfer

$$R_r = \frac{1}{h_r\ A_1} = \frac{1}{\sigma\, A_1\, \mathcal{F}_{1-2}\, (T_w + T_\infty)\,(T_w^2 + T_\infty^2)} \tag{1.50}$$

The total rate of heat transfer by convection and radiation, which occur in parallel, is

$$\begin{aligned} Q &= Q_c + Q_r \\ &= \mathbf{(h_c + h_r)\, A_1\, (T_w - T_\infty)} \end{aligned} \tag{1.51}$$

The equivalent physical system and thermal circuit for heat transfer between two bodies 1 and 2 are shown in Fig. 1.30.

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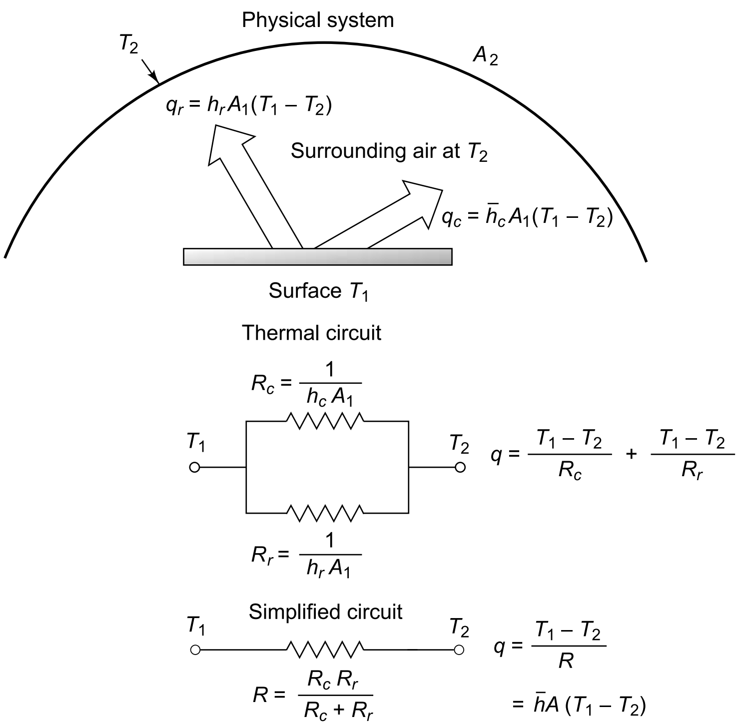


**Fig. 1.30** *Thermal circuit with convection and radiation acting in parallel*

### 1.6.2 Combined Conduction, Convection and Radiation

It is easy to envision cases in which all three modes of heat transfer are present, as in Fig. 1.31. In this case the heat conducted through the plate is removed from the plate surface by a combination of convection and radiation. An energy balance would give

$$-kA\left.\frac{dT}{dy}\right|_{\text{wall}} = hA(T_w - T_\infty) + \sigma A \mathcal{F}_{1-2}(T_w^4 - T_s^4) \tag{1.51a}$$

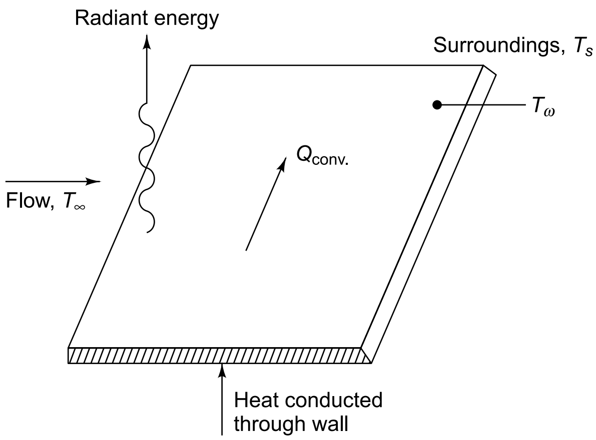


**Fig. 1.31** *Combination of conduction, convection and radiation heat transfer*

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where $Q_{\text{conv.}} = hA(T_w - T_\alpha)$, $T_w$ = wall surface temperature, $T_\alpha$ = fluid temperature and $T_s$ = temperature of surroundings.

## 1.7 THERMAL INSULATION

There are certain situations in engineering design when the objective is to reduce the flow of heat e.g., heat exchangers, building insulation, thermos flask and so on. Thermal insulation materials must have a low thermal conductivity. In most cases this is achieved by trapping air or some other gas inside small cavities in a solid. It uses the inherently low conductivity of a gas to inhibit heat flow. Heat can, however, be transferred by natural convection inside the gas pockets and by radiation between the solid enclosure walls. The conductivity of insulating materials is, therefore, the result of a combination of heat flow mechanisms (Fig. 1.32). It is an effective value, $k_{\text{eff}}$, that changes with temperature, pressure and environmental conditions e.g., moisture.

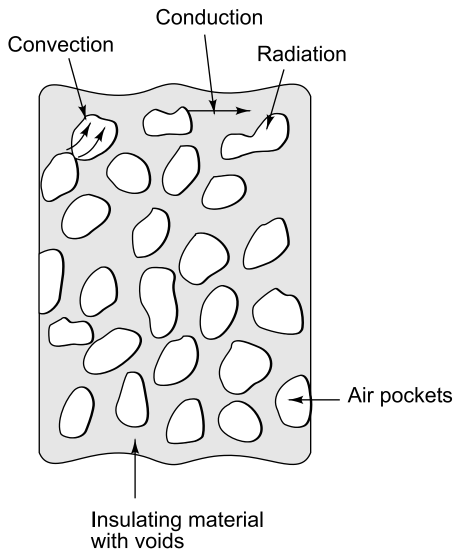


**Fig. 1.32** *Apparent thermal conductivity of an insulating material accounting for conduction through solid material, and conduction or convection through the air space as well as radiation*

There are essentially three types of insulation materials:

1. *Fibrous:* Fibrous materials consist of small-diameter particles or filaments of low density that can be poured into a gap as "loose-fill" or formed into boards or blankets. Fibrous materials have very high porosity (~90%). Mineral wool is a common fibrous material for applications at temperatures below 700°C, and fibreglass is often used for temperatures below 200°C. Between 700°C and 1700°C, one can use refractory fibres such as alumina ($Al_2O_3$) or silica ($SiO_2$).
2. *Cellular:* Cellular insulations are closed- or open-cell materials that are usually in the form of flexible or rigid boards. They can also be foamed or sprayed to achieve desired geometrical shapes. Low density, low heat capacity and good compressive strength are their advantages. Examples are polyurethane and expanded polystyrene foam.
3. *Granular:* Granular insulation consists of small flakes or particles of inorganic materials bonded into desired shapes or used as powders. Examples are perlite powder, diatomaceous silica, and vermiculite.

Fibrous and granular insulation can be evacuated to eliminate convection and conduction, thus decreasing the effective conductivity appreciably. Figure 1.33 shows the ranges of effective thermal conductivity for evacuated and non-evacuated insulation as well as the product of thermal conductivity and bulk density, which is sometimes important in design.

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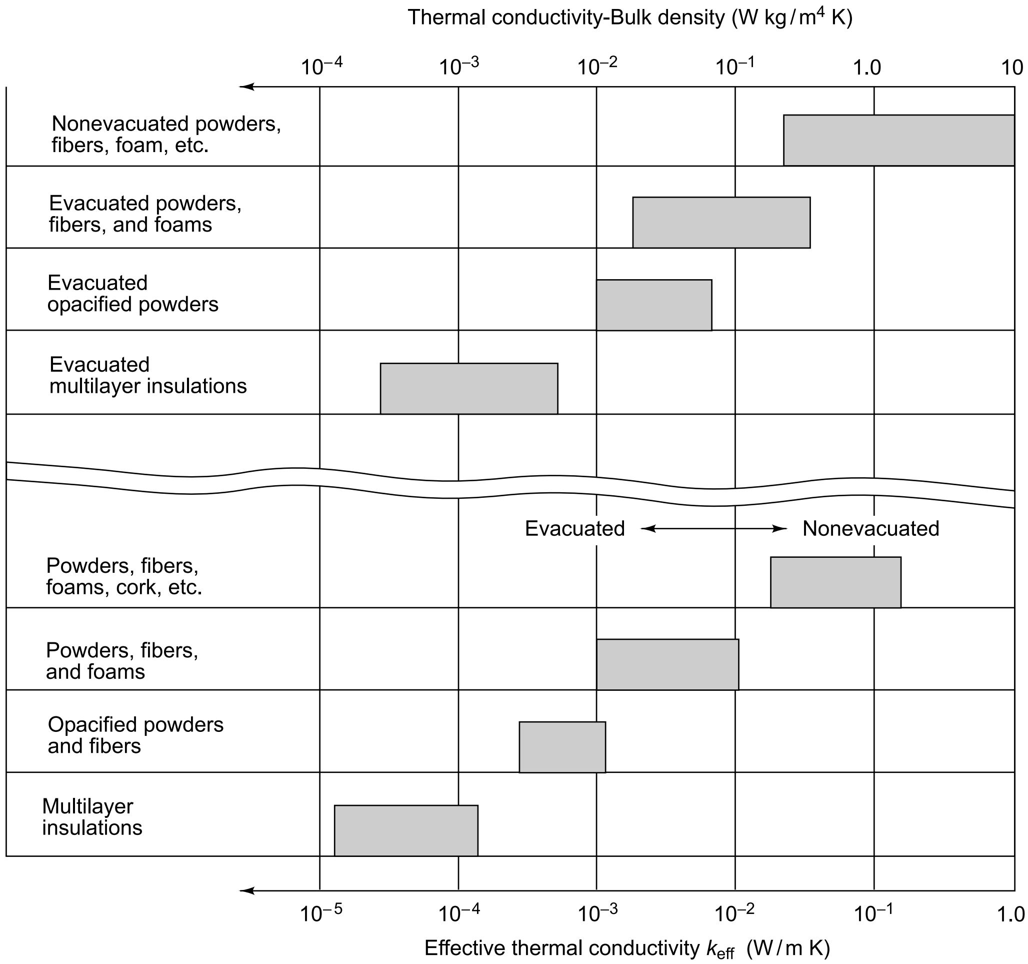


**Fig. 1.33** *Range of thermal conductivities of thermal insulations and products of thermal conductivity and bulk density*

Sometimes reflective sheets are used to provide insulation. Two or more thin sheets of metal with low emittance are placed parallel to each other to reflect radiation back to its source. An example is the thermos flask, in which the space between the reflective surfaces is evacuated to suppress convection and conduction, leaving radiation as the main transfer mechanism.

Apart from low thermal conductivity, insulation material should have structural rigidity, low density, less degradation, good chemical stability and low cost.

Figure 1.34 shows ranges of thermal conductivities for several common low-temperature fibrous and cellular insulation materials. All of the values are for new materials. Polyurethane and polystyrene generally lose between 20 and 50% of their insulation quality during the first year of use; moisture uptake or loss of vacuum also reduces insulating property. Except for cellular glass, cellular insulating materials are plastics that are inexpensive and of light weight.

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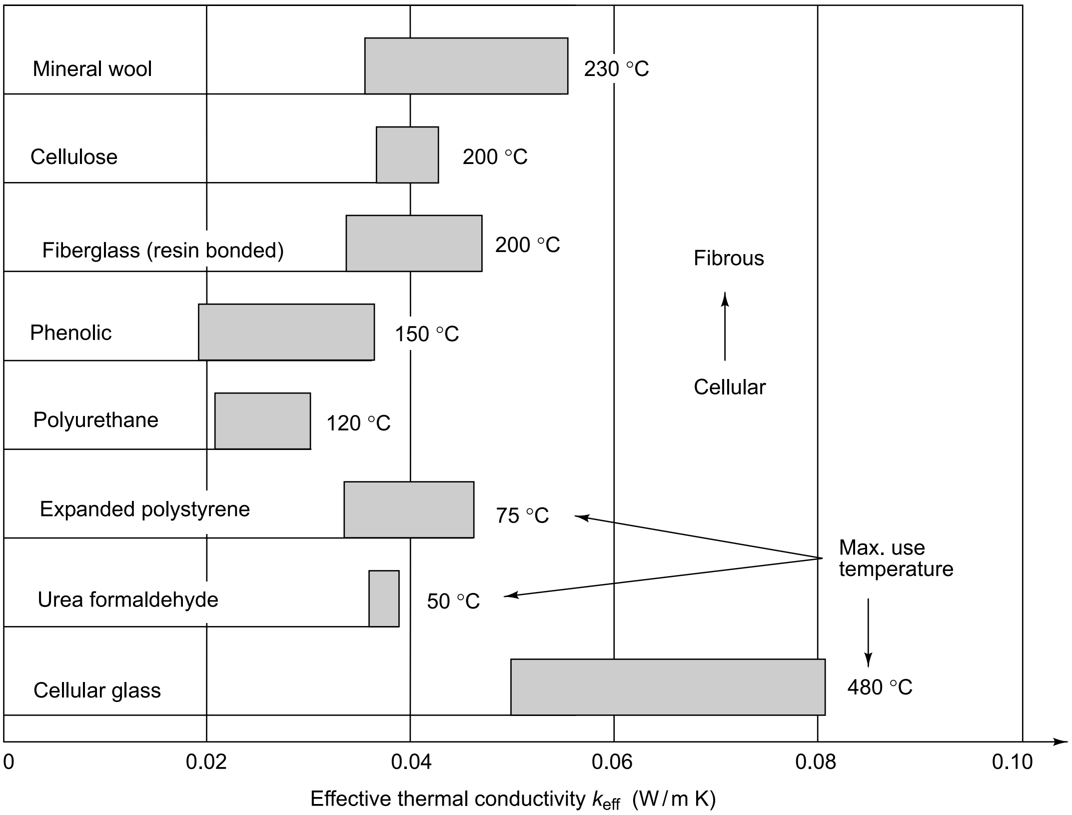


**Fig. 1.34** *Thermal conductivity ranges for fibrous and cellular insulations with maximum-use temperatures*

For high temperature applications, refractory materials are used. They can be in the form of bricks which can withstand temperatures upto 1700°C. The effective conductivity varies from 1.5 W/m K for fire-clay to about 2.5 W/m K for zirconia. Loose-fill insulation has a much lower thermal conductivity, as shown in Fig. 1.35, and it can be used only below 900°C. The effectiveness of insulating materials is often expressed in terms of their *R-value*, which is the thermal resistance of the material for a unit area, i.e. $R_{value} = L/k$, $L$ being the thickness and $k$ the thermal conductivity.

Insulation pays for itself from the energy it saves. Insulating a surface properly requires a one-time capital investment, but its effects are dramatic and long term. The payback period of insulation is usually under 2 years.

## 1.8 DIFFUSION AND MASS TRANSFER

Diffusional mass transfer occurs at a microscopic or molecular level which deals with the transport of one constituent of a fluid solution or gas mixture from a region of higher concentration to a region of lower concentration (mol/m$^3$). Heat is transferred in a direction which reduces an existing temperature gradient, and mass is transferred in a direction which reduces an existing concentration gradient.

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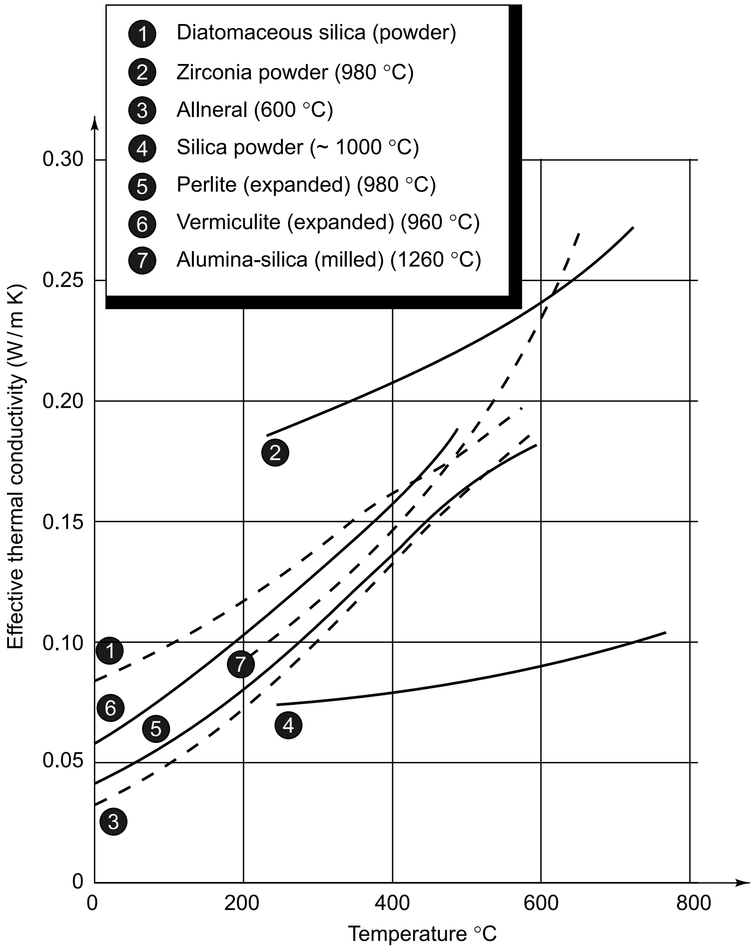


**Fig. 1.35** *Thermal conductivity varying with temperature for some high temperature insulations with maximum use temperature in parenthesis*

The rate of molecular diffusion is proportional to the concentration gradient

$$\frac{N_A}{A} \propto \frac{\mathrm{d}C_A}{\mathrm{d}y}$$

where $N_A/A$ is the diffusion rate per unit area (kg mol/m$^2$s) of the diffusing species $A$, and $dC_A/dy$ is the concentration gradient (kg mol/m$^4$) in the direction of diffusion $Y$, $C_A$ being the concentration of $A$ in kg mol/m$^3$

$$\frac{\mathbf{N_A}}{\mathbf{A}} = -\mathbf{D}\frac{\mathbf{dC_A}}{\mathbf{dy}} \tag{1.52}$$

where $D$ is the constant of proportionality, called the diffusivity (m$^2$/s). It is called the *Fick's law of diffusion*, which is similar to Fourier's law of heat conduction.

Whenever there is concentration gradient there will be mass transfer, till the concentration of that particular constituent becomes uniform.

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Figure 1.36 shows a sugar cube or lump dissolving in a cup of tea. The concentration of dissolved sugar adjacent to the lump is higher than in the bulk tea, and the dissolved sugar moves down its concentration gradient by the process known as *ordinary diffusion*. Ordinary diffusion is analogous to heat conduction, which may be viewed as diffusion of thermal energy or heat down its temperature gradient. If the tea is stirred, the fluid motion transports dissolved sugar away from the lump by the process known as *mass convection*. Mass convection is exactly analogous to heat convection: the fluid can transport both energy and chemical species by virtue of motion. The transport of perfume or noxious odours in the air surrounding us similarly involves the processes of mass diffusion and mass convection.

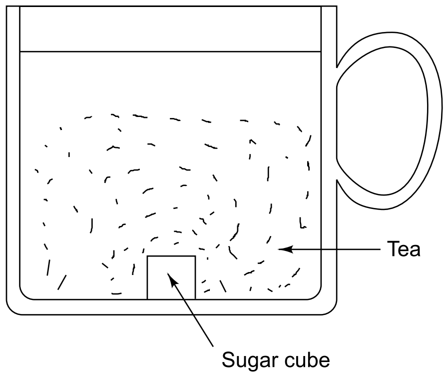


**Fig. 1.36** *A sugar lump dissolving in a cup of tea: the dissolved sugar moves away from the lump by diffusion in the direction of decreasing sugar concentration*

We often encounter processes involving the evaporation of water into air e.g., from a hot tub or swimming pool, or when we sweat while doing physical exercise, the air adjacent to the water surface is saturated with water vapour, and the corresponding water vapour concentration is usually higher than that in the surrounding air: water vapour diffuses away from the surface and is replenished by evaporation of the liquid water. The latent heat required to evaporate the water is supplied from the bulk water or human body, causing the cooling effect. This is known as sweat cooling (evaporative cooling) as it keeps the surface wet. We welcome a breeze when sweating, the mass convection associated with air motion increases the rate of evaporation and the cooling effect. Sweat cooling includes *simultaneous heat and mass transfer*. A wet cooling tower cools water from the condenser of a power plant. All combustion processes involve simultaneous mass transfer of the reactants and products, and heat transfer associated with the release of heat of combustion.

Mass transfer occurs in a variety of equipment, particularly those required to control pollution of the environment by exhaust gases from combustion processes like the exhaust from automobiles or stack gases from power plants. A catalytic converter on an automobile is a *mass exchanger* that removes carbon monoxide, unburnt hydrocarbons, and nitrogen oxides from the engine exhaust. The coal-fired power plants are often required to have mass exchangers that remove sulphur oxides (causing acid rain) as well as nitrogen oxides and particulate matter from the furnace exhaust.

Fick's law of diffusion governing ordinary diffusion, analogous to Fourier's law of heat conduction, states that the local mass flux of a chemical species is proportional to the negative of local concentration gradient. The mass fraction of chemical species is defined as

$$\begin{aligned} m_i &= \text{mass fraction of a species } i \\ &= \frac{\text{Partial density of species } i}{\text{Density of the mixture}} = \frac{\rho_i}{\rho} \end{aligned} \tag{1.53}$$

where $\rho = \sum_{i=1}^{n} \rho_i$ for a mixture of $n$ species.

Fick's law then gives the *diffusion mass flux* $j_1$ (kg/m$^2$ s) of species 1 in a binary mixture of species 1 and 2 as

$$j_1 \propto -\frac{dm_1}{dx}$$

for one-dimensional diffusion in the $x$-direction.

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$$\therefore \qquad \mathbf{j_1 = -\rho\, D_{1\text{–}2} \frac{dm_1}{dx}} \tag{1.54}$$

where $\rho D_{1\text{–}2}$ is the constant of proportionality, $\rho(\text{kg/m}^3)$ is the local mixture density and $D_{1\text{–}2}$ is the binary diffusion coefficient or mass diffusivity ($\text{m}^2$/s).

Mass convection is essentially identical to heat convection, and similar considerations apply. The flow may be forced or natural, internal or external, and laminar or turbulent. Referring to Fig. 1.37, analogous to Newton's law of cooling (Eq. 1.27), we may write

$$j_{1,\,w} = J_{m1}\, \Delta m_1 \tag{1.55}$$

where, $\quad \Delta m_1 = m_{1,\,w} - m_{1,\,e}$

Here, $J_{m1}$ (kg/m$^2$ s) is the mass transfer coefficient. The mass transfer for laminar flow in a tube like heat convection (Eq. 1.32),

$$J_{m1} = 3.66 \frac{\rho D_{1-2}}{D}$$

where $D$ is the tube diameter. The average mass transfer coefficient is

$$J_{ml} = \frac{1}{A} \int_6^A j_{\,m1} \cdot dA$$

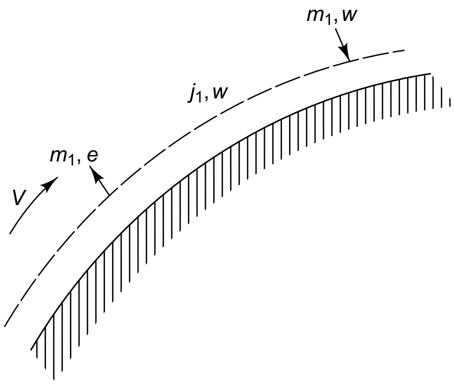


**Fig. 1.37** *Convective mass transfer in an external flow*

As for natural heat convection, natural mass convection is driven by buoyancy forces arising from a density difference caused by concentration difference or temperature difference. Convective heat and mass problems will further be discussed in Chapter 10. There is no mass transfer analog to radiation heat transfer.

## 1.9 UNITS AND DIMENSIONS

SI system of units will be used throughout, in which the fundamental units are metre (L), kilogram (M), second (T) and kelvin (K). Force and energy or heat are derived units. The force is in newton, 1 N = 1 kg m/s$^2$, and the energy is in joule or newton-metre (1 J = 1 Nm). Power is in watts (W, kW or MW); 1 W = 1 J/s. The pressure is in pascals, 1 Pa = 1 N/m$^2$, or in bar.

$$\mathbf{1\ bar = 10^5\ Pa = 100\ kPa = 0.1\ MPa}$$

and $\quad$ $\mathbf{1\ atm = 1.01325\ bar = 101.325\ kPa.}$

## Solved Examples

**Example 1.1** *Given:* Sheets of brass and steel, each 1 cm thick, are in contact. The outer surface of brass is at 100°C and that of steel is at 0°C, and $k_b/k_s = 2$.

*To find:* Interface temperature of the sheets.

**Solution** The steady state temperature distribution through a plane wall is given by Eq. (1.6),

$$Q_k = kA \frac{T_1 - T_2}{L}$$

Let $T$ be the interface temperature. Since the heat flow through the sheets is in series,

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$$Q = k_b A \frac{100 - T}{L} = k_s A \frac{T - 0}{L}$$

$$\therefore \quad \frac{T}{100 - T} = \frac{k_b}{k_s} = 2$$

$$\therefore \quad T = 66.7°\text{C } Ans.$$

**Example 1.2** *Given:* Temperature of air over the surface of a lake = – 6°C. $k_i$ = 1.675 W/mK, $\rho_i$ = 920 kg/m$^3$, $l_{fu}$ = 325 kJ/kg.
*To find:* The time required to form a thickness of 4 cm of ice on the surface of the lake.

**Solution** To form ice of differential thickness $dy$, let $d\tau$ be the time required. By energy balance,

$$\rho A\, dy\, l_{\text{fu}} = k_i A \frac{T_2 - T_1}{y} d\tau$$

$$\therefore \quad \int_0^\tau d\tau = \int_0^y \frac{\rho l_{\text{fu}}}{k_i (T_2 - T_1)} y\, dy$$

$$\therefore \quad \tau = \frac{\rho l_{\text{fu}}}{k_i (T_2 - T_1)} \cdot \frac{y^2}{2}$$

Substituting the given values,

$$\tau = \frac{920 \times 325 \times (0.04)^2 \times 10^3}{1.675[0 - (-6)] \times 2}$$

$$= 24533.33 \text{ s} = 6.815 \text{ h} = 6 \text{ h } 49 \text{ min. } Ans.$$

**Example 1.3** *Given:* The convective heat transfer coefficient, $h_c$ = 2.512 $(\Delta T)^{1/4}$ W/m$^2$K. A hot plate of $A$ = 0.2 m$^2$ at 59°C loses heat to a room at temperature 20°C.
*To find:* The fraction of heat lost by natural convection, when heat is transferred from the plate steadily at the rate of 100 W.

**Solution** The rate of heat transfer by convection, $Q_c = h_c A\, (\Delta T)$,

where $\quad h_c = 2.512\,(59 - 20)^{1/4} = 6.277$ W/m$^2$K

$$\therefore \quad Q_c = 6.277 \times 0.2 \times 39 = 48.96 \text{ W}$$

Fraction of the supplied heat lost by convection is 48.96/100 or 0.4896 or 48.96 or 49%.
The remaining 51% heat transfer occurs by radiation.

**Example 1.4** *Given:* A jet aircraft compartment is assumed to be a cylindrical tube of 3-m diameter and 20-m length. It is lined inside with 3 cm of insulating material of $k$ = 0.042 W/mK. It is flying at a height where the average outside temperature is – 30°C.
*To find:* The rate of heating required to maintain the compartment at 20°C for passenger comfort.

**Solution** The rate of heat loss from the cylindrical compartment is given by Eq. (1.21)

$$Q = \frac{2\pi k L (T_1 - T_2)}{\ln(r_2 / r_1)}$$

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$$= \frac{2\pi \times 0.042 \times 20[20-(-30)]}{\ln(300/294)} = 13060 \text{ W}$$

$$= 13.06 \text{ kW} \quad Ans.$$

This is the heat which must be supplied to the compartment to make up for the heat loss through the walls.

**Example 1.5** *Given:* A wire 0.5 mm in diameter is stretched along the axis of a cylinder 50 mm in dia and 250 mm long. The temperature of the wire is 750 K while the cylinder is at 25 K and the gas in it has a $k$ = 0.0251 W/mK.
*To find:* The rate of heat transfer througth the gas by conduction and by radiation if the wire is black.

**Solution** The rate of heat transfer through the gas in the cylinder is given by Eq. (1.21),

$$Q_k = \frac{2\pi kL(T_2 - T_1)}{\ln(r_2/r_1)}$$

$$= \frac{2\pi \times 0.0251 \times 0.25(750-250)}{\ln(25/0.25)}$$

$$= 4.28 \text{ W} \quad Ans.$$

Heat lost by radiation

$$Q_r = \sigma A_1 (T_1^4 - T_2^4)$$

$$= 5.67 \times 10^{-8} \times 2\pi \times 0.25 \times 10^{-3} \times 0.25\,(750^4 - 250^4)$$

$$= 6.958 \text{ or } 7 \text{ W. } Ans.$$

**Example 1.6** *Given:* The ratio of radius of the earth's orbit to that of sun is 216. The solar insolation on the earth is 1.4 kW/m$^2$.
*To find:* The surface temperature of the sun if it is assumed to be an ideal radiator (black body).

**Solution** Total radiation from the sun

$$Q_r = 1.4 \times 4\pi R^2$$

where $R$ is the radius of the earth's orbit.

Total radiation emitted by the sun

$$Q_r = \sigma\, 4\pi r^2\, T^4$$

where $r$ is the radius of the sun and $T$ is the surface temperature of the sun. Therefore,

$$\sigma 4\pi r^2\, T^4 = 1.4 \times 4\pi R^2$$

$$\therefore \quad T^4 = \frac{1.4 \times 10^3}{5.67 \times 10^{-8}} \times (216)^2 = 0.1152 \times 10^{16} \text{ K}^4$$

$$\therefore \quad T = 5826 \text{ K. } Ans.$$

**Example 1.7** *Given:* A furnace wall has the inside surface temperature of 1100°C, while the ambient air temperature is 25°C. The wall consists of 125 mm thick refractory bricks ($k$ = 1.6 W/mK), 125 mm thick firebricks ($k$ = 0.3 W/mK) and 12 mm thick plaster ($k$ = 0.14 W/mK). There is an air gap which offers a thermal resistance of 0.16 K/W. The heat transfer coefficient on the outside wall to the air is 17 W/m$^2$K.
*To find:* (a) The rate of heat loss per unit area of wall surface, (b) the interface temperatures throughout the wall, and (c) the temperature of the outside surface of the wall.

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**Solution** There are a number of thermal resistances in series to heat flow from the inside furnace wall to the ambient air. The rate of heat flow $Q$ is given by,

$$Q_k = \frac{T_1 - T_2}{\Sigma R}$$

where $T_1 = 1100°\text{C}$, $T_2 = 25°\text{C}$ and

$$\Sigma R = R_1 + R_2 + R_3 + R_4 + R_5$$

and $R_1$ = Resistance of the refractory brick $= \dfrac{x_1}{k_1 A} = \dfrac{0.125}{1.6 \times 1} = 0.0781$ K/W

$R_2$ = Resistance of the insulating firebricks $= \dfrac{0.125}{0.3 \times 1} = 0.417$ K/W

$R_3$ = Resistance of plaster $= \dfrac{0.012}{0.14 \times 1} = 0.0857$ K/W

$R_4$ = Resistance of air film on outside surface $= \dfrac{1}{h_c A} = \dfrac{1}{17 \times 1} = 0.0588$ K/W

$R_5$ = Resistance of air gap = 0.16 K/W

$\Sigma R$ = Total resistance

$= 0.0781 + 0.417 + 0.0857 + 0.0588 + 0.16$

$= 0.7996 \cong 0.8$ K/W

$\therefore$ Rate of heat loss per unit area

$$Q_k = \frac{1100 - 25}{0.8} = 1344 \text{ W} = 1.344 \text{ kW} \quad \textit{Ans. (a)}$$

The interface temperatures are $T_3$, $T_4$ and $T_5$ and the outside surface temperature is at $T_6$ (Fig. Ex. 1.7).

$$Q_k = 1344 = \frac{1100 - T_3}{0.0781}$$

$\therefore$ $T_3 = 995°\text{C}$ *Ans.* (b)

$$Q_k = 1344 = \frac{T_3 - T_4}{R_5} = \frac{995 - T_4}{0.16}$$

$\therefore$ $T_4 = 780°\text{C}$ *Ans.* (b)

$$Q_k = 1344 = \frac{T_4 - T_5}{R_2} = \frac{780 - T_5}{0.417}$$

$\therefore$ $T_5 = 220°\text{C}$ *Ans.* (b)

$$Q_k = 1344 = \frac{T_5 - T_6}{R_3} = \frac{220 - T_6}{0.0857}$$

$\therefore$ $T_6 = 104.1°\text{C}$ *Ans.* (c)

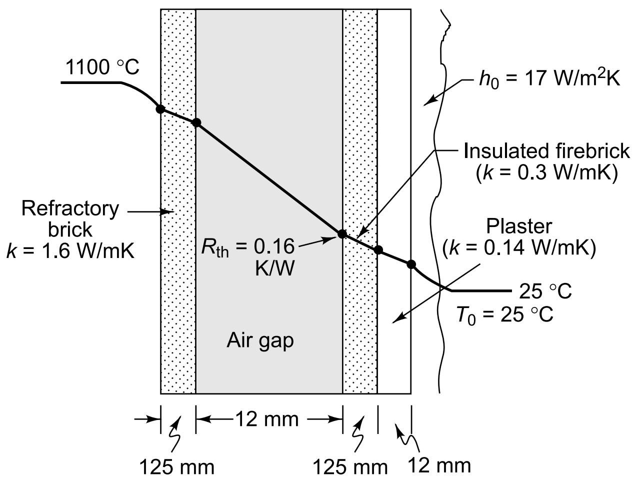


**Fig.** *Ex. 1.7*

However, the outside furnace wall surface temperature should be below 60°C to avoid any injury to the person touching the wall.

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**Example 1.8** *Given:* A steam pipe made of steel ($k$ = 58 W/mK) has i.d. of 160 mm and o.d. of 170 mm. The saturated steam flowing through it is at 300°C, while the ambient air is at 50°C. It has two layers of insulation, the inner layer ($k$ = 0.17 W/mK) is 30 mm thick and the outer layer ($k$ = 0.023 W/mK) is 50 mm thick, the heat transfer coefficients on the inside and outside walls are 30 and 5.8 W/m$^2$K respectively.
*To find:* The rate of heat loss per unit length of the pipe.

**Solution** Here too there are a number of thermal resistances in series through which heat flows from the steam pipe to the ambient air (Fig. Ex. 1.8). The rate of heat flow per unit length

$$Q/L = \frac{T_1 - T_0}{R_1 + R_2 + R_3 + R_4 + R_5}$$

$$= \frac{2\pi(T_1 - T_0)}{\frac{1}{h_i r_i} + \frac{\ln(r_2/r_1)}{k_{w1}} + \frac{\ln(r_3/r_2)}{k_{w2}} + \frac{\ln(r_4/r_3)}{k_{w3}} + \frac{1}{h_0 r_4}}$$

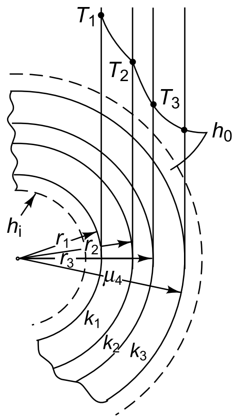


**Fig.** *Ex. 1.8*

where $r_1$ = 80 mm, $r_2$ = 85 mm, $r_3$ = 115 mm and $r_4$ = 165 mm

$$\therefore \qquad Q/L = \frac{2\pi(300-50)}{\frac{1}{30 \times 0.08} + \frac{\ln(85/80)}{58} + \frac{\ln(115/85)}{0.17} + \frac{\ln(165/115)}{0.023} + \frac{1}{58 \times 0.16}}$$

$$= \frac{500\pi}{0.4167 + 0.00105 + 1.738 + 15.69 + 0.108}$$

$= 82.94$ W/m *Ans.*

**Example 1.9** *Given:* An aluminium ($k$ = 185 W/mK) pipe of i.d. 10 cm and o.d. 12 cm carrying steam at 110°C loses heat to the room air at 30°C having $h$ = 15 W/m$^2$K.
*To find:* (i) The rate of heat transfer, (ii) the percentage reduction in heat transfer if an insulation ($k$ = 0.2 W/mK) of 5 cm thickness covers the pipe.

**Solution** If the convective resistance of steam is neglected, the inside surface temperature of the pipe is the same as the steam temperature. Since the wall conduction resistance and the air convective resistance are in series, the rate of heat transfer per unit length is given by

$$\frac{Q}{l} = \frac{T_s - T_\infty}{\frac{\ln(r_2/r_1)}{2\pi k} + \frac{1}{2\pi r_2 h_0}}$$

$$= \frac{110-30}{\frac{\ln(6/5)}{2\pi \times 185} + \frac{1}{2\pi \times 0.06 \times 15}}$$

$$= \frac{80}{1.57 \times 10^{-4} + 0.177} = 452 \text{ W/m } Ans.$$

For the insulated pipe,

$$\frac{Q}{l} = \frac{T_s - T_\infty}{\frac{\ln(r_2/r_1)}{2\pi k_p} + \frac{\ln(r_3/r_2)}{2\pi k_i} + \frac{1}{2\pi r_3 h_0}}$$

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$$= \frac{110-30}{\dfrac{\ln 6/5}{2\pi\times 185}+\dfrac{\ln(11/6)}{2\pi\times 0.2}+\dfrac{1}{2\pi\times 0.11\times 15}}$$

$$= \frac{80}{1.57\times 10^{-4}+0.182+0.096} = 138 \text{ W/m}$$

Reduction of heat loss from the pipe by insulation $= \dfrac{452-138}{452} = 0.695$ or 69.5% *Ans.*

The resistance of aluminium pipe can be neglected in both the cases without much loss of accuracy.

**Example 1.10** *Given:* A plastic pipe ($k$ = 0.5 W/mK) of i.d. 3 cm and o.d. 4 cm carries a fluid of average temperature 100°C and $h$ = 300 W/m$^2$K. The rate of heat transfer per unit length is 500 W/m.
*To find*: (i) The outside surface temperature of pipe, (ii) the overall heat transfer coefficient based on outside area.

**Solution** The rate of heat transfer per unit length of the plastic pipe is given by

$$\frac{Q}{l} = \frac{T_1 - T_2}{\dfrac{1}{h_1 2\pi r_1}+\dfrac{\ln(r_2/r_1)}{2\pi k}}$$

$$500 = \frac{100 - T_2}{\dfrac{1}{300\times 2\pi\times 0.015}+\dfrac{\ln(2/1.5)}{2\times 0.5}}$$

$\therefore$ $T_2$ = outside surface temperature of pipe = 36.5°C *Ans.*

Now, $$\frac{1}{U_0 A_0} = \frac{1}{h_1 2\pi r_1 l}+\frac{\ln(r_2/r_1)}{2\pi k_1 l}$$

$\therefore$ $$U_0 = \frac{1}{r_2/r_1 h + \dfrac{r_2\ln(r_2/r_1)}{k_1}}$$

$\therefore$ $$U_0 = \frac{1}{\dfrac{2}{1.5\times 300}+\dfrac{0.02\ln(2/1.5)}{0.5}} = 62.69 \text{ W/m}^2\text{K. } \textit{Ans.}$$

As a check, $Q = U_0 A_0 (T_1 - T_2) = 62.69 \times 2\pi \times 0.02 \times (100 - 36.5) = 500$ W/m

**Example 1.11** *Given:* A 60 W lamp buried in soil ($k$ = 0.83 W/mK) at 0°C is switched on.
*To find:* To find the soil temperature 0.3 m away from the lamp at steady state.

**Solution** By energy balance in a spherical shell of thickness d$r$ at a radius $r$

$$Q = -k\,4\pi\, r_2 \frac{dT}{dr}$$

$$\frac{Q}{4\pi K}\int_{r=r_1}^{r=\infty} r^{-2}dr = -\int_{T_1}^{0} dT$$

$$\frac{Q}{4\pi k}\left(-\frac{1}{r}\right)_{r_1}^{\infty} = T_1$$

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$$Q = 4\pi k\, r_1 T_1$$

$$\therefore \qquad T_1 = \frac{60}{4\pi \times 0.83 \times 0.3} = 19°\text{C } Ans.$$

**Example 1.12** *Given:* Two large aluminium ($k$ = 240 W/mK), each 2 cm thick, with 10 µm surface roughness are placed in contact at $10^5$ N/m$^2$ pressure (Fig. Ex. 1.12) with the outside surface temperatures of 390°C and 406°C. The thermal contact resistance is 2.75 × $10^{-4}$ m$^2$K/W.
*To find*: (i) The heat flux, (ii) the temperature drop due to contact resistance, and (iii) the contact temperatures.

**Solution** The rate of heat flow per unit area

$$q = \frac{T_1 - T_2}{R_1 + R_2 + R_3} = \frac{T_1 - T_2}{(L/K)_1 + R_C + (L/K)_2}$$

where $R_C$ = 2.75 × $10^{-4}$ m$^2$K/W and each of the other two resistances is equal to

$$(L/K) = \frac{0.02}{240} = 8.34 \times 10^{-5}\, \text{m}^2\text{K/W}$$

$$\text{Heat flux } q = \frac{406 - 390}{8.34\times10^{-5} + 2.75\times10^{-4} + 8.34\times10^{-5}}$$

$$= \frac{16}{4.418 \times 10^{-4}} = 3.67 \times 10^4 \text{ W/m}^2 \; Ans. \text{ (i)}$$

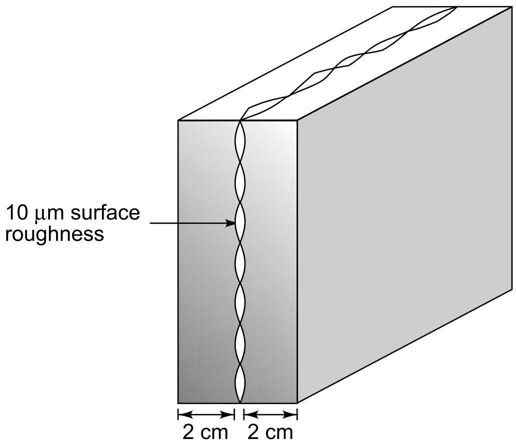


**Fig.** *Ex. 1.12*

(ii) The temperature drop in each section is proportional to the resistance. The fraction of contact resistance is

$$\frac{R_C}{R} = \frac{2.75}{4.418} = 0.622$$

Temperature drop = 0.622 × 16 = 9.95°C *Ans.* (ii)

(iii) The temperature drop in each aluminium plate = $\dfrac{16 - 9.95}{2}$ = 3.025°C

$$\therefore \quad T_{C_1} = 406 - 3.025 = 402.975°\text{C}$$

$$T_{C_2} = 402.975 - 9.95 = 393.025°\text{C}$$

These are the contact temperatures. *Ans.* (iii)

**Example 1.13** *Given:* A small hot surface at 425 K having an emissivity 0.85 dissipates heat by radiation to the surrounding air at 400 K.
*To find:* The radiation heat transfer coefficient.

**Solution** The rate of radiant heat transfer

$$Q_r = \sigma A_1 \varepsilon_1 (T_1^4 - T_2^4)$$

$$= \sigma A_1 \varepsilon_1 (T_1^2 + T_2^2)(T_1 + T_2)(T_1 - T_2)$$

Since $(T_1 - T_2) << T_1$, the above equation can be written as

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$$Q_r = \sigma A_1 \varepsilon_1 4\, T_1^3\, (T_1 - T_2) = h_r A\, (T_1 - T_2)$$

where $\quad h_r$ = the radiation heat transfer coefficient

$$= 4\sigma\, \varepsilon_1\, T_1^4$$
$$= 4 \times 5.67 \times 10^{-8} \times 0.85 \times (4.25)^3 \times 10^6$$
$$= 14.6 \text{ W/m}^2\text{K } \textit{Ans.}$$

**Example 1.14** *Given:* A steam pipe ($\varepsilon$ = 0.9) of 0.4 m diameter has a surface temperature of 500 K and is located in a large room at 27°C where $h_c$ = 25 W/m$^2$K.
*To find*: (i) The combined heat transfer coefficient, (ii) the rate of heat loss per unit length.

**Solution** The rate of radiant heat transfer

$$Q_r = \sigma A_1 \mathscr{F}_{12}(T_1^4 - T_2^4) = h_r A_1\, (T_1 - T_2)$$

$$\therefore \quad h_r = \sigma\, \varepsilon (T_1 + T_2)\, (T_1^2 + T_2^2)$$
$$= 5.67 \times 10^{-8} \times 0.9\, (500 + 300)\, (500^2 + 300^2)$$
$$= 13.88 \text{ W/m}^2\text{K}.$$

Combined heat tansfer coefficient

$$h = h_c + h_r = 25 + 13.88 = 38.88 \text{ W/m}^2\text{K } \textit{Ans.} \text{ (i)}$$

Rate of heat loss per unit length

$$Q = h\pi d(T_1 - T_2) = 38.88 \times \pi \times 0.4 \times 200 = 9771.6 \text{ W}$$
$$L = 9.77 \text{ kw } \textit{Ans.} \text{ (ii)}$$

**Example 1.15** *Given:* A 0.8 m high, 1.5 m wide double-pane window consists of two 4 mm thick layers of glass ($k$ = 78 W/mK) and is separated by a 10 mm wide stagnant air space ($k$ = 0.026 W/mK). The room is at 20°C and the outside air is at –10°C. The heat transfer coefficients are $h_i$ = 10 and $h_o$ = 40 W/m$^2$K.
*To find:* (i) The rate of heat transfer through the window, (ii) the inside surface temperature.

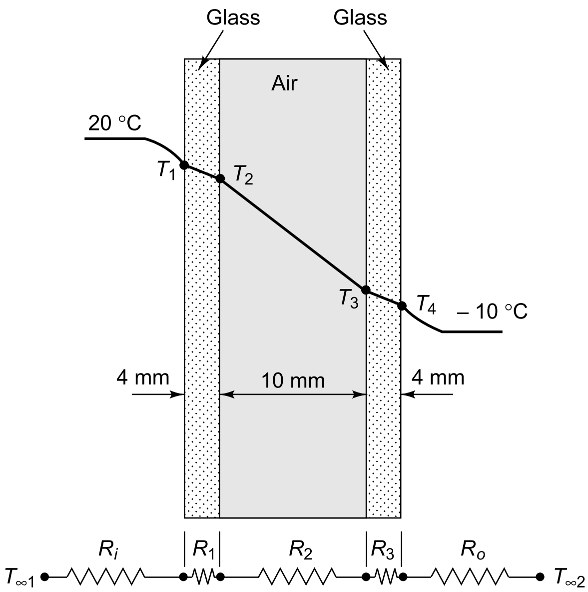


**Fig.** *Ex. 1.15*

**Solution** As shown in Fig. Ex. 1.15,

$$R_i = \frac{1}{h_i A} = \frac{1}{10 \times (0.8 \times 1.5)} = 0.08333 \text{ K/W}$$

$$R_i = \frac{L_1}{k_1 A} = \frac{0.004}{0.78 \times 1.2} = 0.00427 \text{ K/W}$$

$$R_2 = \frac{L_2}{k_2 A} = \frac{0.01}{0.026 \times 1.2} = 0.3205 \text{ K/W}$$

$$R_o = \frac{1}{h_0 A} = \frac{1}{40 \times 1.2} = 0.02083 \text{ K/W}$$

$$\therefore \quad R_{\text{total}} = R_i + R_1 + R_2 + R_3 + R_o$$
$$= 0.08333 + 0.00427 + 0.3205 + 0.00427 + 0.02083$$
$$= 0.4332 \text{ K/W}$$

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$\therefore$ Rate of heat transfer $= \dfrac{T_1 - T_2}{R_{\text{total}}} = \dfrac{20-(-10)}{0.4332} = 69.2$ W *Ans.*

The inner surface temperature of the window is obtained from

$$T_{\infty 1} - T_1 = QR_1 = 69.2 \times 0.08333 = 5.8°\text{C}$$

$$\therefore \quad T_1 = 20 - 5.8 = 14.2°\text{C} \ \textit{Ans.}$$

**Example 1.16** *Given:* Steam at 350°C flowing in a pipe ($k$ = 80 W/mK) 5 cm i.d., 5.6 cm o.d. is covered with 3 cm thick insulation ($k$ = 0.05 W/mK). Heat is lost to the surroundings at 5°C by natural convection and radiation with combined $h$ = 20 W/m$^2$K and $h_i$ = 60 W/m$^2$K.
*To find*: (i) The rate of heat loss from the pipe per unit length, (ii) the temperature drops across the pipe and the insulation.

**Solution** For steady one-dimensional heat transfer through the pipe, the thermal resistances in series are given in Fig. Ex. 1.16.

$$A_1 = 2\pi r_1 L = 2\pi \times 0.025 \times 1 = 0.157 \text{ m}^2$$

$$A_2 = 2\pi r_2 L = 2\pi \times 0.058 \times 1 = 0.364 \text{ m}^2$$

$$R_i = \frac{1}{h_i A_i} = \frac{1}{60 \times 0.157} = 0.106 \text{ K/W}$$

$$R_1 = \frac{\ln(r_2/r_1)}{2\pi k_1 L} = \frac{\ln(2.8/2.5)}{2\pi \times 80 \times 1} = 0.00023 \text{ K/W}$$

$$R_2 = \frac{\ln(r_3/r_2)}{2\pi k_2 L} = \frac{\ln(5.8/2.8)}{2\pi \times 0.05 \times 1} = 2.318 \text{ K/W}$$

$$R_o = \frac{1}{h_o A_3} = \frac{1}{20 \times 0.364} = 0.137 \text{ K/W}$$

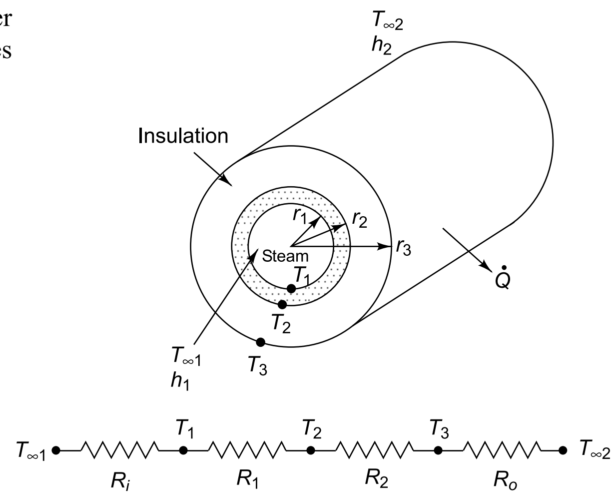


**Fig.** *Ex. 1.16*

$$\begin{aligned} R_{\text{total}} &= R_i + R_1 + R_2 + R_o \\ &= 0.106 + 0.00023 + 2.318 + 0.137 \\ &= 2.56123 \text{ K/W} \end{aligned}$$

(i) Rate of heat transfer $Q = \dfrac{T_1 - T_2}{R_{\text{total}}} = \dfrac{350-5}{2.56123} = 134.7$ W *Ans.* (i)

(ii) $\Delta T_{\text{pipe}} = QR_1 = 134.7 \times 0.00023 = 0.03°\text{C}$

$\Delta T_{\text{insulation}} = QR_2 = 134.7 \times 2.318 = 312.2°\text{C}$ *Ans.* (ii)

**Example 1.17** *Given:* A brick ($k$ = 1.2 W/mK) wall 0.15 m thick separates comustion gases in a furnace from the atmospheric air at 30°C. The outside surface temperature is 100°C while its ε = 0.8 and $h_0$ = 20 W/m$^2$K.
*To find*: The inner surface temperature of the brick wall.

**Solution** At steady state,

Heat conducted through the brick wall = Heat dissipated to the surroundings by natural convection and radiation

$$Q_k = Q_c + Q_r$$

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$$k\,\frac{T_1 - T_2}{L} = h\,(T_2 - T_\infty) + \sigma\varepsilon\,(T_2^4 - T_\infty^4)$$

$$1.2\,\frac{T_1 - 373}{0.15} = 20\,(373 - 298) + 0.8 \times 5.67 \times 10^{-8} \times (373^4 - 298^4) = 1500 + 520 = 2020 \text{ W/m}^2$$

$$\therefore \qquad T_1 = \frac{2020 \times 0.15}{1.2} + 373 = 625 \text{ K} = 352°\text{C } \textit{Ans.}$$

**Example 1.18** *Given:* A refrigerated container in the form of a cube with 2 m sides and 5 mm thick aluminium walls ($k$ = 204 W/mK) is insulated with a 0.1 m layer of cork ($k$ = 0.043 W/mK) and the surface temperatures are $T_i$ = –5°C and $T_0$ = 20°C.
*To find:* The cooling load of the refrigerator.

**Solution** As shown in Fig. Ex. 1.18,

$$Q = \frac{\Delta T}{R_A + R_B},$$

where
$$R_A = \frac{L_A}{k_A A} = \frac{0.005}{204 \times 4} = 6.13 \times 10^{-6} \text{ K/W}$$

$$R_B = \frac{L_B}{k_B B} = \frac{0.10\,\text{m}}{0.043 \times 4} = 0.581 \text{ K/W}$$

Since $R_A << R_B$, $R_A$ can be neglected.

$$\Delta T = 20 - (-5) = 25 \text{ K.}$$

$$\therefore \qquad Q = \frac{25 \text{ K}}{0.581 \text{ K/W}} = 43 \text{ W}$$

$\therefore$ For six sides, the total cooling load on the refrigerator = 6 × 43 = 258 W *Ans.*

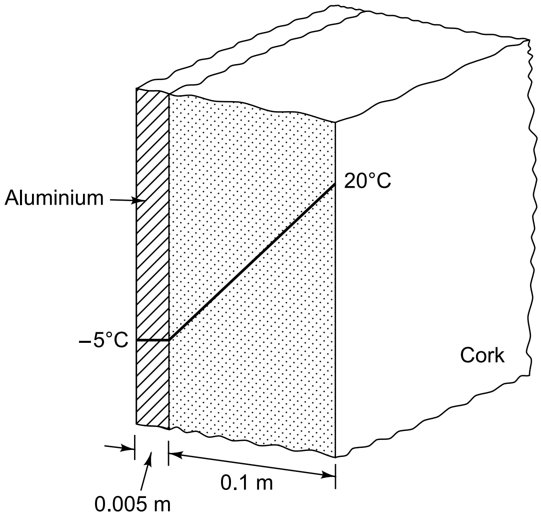


**Fig.** *Ex. 1.18*

**Example 1.19** Air flows through a pipe with a diameter $D$. The velocity distribution in the pipe is approximately given by $u = 20\,[(R - r)/R]^{1/7}$ m/s, $r$ being the radial distance to the point at which $u$ is the velocity, and $R$ is the radius of the pipe. The temperature distribution in the flow is approximately given by $70 - 40\,[(R - r)/R]^{1/7}$ °C. Find the mean temperature in the flow.

**Solution** The bulk temperature of the fluid is given by

$$T_b = \frac{\int_A u\,T\,\mathrm{d}A}{\int_A u\,\mathrm{d}A}$$

Putting
$$\mathrm{d}A = 2\pi r\,\mathrm{d}\,r$$

$$T_b = \frac{\int_0^R u T\; 2\pi r\,\mathrm{d}r}{\int_0^R u\, 2\pi r\,\mathrm{d}r}$$

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or,
$$T_b = \frac{\int_0^1 uT\left(\frac{r}{R}\right) \mathrm{d}\left(\frac{r}{R}\right)}{\int_0^1 u\left(\frac{r}{R}\right) \mathrm{d}\left(\frac{r}{R}\right)}$$

Using the given expressions for the velocity and temperature distributions,

$$T_b = 70 - 20 \times 40 \frac{\int_0^1 \left[1 - \frac{r}{R}\right]^{2/7} \left(\frac{r}{R}\right) \mathrm{d}\left(\frac{r}{R}\right)}{20 \int_0^1 \left(1 - \frac{r}{R}\right)^{1/7} \left(\frac{r}{R}\right) \mathrm{d}\left(\frac{r}{R}\right)}$$

Putting $Y = 1 - \frac{r}{R}$,

$$T_b = 70 - 20 \times 40 \frac{\int_1^0 Y^{2/7}(1 - Y)\, \mathrm{d}Y}{20 \int_1^0 Y^{1/7}(1 - Y)\, \mathrm{d}Y}$$

$$= 70 - \frac{20 \times 40\left[\frac{7}{9} - \frac{7}{16}\right]}{20\left[\frac{7}{8} - \frac{7}{15}\right]}$$

$\therefore$ $T_b = 36.7°\text{C}$ *Ans.*

**Example 1.20** *Given:* A pipe 2 cm in dia. at 40°C is placed in (i) an air flow at 50°C, with $h$ = 20 W/m$^2$K and in (ii) water at 30°C with $h$ = 70 W/m$^2$K.
*To find*: The heat transfer rate per unit length of the pipe.

**Solution** The definition of the mean heat transfer coefficient gives

$$Q = hA(T_w - T_\infty)$$

Here $T_w$ = 40°C, and since 1 m length of pipe is being considered

$$A = \pi DL = \pi \times 0.02 \text{ m}^2$$

$$\therefore \quad Q = h\pi \times 0.02 \times (40 - T_\infty)$$

For case (i),

$$Q = 20 \times \pi \times 0.02 \times (40 - 50)$$
$$= -12.57 \text{ W}$$

The negative sign indicates that the heat transfer is from the air to the cylinder. *Ans.*
For case (ii),

$$Q = 70 \times \pi \times 0.02 \times (40 - 30)$$
$$= 43.98 \text{ W}$$

This result is positive which indicates the heat transfer to be occurring from the cylinder to the water. *Ans.*

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**Example 1.21** *Given:* An electronic transistor capsule of 2 cm dia spherical shape is kept in an evacuated space with black walls at 30°C. Heat loss at the rate of 300 mW from the capsule to the case walls takes place only by radiation.
*To find:* The capsule temperature if it is (i) bright aluminium ($\varepsilon$ = 0.035) and (ii) black anodized aluminium ($\varepsilon$ = 0.80).

**Solution** A 2 cm diameter transistor capsule dissipating 300 mW is a small gray body in a black large enclosure.

$$\therefore \quad Q_{12} = 300 \text{ mW} = 0.3 \text{ W}$$

$$T_2 = 30°\text{C} = 303 \text{ K}$$

$$Q_{12} = \sigma\varepsilon_1 A_1(T_1^4 - T_2^4)$$

$$= 5.67 \times 10^{-8}\, \varepsilon_1\, (\pi \times 0.02^2)\, (T_1^4 - 303^4)$$

$$= 2.268 \times 10^{-3} \times \pi\varepsilon_1 \left[\left(\frac{T_1}{100}\right)^4 - (3.03)^4\right]$$

$$\therefore \quad 0.3 = 7.125 \times 10^{-3}\ \varepsilon_1 \left[\left(\frac{T_1}{100}\right)^4 - 84.29\right]$$

(i) When $\varepsilon_1 = 0.035$,

$$42.105 = \varepsilon_1 \left[\left(\frac{T_1}{100}\right)^4 - 84.29\right]$$

$$1203 = \left(\frac{T_1}{100}\right)^4 - 84.29$$

$$\therefore \quad T_1 = 599 \text{ K or } 326°\text{C} \quad \textit{Ans. (i)}$$

(ii) When $\varepsilon_2 = 0.80$

$$\frac{42.105}{0.8} = \left[\left(\frac{T_1}{100}\right)^4 - 84.29\right]$$

$$\therefore \quad T_1 = 342 \text{ K or } 69°\text{C} \quad \textit{Ans. (ii)}$$

Thus, anodized aluminium gives a satisfactory operating temperature of 69°C, whereas a bright aluminium capsule could not be used, since 326°C is far in excess of allowable operating temperature for semiconductor devices.

**Example 1.22** *Given:* The walls of a cabin (Fig. Ex. 1.22) consist of two layers of pine-wood ($k$ = 0.1 W/mK), each 2 cm thick, sandwiching 5 cm of fibreglass ($k$ = 0.038 W/mK). The cabin inside temperature is 20°C, while the outside temperature $T_0$ is 2°C. The convective heat transfer coefficients are $h_i$ = 3 and $h_0$ = 6 W/m$^2$K. The exterior surface is coated with white acrylic paint ($\varepsilon$ = 0.9).
*To find:* The heat flux through the wall.

**Solution** The heat flux through the wall

$$Q = UA\,(T_i - T_0)$$

where

$$\frac{1}{U} = \frac{1}{h_{c,i}} + \frac{L_A}{k_A} + \frac{L_B}{k_B} + \frac{L_C}{k_C} + \frac{1}{(h_{c,0} + h_{r,0})}$$

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The exterior radiation heat transfer coefficient is given by Eq. (1.47),

$$h_{r,0} = 4\sigma\varepsilon T_m^3$$

$$= 4 \times 5.67 \times 10^{-8} \times 0.9 \times (275)^3 = 4.2 \text{ W/mK}$$

$$\therefore \quad \frac{1}{U} = \frac{1}{3} + \frac{0.02}{0.10} + \frac{0.05}{0.038} + \frac{0.02}{0.10} + \frac{1}{6+4.2}$$

$$= 2.15 \text{ (W/m}^2\text{K)}^{-1}$$

$$\therefore \quad U = 0.466 \text{ W/m}^2\text{K}$$

$$q = \frac{Q}{A} = U(T_i - T_0) = 0.466\,(20 - 2)$$

$$= 8.38 \text{ W/m}^2 \; Ans.$$

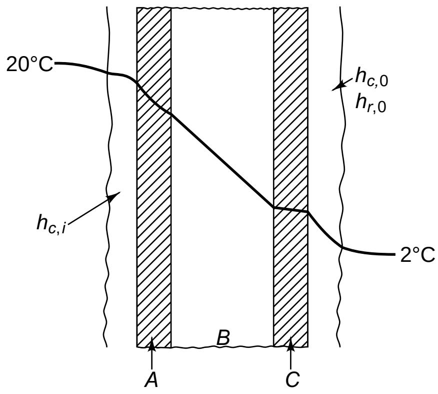


**Fig.** *Ex. 1.22*

The outside resistance is seen to be 0.098/2.15 = 5% of the total resistance. Hence, the outside wall of the cabin is only about 1 K above the ambient air, and our assumption of $T_m$ = 275 K for the evaluation of $h_{r,o}$ is adequate.

**Example 1.23** *Given:* The outer surface temperature of a refrigerator is 16°C where $h$ = 10 W/m$^2$K and the room temperature is 20°C. The sides are 30 cm thick and $k$ = 0.1 W/mK.
*To find*: The net heat flow and inside temperature of the refrigerator.

**Solution** Convective heat flux to the surface

$$q = \frac{Q}{A} = h(T_{s,0} - T_\infty)$$

$$= 10(16 - 20) = -40 \text{ W/m}^2 \; Ans.$$

Since this must be equal to the heat conducted through the sides,

$$q = -k\frac{dT}{dx} = -k\frac{T_{s,0} - T_{s,i}}{L}$$

$$\therefore \quad T_{s,i} = \frac{qL}{k} + T_{s,0} = -\frac{40 \times 0.03}{0.1} + 16$$

$$= 4°\text{C} \; Ans.$$

**Example 1.24** *Given:* A hollow spherical shell ($r_i$ = 0.5 m, $r_0$ = 0.7 m, $k$ = 40 (1 + 0.001 $T$) stores a liquid at 250°C, while the outside surface temperature of the sphere is 100°C. The ambient air is at 30°C.
*To find*: The heat flux through the shell and the outside heat transfer coefficient $h_0$.

**Solution** The rate of heat conduction is

$$Q = -kA\,\frac{dT}{dr}$$

$$= -40(1 + 0.001T)\,4\pi r^2\,\frac{dT}{dr}$$

$$\therefore \quad \frac{Q}{160\pi} \cdot \int_{r_1=0.5\text{m}}^{r_2=0.7\text{m}} \frac{dr}{r^2} = \int_{250°\text{C}}^{100°\text{C}} -(1 + 0.001T)\,dT$$

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$$\frac{Q}{160\pi}\left(\frac{1}{0.5}-\frac{1}{0.7}\right)=(250-100)+\frac{0.001(250^2-100^2)}{2}=176.25$$

$$\therefore \qquad Q = 176.25 \times 160\pi \times \frac{0.35}{0.20} = 155 \text{ kW } Ans.$$

$$Q = h_0 A_0 \Delta T$$

$$155 \times 10^3 = h_0 \times 4\pi(0.7)^2 (100 - 30)$$

$$\therefore \qquad h_0 = 360 \text{ W/m}^2\text{K } Ans.$$

**Example 1.25** *Given:* A steam pipe (o.d. = 10 cm, $T_s$ = 500 K, $\varepsilon$ = 0.8) passing through a large room at 300 K. The pipe loses heat by natural convection ($h$ = 15 W/m$^2$K) and radiation.
*To find*: (i) The surface emissive power of the pipe, (ii) the total radiation falling upon the pipe, and (iii) the total rate of heat loss from the pipe.

**Solution** Surface emissive power of the pipe,

$$E = \varepsilon\,\sigma T^4 = 0.8 \times 5.67 \times 10^{-8} \times (500)^4$$
$$= 2834 \text{ W/m}^2.\ Ans. \text{ (i)}$$

Total radiation falling upon the pipe surface

$$G = \sigma T^4 = 5.67 \times 10^{-8} \times (300)^4 = 459.1 \text{ W/m}^2\ Ans. \text{ (ii)}$$

Heat loss from the pipe by radiation

$$Q_r = \varepsilon A \sigma (T_5^4 - T_\infty^4)$$
$$= 0.8 \times \pi \times 0.1 \times 5.67 \times 10^{-8} \times (500^4 - 300^4) = 775 \text{ W/m}^2$$

Heat loss by natural convection

$$Q_c = h_c A(\Delta T)$$
$$= 15 \times \pi \times 0.1 \times (500 - 300) = 942.6 \text{ W/m}$$

Thus, total rate of heat loss

$$Q = Q_c + Q_r = 942.6 + 775 = 1717.6 \text{ W/m}.\ Ans. \text{ (iii)}$$

## Summary

The essential difference between thermodynamics and heat transfer is explained. Thermodynamics deals with systems in equilibrium and calculates the energy transferred to change a system from one equilibrium state to another. The science of heat transfer not only deals with the rate at which heat flows, but also the temperature distribution in the medium, and the surface area required to accomplish a certain heat duty. First law energy equations for a closed system and a steady flow open-system are stated. A brief reference is made regarding the second law of thermodynamics.

The three modes of heat transfer, viz., conduction, convection and radiation are introduced. Fourier's law of heat conduction, Newton's law of convection cooling, and Stefan-Boltzmann equation of thermal radiation are explained with the help of electrical circuits leading to the definition of overall heat transfer coefficient. An outline of forced and natural convection, laminar and turbulent flows, and the boundary layer concepts are provided. Explanations of contact resistance, thermal conductivity, thermal insulation properties and critical radius of insulation are given.

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A brief introduction to both diffusional and convective mass transfer is provided. Fick's law of diffusion and mass diffusivity are explained, bringing in a close analogy of heat and mass transfer. Examples of simultaneous heat and mass transfer are illustrated.

## Important Formulae and Equations

| Equation number | Equation | Remarks |
|---|---|---|
| (1.5) | $q_k = -k\dfrac{dT}{dx}$ | One-dimensional steady state heat conduction |
| (1.6) | $Q_k = kA\dfrac{T_1 - T_2}{x}$ | Heat transfer through a plane wall of constant $k$ |
| (1.8) | $Q = \dfrac{T_1 - T_2}{R_{th}}, R_{th} = \dfrac{x}{kA}$ | Thermal resistance of a wall |
| (1.10) | $K_k = \dfrac{kA}{L}$ | Thermal conductance |
| (1.11) | $k(T) = k_0(1 + \beta_k T)$ | Thermal conductivity varying with temperature |
| (1.14) | $Q = \dfrac{T_1 - T_4}{R} = \dfrac{T_1 - T_4}{\dfrac{L_1}{k_1 A} + \dfrac{L_2}{k_2 A} + \dfrac{L_3}{k_3 A}}$ | Heat conduction through a composite wall |
| (1.20) | $Q_k = \dfrac{2\pi k L(T_1 - T_2)}{\ln(r_2/r_1)}$ | Steady radial heat conduction in a cylindrical wall |
| (1.21) | $Q_k = -kA_{ln}\dfrac{T_2 - T_1}{x_m}, A_m = \dfrac{A_2 - A_1}{\ln\dfrac{A_2}{A_1}}$ | Heat conduction through a cylindrical wall |
| (1.23) | $T = C_1 \ln r + C_2$ | Temperature variation in a cylindrical wall |
| (1.24) | $Q_k = \dfrac{4\pi k(T_1 - T_2)r_1 r_2}{r_2 - r_1}$ $= -kA_{gm}\dfrac{T_2 - T_1}{x_w}, A_{gm} = \sqrt{A_1 A_2}$ | Steady radial heat conduction in a spherical wall |
| (1.27) | $Q_c = h_c A(T_w - T_\infty), h_c = \dfrac{k_f}{\delta}$ | Convective heat transfer from a wall |
| (1.28) | $R_c = \dfrac{1}{h_c A}$ | Convective resistance |

*(Contd)*

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| *Equation number* | *Equation* | *Remarks* |
|---|---|---|
| (1.30) | $T_b = \dfrac{\int_A uT\mathrm{d}A}{\int_A u\mathrm{d}A}$ | Bulk temperature of a fluid through a pipe |
| (1.32) | $h_c = 3.66\dfrac{k}{D}$ | Heat transfer coefficient for laminar flow through a pipe having uniform wall temperature |
| (1.33) | $h_c = 0.023\dfrac{V^{0.8}(\rho c_p)^{0.4}(k)^{0.6}}{D^{0.2}\,\nu^{0.4}}$ | Heat transfer coefficient for turbulent flow through a pipe |
| (1.34) | $h_c = 1.07\,(\Delta T/x)^{1/4}$ | Natural convection heat transfer coefficient in laminar flow |
| (1.35) | $h_c = 1.3\,(\Delta T)^{1/3}$ | Natural convection heat transfer coefficient in turbulent flow |
| (1.36) | $Q_c = \dfrac{T_h - T_c}{\dfrac{1}{h_{c_1}A} + \dfrac{x}{kA} + \dfrac{1}{h_{c_2}A}} = UA(T_h - T_c)$ | Overall heat transfer coefficient for heat transfer from a hot to a cold fluid through a plane wall |
| (1.39) | $\dfrac{1}{U_0 A_0} = \dfrac{1}{h_i A_i} + \dfrac{x_\omega}{k_w A_{\mathrm{lm}}} + \dfrac{1}{h_0 A_0}$, $Q = U_0 A_0 (T_h - T_c)$ | Overall heat transfer coefficient for heat transfer from a hot to a cold fluid through a cylindrical wall |
| (1.42) | $Q_r = \sigma A T^4$ | Stefan–Boltzmann law for radiative heat transfer from a black body |
| (1.43) | $Q_{1-2} = \sigma A_1 \mathscr{F}_{12}(\mathcal{T}_1^4 - T_2^4)$ | Radiant heat exchange between two gray bodies |
| (1.44) | $F_{12} = \dfrac{1}{\left(\dfrac{1}{\varepsilon_1} - 1\right) + \dfrac{1}{F_{12}} + \dfrac{A_1}{A_2}\left(\dfrac{1}{\varepsilon_2} - 1\right)}$ | View factor for two gray bodies exchanging radiation |
| (1.45) | $A_1 F_{12} = A_2 F_{21}$ | Reciprocity theorem for radiation heat transfer |
| (1.51) | $Q = (h_c + h_r)\,A_1\,(T_w - T_\infty)$ | Heat transfer from a hot body by combined convection and radiation |
| (1.52) | $\dfrac{N_A}{A} = -D\dfrac{\mathrm{d}C_A}{\mathrm{d}y}$ | Fick's law of diffusion for molecular diffusion through a concentration gradient |
| (1.54) | $j_1 = -\rho D_1 \dfrac{\mathrm{d}m_1}{\mathrm{d}x}$ | Diffusion mass flux in $x$-direction |

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## Review Questions

1.1 Explain the scope of study of heat transfer.
1.2 How is the subject of heat transfer different from that of thermodynamics?
1.3 What are the three modes of heat transfer? Explain their differences.
1.4 What does conduction refer to? State Fourier's law of heat conduction. Why is the negative sign used?
1.5 What are the inferences drawn from the temperature gradient $dT/dx = -q_k/k$?
1.6 Explain the resistance concept to illustrate the analogy of heat flow and the flow of electricity.
1.7 How are Fourier's law and Ohm's law similar?
1.8 How do the temperature distributions in a solid vary if its thermal conductivity varies linearly with temperature?
1.9 How does conduction occur in a composite wall with different materials put in (a) series and (b) parallel?
1.10 What do you understand by thermal contact resistance? On what parameters does this resistance depend?
1.11 Explain the effect of contact pressure on thermal contact resistance. What is thermal grease?
1.12 Define thermal conductivity. How can it be determined experimentally? What is the difference between thermal conductivity and thermal conductance?
1.13 Explain the mechanisms of heat conduction in gases, liquids and solids.
1.14 Why are metals good thermal conductors, while non-metals are poor conductors of heat?
1.15 Show that the radial heat conduction through a hollow cylinder depends on the logarithmic mean area of the inside and outside surfaces.
1.16 How does the temperature in a cylindrical wall vary?
1.17 Show that the rate of heat conduction through a hollow sphere is given by

$$Q_k = -kA_{gm}\frac{T_2 - T_1}{x_w}$$

where $A_{gm} = (A_1 A_2)^{1/2}$, $A_1$ and $A_2$ being the areas of inside and outside surfaces of the sphere and $x_w$ = wall thickness.
1.18 What is convection? Why is it regarded as a mode of heat transfer?
1.19 What are the differences between natural and forced convection?
1.20 Define a boundary layer. How are hydrodynamic and thermal boundary layer thicknesses different?
1.21 Explain the velocity and temperature profiles for natural and forced convection heat transfer.
1.22 State the Newton's law of cooling. Define heat transfer coefficient. On what factors does it depend?
1.23 Define overall heat transfer coefficient. Comment on the relative magnitudes of the thermal resistances. What is the difference between $U_i$ and $U_o$?
1.24 What is the mode of heat transfer in vacuum? Define absorptivity, reflectivity and transmissivity.
1.25 How can the absorptivity of an opaque body be improved?
1.26 What is a black body? Define emissivity and a gray body.
1.27 State Stefan-Boltzmann law. On what factors does the radiant heat exchange between two gray bodies depend?
1.28 How is radiation heat transfer coefficient defined? What is combined convection and radiation coefficient?
1.29 Explain the characteristics of thermal insulating materials?
1.30 What are the different types of insulating materials? Give a comparative estimate of fibrous, cellular and granular materials in providing insulation.
1.31 What are the insulating materials used in high temperature applications?
1.32 What is the $R$-value of insulation?
1.33 What is diffusional mass transfer? How is it different from convection mass transfer?
1.34 State and explain the Fick's law of diffusion. What is mass diffusivity? What is its dimension?

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## Objective Type Questions

1.1 In MLT$\theta$ system, the dimension of thermal conductivity is
(a) $ML^{-1}T^{-1}\theta^{-1}$ (b) $MLT^{-1}\theta^{-1}$
(c) $MLT^{-3}\theta^{-1}$ (d) $MLT^{-2}\theta^{-1}$

1.2 With rise in temperature, thermal conductivity of solid metals
(a) increases
(b) decreases
(c) remains the same
(d) first increases and then decreases

1.3 With decrease in temperature thermal conductivity of non-metallic amarphous solids
(a) decreases (b) increases
(c) remains constant (d) is unpredictable

1.4 Thermal contact resistance is a function of
(a) surface roughness
(b) the pressure holding the two surfaces in contact
(c) the interface fluid and its temperature
(d) all of the above

1.5 Arrange the thermal conductivity of the following materials in ascending order copper, mercury, silver, water
Using the codes given beow:
(a) copper, silver, water, mercury
(b) mercury, water, copper, silver
(c) water, mercury, copper, silver
(d) silver, copper, mercury, water

1.6 The thermal conductivity of a structure like concrete, stone, etc., may vary for different samples because of variation in
(a) structure and porosity
(b) density
(c) composition
(d) all of the above

1.7 Match List 1 with List 2 and choose the answer from the codes given below:

| *List 1* | *List 2* |
|---|---|
| A. Gases | 1. Transport of energy by free electrons |
| B. Liquids | 2. Volumetric density |
| C. Porous solid | 3. Unstable elastic collision |
| D. Metals | 4. Random molecular collisions |

Codes:

| | A | B | C | D |
|---|---|---|---|---|
| (a) | 4 | 2 | 3 | 1 |
| (b) | 4 | 3 | 2 | 1 |
| (c) | 2 | 3 | 4 | 1 |
| (d) | 3 | 4 | 2 | 1 |

1.8 Insulating materials used for low temperature applications are
(a) asbestos (b) glass wool
(c) magnesia (d) diatomaceous earth

1.9 Match List I with List II and select the correct answers using the codes given below:

| *List I* | *List II* |
|---|---|
| A. Granular insulation | 1. Vermiculite |
| B. Semiconductors | 2. Germinium |
| C. Low electrical resistivity | 3. Liquid metals |
| D. Nuclear reactors | 4. Pure metals |

Codes:

| | A | B | C | D |
|---|---|---|---|---|
| (a) | 2 | 1 | 4 | 3 |
| (b) | 1 | 2 | 3 | 4 |
| (c) | 1 | 2 | 4 | 3 |
| (d) | 4 | 3 | 2 | 1 |

1.10 The thermal conductivity of a damp brick is higher than that of dry brick, because
(a) the thermal conductivity of air is less than that of water.
(b) the heat transfer takes place in damp bricks by convection due to capillary motion of water within the porous material.

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(c) both (a) and (b)
(d) none of the above

1.11 If $A_1$ and $A_2$ are the internal and external surface areas of a hollow cylinder, the logarithmic mean area is given by
(a) $(A_2 + A_1)/\ln A_2/A_1$
(b) $(\ln A_2/A_1)/(A_2 + A_1)$
(c) $(A_2 - A_1)/\ln(A_2/A_1)$
(d) $\dfrac{A_1 + A_2}{2}$

1.12 Heat conduction through a spherical wall of thickness $x_\omega$ is given by
$Q = -kA_m \dfrac{\Delta T}{x_\omega}$, where $A_m$ is the
(a) arithmetic mean of inside and outside surfaces
(b) geometric mean of inside and outside surfaces
(c) logarithmic mean of inside and outside surfaces
(d) harmonic mean of inside and outside surfaces

1.13 The overall heat transfer coefficient is used in the problems of
(a) conduction
(b) convection
(c) radiation
(d) combined conduction and convection

1.14 In fluid flow the shear stress is confined in the
(a) boundary layer
(b) inviscid, incompressible flow region
(c) free stream flow
(d) turbulent flow

1.15 The Reynolds number of a fluid flowing through a pipe depends on
(a) the velocity of the fluid
(b) the diameter of the pipe
(c) kinematic viscosity of the fluid
(d) all of the above

1.16 For a fully developed laminar flow through a pipe with uniform wall temperature, the heat transfer coefficient is
(a) not directly proportional to the thermal conductivity of the fluid film
(b) inversely proportional to the pipe diameter
(c) dependent on fluid velocity
(d) dependent on fluid viscosity

1.17 For buoyancy-induced fluid flow and heat transfer, this dimensionless number is significant
(a) Reynolds number (b) Prandtl number
(c) Grashof number (d) Nusselt number

1.18 The absorptivity of thermal radiation by a solid surface can be enhanced
(a) by polishing the surface
(b) by roughening the surface
(c) by increasing the surface area
(d) by decreasing the surface area

1.19 The radiant heat exchange between two gray bodies is directly proportional
(a) to the difference in the fourth power of the two temperatures, $T_1^4 - T_2^4$
(b) to the view factor
(c) to the surface area of the emitting body
(d) to all of the above

1.20 At room temperature, the ratio of radiation heat transfer coefficient and the surface emissivity is about
(a) 1.5 (b) 2.0
(c) 3.0 (d) 6.0

1.21 Fick's law of diffusion states that the rate of molecular diffusion is proportional to the concentration gradient

$$\frac{N_A}{A} = -D\frac{dC_A}{dy}$$

where the constant of proportionality $D$ is called the diffusivity. The unit of $D$ is
(a) dimensionless (b) m/s
(c) $m^2/s$ (d) $m/s^2$

1.22 There is no mass transfer analog to
(a) conduction heat transfer
(b) convection heat transfer
(c) radiation heat transfer
(d) heat exchanger

1.23 A furnace is made of fire brick having thickness $x = 0.6$ m and $k = 0.8$ W/mK. For the same heat loss (W/m$^2$) and temperature drop, another material having $k = 0.16$ W/mK will have its thickness

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(a) 0.06 m (b) 0.12 m
(c) 0.24 m (d) 0.48 m

1.24 A masonary wall ($k$ = 0.75 W/mK) transmits 80% of the heat loss through another wall ($k$ = 0.25 W/mK, $x$ = 100 mm). If the temperature difference across both the walls is the same, the thickness of the masonary wall would be
(a) 150 mm (b) 240 mm
(c) 300 mm (d) 375 mm

1.25 If $M$ represents the molecular weight and $T$ represents the temperature of a gas, the thermal conductivity of the gas will increase when
(a) both $M$ and $T$ increase
(b) $M$ decreases and $T$ increases
(c) both $M$ and $T$ decrease
(d) $M$ increases and $T$ decreases

1.26 In the heat flow equation $Q = -kA\dfrac{t_1 - t_2}{x}$, $x/kA$ is known as
(a) thermal coefficient
(b) thermal conductivity
(c) thermal resistance
(d) temperature gradient

1.27 It has the least value of thermal conductivity:
(a) Rubber (b) Air
(c) Water (d) Plastic

1.28 The radial heat conduction through a hollow cylinder increases as the ratio of outer radius to inner radius
(a) decreases
(b) increases
(c) remains the same
(d) follows none of the above

1.29 A furnace wall has $x$ = 20 cm and $k$ = 0.1 W/mK with surface temperatures of 600°C and 400°C. The rate of heat conduction is
(a) 200 kW/m$^2$ (b) 100 kW/m$^2$
(c) 50 kW/m$^2$ (d) 300 kW/m$^2$

## Answers

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1.1 | (c) | 1.2 | (b) | 1.3 | (a) | 1.4 | (d) | 1.5 | (c) |
| 1.6 | (d) | 1.7 | (b) | 1.8 | (b) | 1.9 | (c) | 1.10 | (c) |
| 1.11 | (c) | 1.12 | (b) | 1.13 | (d) | 1.14 | (a) | 1.15 | (d) |
| 1.16 | (b) | 1.17 | (c) | 1.18 | (b) | 1.19 | (d) | 1.20 | (d) |
| 1.21 | (c) | 1.22 | (c) | 1.23 | (b) | 1.24 | (d) | 1.25 | (b) |
| 1.26 | (c) | 1.27 | (b) | 1.28 | (a) | 1.29 | (b) | | |

## Open Book Problems

1.1 A plane wall is 150 mm thick and its wall area is 4.5 m$^2$. If its thermal conductivity is 9.35 W/mK and surface temperatures are steady at 150°C and 50°C, determine (a) the heat flow across the plane wall, and (b) the temperature gradient in the flow direction.

*Hints*: Use Eq. (1.5) to find

$$Q = -\,kA\frac{T_2 - T_1}{L} \text{ and } \frac{dT}{dx} = -\frac{Q}{kA}.$$

1.2 During an experiment to determine the thermal conductivity of a material used in a thick cylindrical shell (inner radius 10 cm and outer radius, 20 cm) two thermocouples were inserted, one at a radius equal to 12 cm and the other at a radius equal to 18 cm. The temperatures reacorded were 100°C and 50°C respectively. If the heat transfer per metre length of the shell was 600 W, calculate the thermal conductivity of the material and the temperatures at the inner and outer surfaces of the shell.

*Hints*: From Eq. (1.20),

$$k = Q\,\ln\left(\frac{r_2}{r_1}\right)\Big/2\pi L(T_1 - T_2)$$

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Substitute $Q$ = 600 W, $r_2$ = 18 cm, $r_1$ = 12 cm, $L$ = 1 m, $T_1$ = 100°C and $T_2$ = 50°C, find $k$. Temperature at the inner radius = $T_1 + Q \ln(12/10)/2\pi kL$ and temperature at the outer radius = $T_2 - Q \ln(20/18)/2\pi kL$.

1.3 The temperature at the inner radius ($r_1$ = 5 cm) is 125°C and at the outer radius ($r_2$ = 10 cm) is 60°C in a spherical shell. Calculate the rate of heat flow through, the shell if $k$ of the shell material is 2 W/mK. What would be the temperature of the shell material at a raidus of 7.5 cm?

*Hints*: Use Eq. (1.24) to find

$$Q = \frac{4\pi k(T_1 - T_2)r_1 r_2}{r_2 - r_1}$$

At $r$ = 7.5 cm,

$T = T_1 - Q(r - r_1)/4\pi k r r_1$.

1.4 An electric current is passed through a wire, 1 mm in diameter and 10 cm long. The wire is submerged in liquid water at atmospheric pressure and the current is increased until the water boils. For this situation, $h$ = 5000 W/m$^2$K, and the water temperature is 100°C. How much electrical power must be supplied to the wire to maintain the wire surface at 120°C?

*Hints*: Use Eq. (1.31) to find $Q = h_c A(T_\omega - T_b)$ where $A = \pi dL$, $T_w$ = 120°C, $T_b$ = 100°C and $h_c$ = 5000 W/m$^2$K.

1.5 A horizontal steel pipe of 50-mm diameter is maintained at 60°C in a large room where the air and wall temperature is at 25°C. The surface emissivity of steel is 0.8. If $h_c$ = 6.5 W/m$^2$K, calculate the total heat lost by the pipe per unit length.

*Hints*: From Eq. (1.37)

find $Q_c/L = h_c \pi d(T_\omega - T_b)$

and from Eq. (1.46),

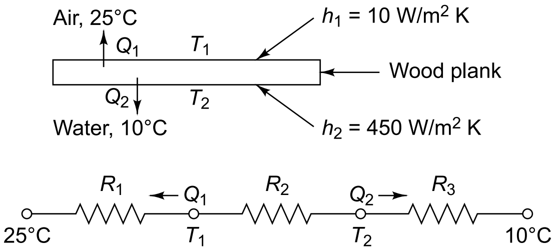
find $Q_r/L = \varepsilon(\pi d_1)\,\sigma(T_1^4 - T_2^4)$,

where $T_1$ = 333 K and $T_2$ = 298 K.

Total heat loss is $Q/L = Q_c/L + Q_r/L$.

1.6 A 0.625 cm thick plank of wood ($k$ = 0.166 W/mK) floats in a large pool of water. The convective heat transfer coefficients between the top surface of the wood and the air, and between the bottom surface of the wood are 10 and 450 W/m$^2$K respectively. The wood absorbs 470 W/m$^2$ of solar radiation. The temperature of the air is 25°C and that of water is 10°C. Ignoring radiation from the wood, calculate the rate of heat transfer from wood to air and water, and the temperature of the bottom surface of the wood.

*Hints*:




The schematic diagram and the equivalent electric circuit are shown above. Find the thermal resistances:

$$R_1 = 1/h_1,\ R_2 = L/k,\ R_3 = \frac{1}{h_2}$$

Then, $Q_1$ = Heat transfer from wood to air = $(T_1 - 25)/R_1$

and $Q_2$ = Heat transfer from wood to water = $(T_1 - 10)/(R_1 + R_2)$

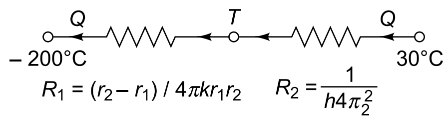
$Q_1 + Q_2$ = 470 W/m$^2$

Find $Q_1$ and $Q_2$, and

$T_2 = T_1 - Q_2 R_2$.

1.7 A spherical vessel 1 m in diameter stores liquid nitrogen at –200°C. Calculate the amount of nitrogen vaporized per day if the container is provided with a 5 cm thick layer of superinsulation having $k$ = 0.00017 W/mK. The outside temperature is 30°C, the convective heat transfer coefficient at the outer surface is 15 W/m$^2$K and 200 kJ of energy is required to vaporize 1 kg of nitrogen at that temperature. Find also the temperature at the outside surface of the superinsulation.

*Hints*:



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The equivalent electrical circuit is shown above.

Find $R_1 = (r_2 - r_1)/4\pi k r_1 r_2$

and $R_2 = \dfrac{1}{h 4\pi r_2^2}$.

Total resistance,

$$R = R_1 + R_2$$

Use $Q = \dfrac{\Delta T}{R}$ watts

Mass of nitrogen vaporized per day

$$= \frac{Q \times 3600 \times 24}{200 \times 1000}$$

Temperature of the outer surface of superinsulation

$$T = 30 - QR_2.$$

1.8 A circular conducting rod, diameter $d$ and length $L$, having an electrical resistance per unit length $R_e$, is in thermal equilibrium with its surroundings. Obtain an expression to compute the variation in temperature of the rod with time, when an electric current I is passed through the rod.

*Hints*: By energy balance,

Energy generated within the rod = Energy convected and radiated away + Energy stored.

$$I^2 R_e L = \sigma \varepsilon \pi d L\,(T^4 - T_\infty^4) +$$

$$h_c \pi d L (T - T_\infty) + p_c \frac{\pi d^2}{4} L \frac{dT}{dt}.$$

Hence, find $\dfrac{dT}{dt}$.

## Problems for Practice

1.1 The heat flow rate through a 3 cm thick wood board for a temperature difference of 30°C between the two surfaces is 120 W/m$^2$. Calculate the thermal conductivity of the wood. (*Ans.* 0.12 W/m K)

1.2 A hot plate maintained at a temperature of 120°C dissipates heat at the rate of 7500 W/m$^2$ to the ambient air at 30°C. Calculate the heat transfer coefficient for convection between the plate and the air. (*Ans.* 83.3 W/m$^2$ K)

1.3 A circular plate of 0.2 m diameter has one of its surfaces insulated, and the other is maintained at 550 K. If the hot surface has an emissivity of 0.9 and is exposed to the air at 300 K, calculate the heat loss by radiation from the plate to the air. (*Ans.* 134.5 W)

1.4 The inside surface of an insulating layer is at 270°C, and the outside surface is dissipating heat by convection into air at 20°C. The insulation layer is 40 mm thick and has a thermal conductivity of 1.2 W/m K. What is the minimum value of the heat transfer coefficient at the outside surface if the outside surface temperature should not exceed 70°C. (*Ans.* 120 W/m$^2$ K)

1.5 A thin metallic plate is insulated at the back surface and is exposed to the sun at the front surface. The front surface absorbs solar radiation at 900 W/m$^2$ and dissipates it mainly by convection to the ambient air at 30°C. If the heat transfer coefficient between the plate and the air is 15 W/m$^2$ K, what is the temperature of the plate? (*Ans.* 90°C)

1.6 A thin plate 500 mm by 500 mm is subjected to 400 W of heating on one surface and dissipates the heat by combined convection and radiation from the other surface into the ambient air at 290 K. If the surface of the plate has an emissivity of 0.9 and the heat transfer coefficient between the surface and the ambient air is 15 W/m$^2$ K, calculate the temperature of the plate. (*Ans.* 89°C)

1.7 A large plane wall is 0.35 m thick. One surface is maintained at 35°C and the other surface is at 115°C. Only two values of thermal conductivity are available for the wall material. At 0°C, $k$ = 26 W/m K and at 100°C, $k$ = 32 W/m K. Determine the heat flux (W/m$^2$) through the wall assuming that

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thermal conductivity varies linearly with temperature. (*Ans.* 6970 W/m$^2$)

1.8 A cubical tank of water of volume 1 m$^3$ is kept at a steady temperature of 65°C by a 1 kW heater. The heater is switched off. How long does the tank take to cool to 50°C if the room temperature is 15°C. (*Ans.* 20.64 h)

1.9 A 1.2 m high and 2 m wide double-pane window consists of two 3 mm thick layers of glass ($k$ = 0.78 W/m K) separated by a 12 mm wide stagnant air gap ($k$ = 0.026 W/m K). Determine the steady rate of heat transfer through this double-paned window and the temperature of its inner surface for a day during which the room is maintained at 24°C while the temperature of the outdoors is – 5°C. Take the convection heat transfer coefficients on the inner and outer surfaces of the window to be $h_1$ = 10 W/m$^2$ K and $h_z$ = 25 W/m$^2$ K, and disregard any heat transfer by radiation.

(*Ans.* 113 W, 19.2°C)

1.10 A 15 cm × 18 cm epoxy glass laminate ($k$ = 0.26 W/m K) of 0.14 mm thickness has 1 mm diameter cylindrical copper fillings ($k$ = 386 W/m K) planted throughout the board with centre-to-centre distance of 3 mm. Determine the thermal resistance of the epoxy board for heat conduction.

(*Ans.* 0.00153 K/W)

1.11 A long cylindrically heated rod, 2 cm in diameter, is installed in a vacuum furnace. The surface of the heating rod has an emisivity of 0.9 and is maintained at 1000 K while the interior walls of the furnace are at 800 K. Calculate the net rate at which heat is lost from the rod per unit length and the radiation heat transfer coefficient.

(*Ans.* 1893 W, 150.6 W/m$^2$ K)

1.12 A fluid at an average temperature of 200°C flows through a plastic pipe ($k$ = 0.5 W/m K) of 4 cm outer diameter and 3 cm inner diameter. If the heat transfer coefficient at the inside is 300 W/m$^2$ K and that at the outer surface is 10 W/m$^2$ K, and if the pipe is located in a room at 30°C, find the heat loss per unit length of pipe.

(*Ans.* 157 W/m)

1.13 Liquid nitrogen is stored in a thin-walled spherical metallic container at 77 K. The container has a diameter of 0.5 m and is covered with an evacuated, reflective insulation system composed of silica powder ($k$ = 0.0017 W/m K). The insulation is 25 mm thick and its outer surface is exposed to ambient air at 300 K. The latent heat of vaporisation of liquid nitrogen is $2 \times 10^5$ J/kg. If the convection coefficient is 20 W/m$^2$ K over the outer surface, determine the rate of heat transfer and the rate of liquid boil-off of nitrogen per hour.

(*Ans.* 13.06 W, 0.235 kg/h)

1.14 Water flows at 50°C inside a 2.5 cm inner diameter tube such that $h_i$ = 3500 W/m$^2$ K. The tube has a wall thickness of 0.8 mm with a thermal conductivity of 16 W/m K. The outside of the tube loses heat by free convection with $h_o$ = 7.6 W/m$^2$ K. Calculate the overall heat transfer coefficient and heat loss per metre length to surrounding air at 20°C.

(*Ans.* $U_o$ = 7.58 W/m$^2$ K, 19 W)

1.15 Two stainless steel bars, each of diameter 2 cm and length 3 cm are pressed together with a pressure of 10 atm. The surface has a roughness of 2.5 $\mu$m for which the contact conductance is 3000 W/m$^2$ K. An overall temperature difference of $\Delta T$ = 100°C is applied across the bars. The interface temperature is about 90°C. Calculate (a) the heat flow rate along the bars and (b) the temperature drop at the interface.

(*Ans.* (a) 9.42 W, (b) 10°C)

1.16 A surface whose temperature is maintained at 400°C is separated from an airflow by a layer of insulation 25 mm thick for which the thermal conductivity is 0.1 W/m K. If the air temperature is 35°C and the convection coefficient between the air and the outer surface of the insulation is 500 W/m$^2$ K, what is the temperature of this outer surface?

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# Conduction Heat Transfer at Steady State

# 2

In conduction mode heat is transferred through a complex submicroscopic mechanism that involves flow of free electrons and lattice vibrations in a solid. From an engineering point of view there is no need to delve into the complexities of the mechanism, because the rate of heat propagation can be predicted from the macroscopic phenomenological relation as depicted by the Fourier's law. Although conduction also occurs in liquids and gases, it is rarely the predominant heat transfer mechanism, except in liquid metals, since once heat begins to flow in a fluid, even if no external force is applied, density gradients are set up and convective currents are set in motion. Heat is then transported on a macroscopic scale as well as on a microscopic scale with convection currents generally being the more effective. Conduction heat transfer has been a fertile field for applied mathematicians for the past 200 years [1]. Laplace and Fourier obtained analytical solutions of the partial differential equations for different heat conduction problems. However, the analytic approach is limited to simple geometrical shapes and boundary conditions. With the advent of the high speed computers it is now possible to solve complex heat conduction problems with relative ease.

## 2.1 GENERAL EQUATION OF HEAT CONDUCTION

Heat flows spontaneously from one body to another body (or from one part of a body to another part of the same body) only when the bodies are at different temperatures. If a temperature difference exists, heat flows in the direction of decreasing temperature. By the first law of thermodynamics the flowing thermal energy is conserved in the absence of heat sources or sinks. Thus a solid may have a temperature distribution which is dependent upon the space coordinates ($x$, $y$, $z$) and time of observation ($t$):

$$T = f(x, y, z, t)$$

We may suppose that within this solid there is a surface such that, when observed at a certain time, each point on it has the same temperature. Such a surface is called an isothermal surface. We can further visualise other isothermal surfaces within this body which differ from one another by being hotter or colder by amounts $\pm\ \delta T$, respectively (Fig. 2.1). These isothermal surfaces never intersect because no point in this solid can exist at two different temperatures at the same time. The solid is thus visualised as being composed of a number of thin isothermal shells that vary with time. We shall consider only isotropic solids, i.e., solids whose properties and constitution in the neighbourhood of any point are invariant with the direction from the point. Thus the heat flow at a point is along a path perpendicular to the isothermal surface through the point (isothermal and adiabatic planes being orthogonal).

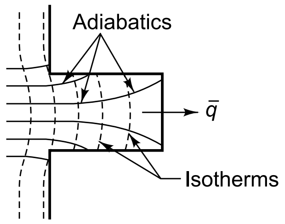

**Fig. 2.1** *Isotherms and adiabatics*

The mathematical formulation of the law of heat conduction (Fourier's law) as interpreted in Fig. 2.2 is expressed as

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$$Q = -kA\frac{\mathrm{d}T}{\mathrm{d}n} \quad \text{or} \quad q_n = -k\frac{dT}{dn} \tag{2.1}$$

The heat flux $Q/A$ or $q$ flows along the normal $n$ of the decreasing temperature. It is a vector quantity. In Fig. 2.3 are shown the isotherms $T$ and $T + \mathrm{d}T$ in a body. The normal to these isotherms is designated by the axis $n$, which is also normal to the differential area d$A$. The heat flux can be calculated in the direction of the normal and in the direction $s$ shown as follows:

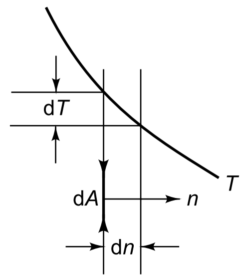


**Fig. 2.2** *Heat conduction system*

$$q_n = \frac{\mathrm{d}Q}{\mathrm{d}A} = -k\frac{\partial T}{\partial n}$$

$$q_s = \frac{\mathrm{d}Q}{\mathrm{d}A\cos\theta} = -k\frac{\partial T}{\partial s}$$

It is seen that $n = s\cos\theta$. Therefore,

$$q_s = -k\frac{\partial T}{\partial n}\cos\theta \tag{2.2}$$

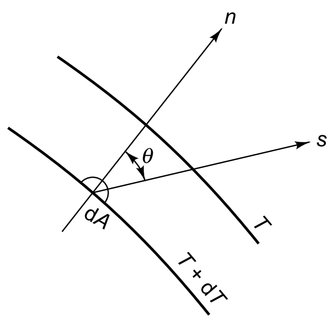


**Fig. 2.3** *Direction of heat flow*

Thus, $q_s$ is a component of the heat flux vector $q_n$. The greatest heat flux will occur when it is calculated along the normal to the isothermal surface ($\theta = 0$). If the component fluxes are related to the planes containing the $x, y, z$ coordinate system, the fluxes are

$$q_x = -k\frac{\partial T}{\partial x}, \quad q_y = -k\frac{\partial T}{\partial y}, \quad q_z = -k\frac{\partial T}{\partial z} \tag{2.3}$$

For an isotropic solid $k_x = k_y = k_z = k$. The fluxes shown in Eq. (2.3) are components of the heat flux vector so that

$$\bar{q} = iq_x + jq_y + kq_z = -k\left(i\frac{\partial T}{\partial x} + j\frac{\partial T}{\partial y} + k\frac{\partial T}{\partial z}\right)$$

$$= -\,\mathbf{k}\ \mathbf{grad}\ \mathbf{T} = -\,\mathbf{k}\,\nabla\,\mathbf{T} \tag{2.4}$$

We will now derive the heat conduction equation in differential form. A solution of this equation, subject to given initial and boundary conditions, yields the temperature distribution in a solid. Once the temperature distribution is known, the heat transfer rate can be evaluated by applying Fourier's law Eq. (1.5). To derive the conduction equation we perform an energy balance on an elemental volume, which includes the possibility of heat generation in the material. Heat generation can result from chemical reactions, electric currents passing through the material or nuclear reactions. Heat transfer problems are classified according to the variables that influence the temperature. If the temperature is a function of time, the problem is classified as *unsteady* or *transient*. If the temperature is independent of time, the problem is called a *steady-state* problem. If the temperature is a function of a single space coordinate, the problem is said to be *one-dimensional*. If it is a function of two or three coordinate dimensions, the problem is *two-* or *three-dimensional* respectively. First we will derive the equation in rectangular coordinate system.

Let us consider an infinitesimal volume element of sides $\delta x$, $\delta y$ and $\delta z$ (Fig. 2.4). The considerations here will include the nonsteady condition of temperature variation with time $t$.

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According to the Fourier heat conduction law, the heat flowing into the left-most face of the element in the $x$-direction

$$dQ_x = -k\,\delta y\,\delta z\,\frac{\partial T}{\partial x}$$

The value of the heat flow out of the right face of the element can be obtained by expanding $dQ_x$ in a Taylor series and retaining only the first two terms as an approximation:

$$dQ_{x+dx} = dQx + \frac{\partial}{\partial x}(dQ_x)\,\delta x + \ldots$$

The net heat flow by conduction in the $x$-direction is therefore

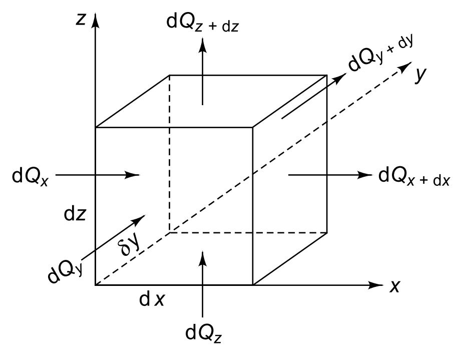


**Fig. 2.4** *Volume element for determining heat conduction equation*

$$dQ_x - dQ_{x+dx} = -\frac{\partial}{\partial x}(dQ_x)\,\delta x = -\frac{\partial}{\partial x}\left(-k\,\delta y\,\delta z\,\frac{\partial T}{\partial x}\right)\delta x$$

$$= k\,\frac{\partial^2 T}{\partial x^2}\,\delta x\;\delta y\;\delta z$$

Similarly, the net heat flows in the $y$- and $z$-direction are

$$dQ_y - dQ_{y+dy} = k\,\frac{\partial^2 T}{\partial y^2}\,\delta x\;\delta y\;\delta z$$

$$dQ_z - dQ_{z+dz} = k\,\frac{\partial^2 T}{\partial z^2}\,\delta x\;\delta y\;\delta z$$

Here, the solid has been assumed to be isotropic and homogeneous with properties uniform in all directions. Let us consider that there is some heat source within the solid, and heat is produced internally as a result of the flow of electrical current or nuclear or chemical reactions. Let $q_G$ is the rate at which heat is generated internally per unit volume (W/m$^3$). Then the total rate of heat generation in the elemental volume is

$$q_G\;\delta x\;\delta y\;\delta z$$

The net heat flow owing to conduction and the heat generated within the element together will increase the internal energy of the volume element. The rate of accumulation of internal energy within the control volume is

$$\rho c\;\;\delta x \cdot \delta y \cdot \delta z\;\frac{\partial T}{\partial t}$$

where $c$ is the specific heat and $\rho$ is the density of the solid.

An energy balance can be achieved on the volume element as:

Rate of energy storage within the solid = Rate of heat influx – Rate of heat outflux + Rate of heat generation

$$\text{or}\quad \rho c \cdot \delta x\;\delta y\;\delta z\;\frac{\partial T}{\partial t} = (dQ_x + dQ_y + dQ_z) - (dQ_{x+dx} + dQ_{y+dy} + dQ_{z+dx}) + q_G\;\delta x\;\delta y\;\delta z$$

$$= k\;\delta x\;\delta y\;\delta z\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + q_G\;\delta x\;\delta y\;\delta z$$

$$\text{or}\qquad \rho c\,\frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + q_G \tag{2.5}$$

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$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{q_G}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{2.6}$$

where $\alpha$ is the thermal diffusivity of the solid given by $k/\rho c$.

If the temperature of a material is not a function of time, the system is in the steady state and does not store any energy. The steady state form of a three-dimensional conduction equation in rectangular coordinates is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{q_G}{k} = 0 \tag{2.7}$$

It is known as *Poisson's equation*.

If the system is in steady state and no heat is generated internally, the conduction equation further simplifies to

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{2.8}$$

Equation (2.8) is known as the *Laplace equation*. It occurs in a number of areas in addition to heat transfer, for example, in diffusion of mass or in electromagnetic fields. The operation of taking the second derivatives of the potential in a field has therefore been given a short symbol, $\nabla^2$, called the *Laplacean* operator. For the rectangular coordinate system Eq. (2.8) becomes

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \nabla^2 T = 0 \tag{2.9}$$

Since the operator $\nabla^2$ is *independent of coordinate system*, the above form will be useful when we study conduction in cylindrical and spherical coordinates. The heat conduction Eq. (2.6) can thus be written as

$$\nabla^2 T + \frac{q_G}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{2.10}$$

### 2.1.1 Vector Method

Equation (2.10) can also be derived vectorially. Let us consider a control surface $S$ enclosing a volume $V$ (Fig. 2.5).

The net rate of heat outflow across the surface $S$ is given by $\oint_s q \cdot n \, dS$ where $n$ is the normal direction. Converting the surface integral to volume integral

$$\oint_s qn \, dS = \oint_v \operatorname{div} q \, dV \tag{2.11}$$

where $\bar{q}$ is the heat flux per unit area.

The net rate of *heat inflow* to the control volume (CV) is $-\oint_v \operatorname{div} q \, dV$

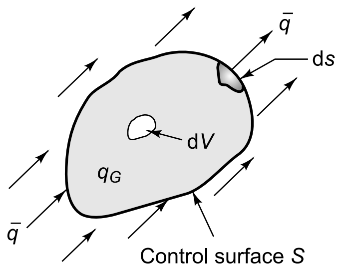

**Fig. 2.5** *Heat conduction through a volume element*

If there is a volumetric heat source inside the CV, the rate of heat generation is

$$\oint_v q_G \, dV$$

The rate of energy accumulation within the CV is

$$\frac{\partial}{\partial t}\oint_v e\rho \, dV = \frac{\partial}{\partial t}\oint_v cT \rho \, dV$$

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$e$ being the specific energy (J/kg)

By energy balance,

$$\frac{\partial}{\partial t}\oint_v \rho c T \, \mathrm{d}V = \oint_v q_G \, \mathrm{d}V - \oint_v \operatorname{div} \bar{q} \, \mathrm{d}V \tag{2.12}$$

This is the energy equation for the entire CV. Writing the energy equation for the elemental volume d$V$ within the CV

$$\rho c \frac{\partial T}{\partial t} \mathrm{d}V = q_G \, \mathrm{d}V - \operatorname{div} \bar{q} \, \mathrm{d}V$$

Since d$V$ is now independent, it can be removed from the above equation. Therefore,

$$\rho c \frac{\partial T}{\partial t} = q_G - \operatorname{div} \bar{q} \tag{2.13}$$

Now, $\operatorname{div} \bar{q} = \nabla \cdot \bar{q}$

and from Eq. (2.4)

$$\bar{q} = -k \nabla T$$

$\therefore$ $\operatorname{div} \bar{q} = \nabla \cdot (-k \nabla T) = -k \nabla^2 T$, for constant $k$

Substituting in Eq. (2.13)

$$\rho c \frac{\partial T}{\partial t} = q_G + k \nabla^2 T$$

Therefore, $\mathbf{\nabla^2 T + \frac{q_G}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}}$

which is the same as Eq. (2.10).

### 2.1.2 Cylindrical Coordinates

For a general transient three-dimensional heat conduction problem in the cylindrical coordinates with $T = T(r, \theta, z, t)$, let us consider an elementary volume d$V$ = d$r r$ d$\theta$ d$z$ (Fig. 2.6).

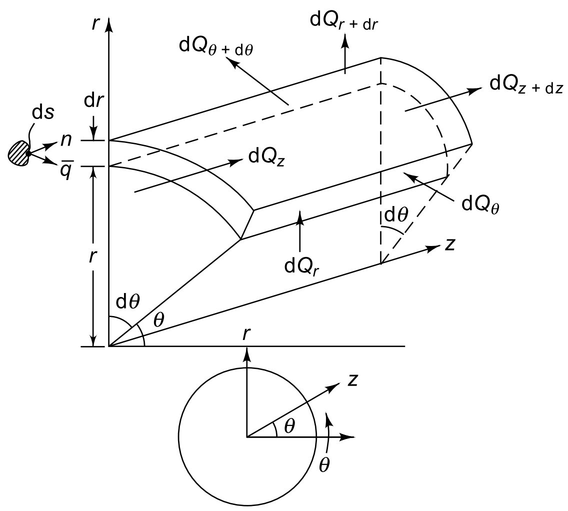


**Fig. 2.6** *Heat conduction in a cylindrical volume element (r, θ, z)*

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By Fourier's law,

$$dQ_r = -k(r\, d\theta\, dz)\frac{\partial T}{\partial r}$$

$$dQ_{r+dr} = dQ_r + \frac{\partial}{\partial r}(dQ_r)\, dr$$

$$dQ_r - dQ_{r+dr} = -\frac{\partial}{\partial r}\left(-kr\, d\theta\, dz\frac{\partial T}{\partial r}\right)dr$$

$$= kr\, d\theta\, dz\, dr\frac{\partial^2 T}{\partial r^2} + k\, d\theta\, dz\, dr\frac{\partial T}{\partial r} \tag{2.14}$$

Similarly, $$dQ_\theta = -k\, dz\, dr\frac{\partial T}{r\partial \theta}$$

$$dQ_{\theta+d\theta} = dQ_\theta + \frac{\partial}{r\partial\theta}(dQ_\theta)\, r\, d\theta$$

$$dQ_\theta - dQ_{\theta+d\theta} = -\frac{\partial}{\partial\theta}\left(-k\, dz\, dr\frac{\partial T}{r\partial\theta}\right)d\theta$$

$$= k\, dz\, dr\, d\theta\frac{1}{r}\frac{\partial^2 T}{\partial\theta^2} \tag{2.15}$$

$$dQ_z = -k\, dr\, r\, d\theta\frac{\partial T}{\partial z}$$

$$dQ_{z+dz} = dQ_z + \frac{\partial}{\partial z}(dQ_z)\, dz$$

$$dQ_z - dQ_{z+dz} = -\frac{\partial}{\partial z}\left(-k\, dr\, r\, d\theta\frac{\partial T}{\partial z}\right)dz$$

$$= k\, dr r\, d\theta\, dz\frac{\partial^2 T}{\partial z^2} \tag{2.16}$$

Rate of heat generation from an internal heat source

$$= q_G\, dr r\, d\theta\, dz \tag{2.17}$$

Rate of energy accumulation within the CV

$$= \rho(dr\, r\, d\theta\, dz)\, c\frac{\partial T}{\partial t} \tag{2.18}$$

By energy balance, from Eqs (2.14)–(2.18),

$$\rho dr\, r\, d\theta\, dz\, c\frac{\partial T}{\partial t} = kr\, d\theta\, dz\, dr\frac{\partial^2 T}{\partial r^2} + k\, d\theta\, dz\, dr\frac{\partial T}{\partial r} + k\, dz\, dr\, d\theta\frac{\partial^2 T}{\partial\theta^2} + k\, dr\, r\, d\theta\, dz\frac{\partial^2 T}{\partial z^2} + q_G\, dr\, r\, d\theta\, dz$$

$$\rho c\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial r^2} + k\frac{1}{r}\frac{\partial T}{\partial r} + k\frac{1}{r^2}\frac{\partial^2 T}{\partial\theta^2} + k\frac{\partial^2 T}{\partial z^2} + q_G$$

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or
$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} + \frac{q_G}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

or
$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} + \frac{q_G}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{2.19}$$

This is the general heat conduction equation in cylindrical coordinates. If we compare this equation with Eq. (2.10), the Laplacian is

$$\nabla^2 T = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \tag{2.20}$$

If heat flows only in radial direction, $T = T(r, t)$, Eq. (2.19) reduces to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{q_G}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{2.21}$$

If the temperature distribution does not vary with time, then at steady state,

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{q_G}{k} = 0 \tag{2.22}$$

In this case the equation for the temperature contains only a single variable $r$ and is therefore an ordinary differential equation.

When there is no volumetric energy generation and the temperature is a function of the radius only, the steady-state conduction for cylindrical coordinates is

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0 \tag{2.23}$$

### 2.1.3 Spherical Coordinates

For spherical coordinates, as shown in Fig. 2.7(a), the temperature is a function of the three space

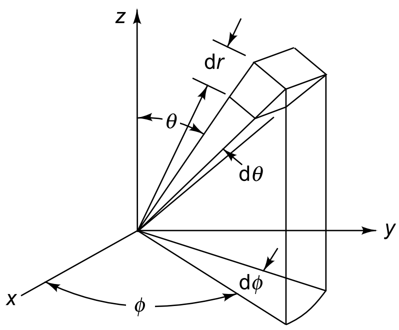


**Fig. 2.7(a)** *Spherical coordinate system for the general conduction equation*

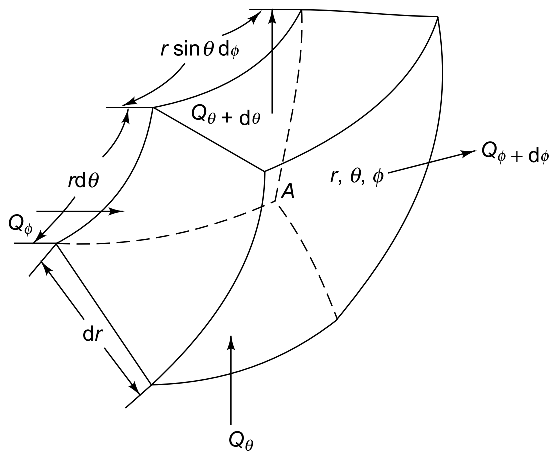


**Fig. 2.7(b)** *Elemental spherical volume*

coordinates $r$, $\theta$ and $\phi$ and time $t$, i.e., $T = T(r, \theta, \phi, t)$. The general form of the conduction equation in spherical coordinates can be found as

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$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2}+\frac{q_G}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{2.24}$$

where the Laplacian includes the first three terms of the above equation in spherical coordinates.

Let us consider an elemental volume having the coordinates ($r$, $\phi$, $\theta$) for three dimensional heat conduction analysis [Fig. 2.7(b)].

The volume of the element = $\mathrm{d}r \cdot r\mathrm{d}\theta \cdot r\sin\theta\,\mathrm{d}\phi$.

**(A) Net heat accumulated in the element due to conduction of heat from all the coordinate directions**

Heat flow through $r-\theta$ plane, $\phi$-direction:

Heat inflow, $$Q_\phi = -k\,\mathrm{d}r\cdot r\mathrm{d}\theta\,\frac{\partial T}{r\sin\theta\,\partial\phi}\,\mathrm{d}t$$

Heat outflow, $$Q_{\phi+\mathrm{d}\phi} = Q_\phi + \frac{\partial Q_\phi}{r\sin\theta\,\partial\phi}\,r\sin\theta\,\mathrm{d}\phi$$

$\therefore$ Heat accumulated in the element for heat flow in $\phi$ direction

$$\mathrm{d}Q_\phi = Q_\phi - Q_{\phi+\mathrm{d}\phi} = -\frac{1}{r\sin\theta}\frac{\partial Q_\phi}{\partial\phi}\,r\sin\theta\,\mathrm{d}\phi$$

$$= -\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left[-k\mathrm{d}r\,r\mathrm{d}\theta\frac{1}{r\sin\theta}\frac{\partial T}{\partial\phi}\cdot\mathrm{d}t\right]r\sin\theta\,\mathrm{d}\phi$$

$$= k(\mathrm{d}r\cdot r\mathrm{d}\theta\cdot r\sin\theta\cdot\mathrm{d}\phi)\,\frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2}\,\mathrm{d}t \tag{2.25}$$

Heat flow in $r$-$\phi$ plane, $\theta$-direction:

Heat inflow, $$Q_\theta = -k(\mathrm{d}r\times r\sin\theta\,\mathrm{d}\phi)\,\frac{\partial T}{r\,\partial\theta}\cdot\mathrm{d}t$$

Heat outflow, $$Q_{\theta+\mathrm{d}\theta} = Q_\theta + \frac{\partial}{r\,\partial\theta}\,(Q_\theta)\cdot r\mathrm{d}\theta$$

Heat accumulated in the element for heat flow in $\theta$-direction:

$$\mathrm{d}Q_\theta = Q_\theta - Q_{\theta+\mathrm{d}\theta}$$

$$= -\frac{\partial}{r\partial\theta}(Q_\theta)\,r\,\mathrm{d}\theta$$

$$= -\frac{\partial}{r\partial\theta}\left[-k\,\mathrm{d}r.r\sin\theta\,\mathrm{d}\phi\frac{\partial T}{r\partial\theta}.\mathrm{d}t\right]r\mathrm{d}\theta$$

$$= k\,\mathrm{d}r\cdot r\mathrm{d}\theta\cdot r\sin\theta\,\mathrm{d}\phi\,\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial T}{\partial\theta}\right]\mathrm{d}t \tag{2.26}$$

Heat flow in $\theta$-$\phi$ plane, $r$-direction:

Heat inflow, $$Q_r = -k(r\mathrm{d}\theta\cdot r\sin\theta\,\mathrm{d}\phi)\,\frac{\partial T}{\partial r}\,\mathrm{d}t$$

Heat outflow, $$Q_{r+\mathrm{d}r} = Q_r + \frac{\partial}{\partial r}(Q_r)\cdot\mathrm{d}r$$

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Heat accumulation in the element due to heat flow in $r$-direction

$$\mathrm{d}Q_r = Q_r - Q_{r+\mathrm{d}r}$$

$$= -\frac{\partial Q_r}{\partial r}\cdot \mathrm{d}r$$

$$= -\frac{\partial}{\partial r}\left[-k(r\mathrm{d}\theta\cdot r\sin\theta\;\mathrm{d}\phi)\frac{\partial T}{\partial r}\cdot\mathrm{d}r\right]\mathrm{d}t$$

$$= k\mathrm{d}r\cdot r\mathrm{d}\theta\cdot r\sin\theta\;\mathrm{d}\phi\;\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial T}{\partial r}\right]\mathrm{d}t \tag{2.27}$$

Net heat accumulated in the element

$$\mathrm{d}Q = \mathrm{d}Q_\phi + \mathrm{d}Q_\theta + \mathrm{d}Q_r$$

$$= k\mathrm{d}r\cdot r\mathrm{d}\theta\cdot r\sin\theta\;\mathrm{d}\phi\cdot\left[\frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2}+\frac{1}{r^2\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)+\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)\right]\mathrm{d}t \tag{2.28}$$

**(B) Heat generated within the element**

$$Q_G = q_G\,(\mathrm{d}r\cdot r\mathrm{d}\theta\cdot r\sin\theta\;\mathrm{d}\phi)\mathrm{d}t \tag{2.29}$$

**(C) Energy stored in the element**

$$\rho\mathrm{d}r\cdot r\mathrm{d}\theta\cdot r\sin\theta\;\mathrm{d}\phi\cdot c\,\frac{\partial T}{\partial z}\cdot\mathrm{d}t$$

By energy balance,

$$(\mathrm{A}) + (\mathrm{B}) = (\mathrm{C})$$

$$\therefore\quad k\mathrm{d}r\cdot r\mathrm{d}\theta\cdot r\sin\theta\;\mathrm{d}\phi\cdot\left[\frac{1}{r^2\sin^2\theta}\cdot\frac{\partial^2 T}{\partial\phi^2}+\frac{1}{r^2\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)+\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)\right]\mathrm{d}t + q_G\,\mathrm{d}r\cdot r\mathrm{d}\theta\cdot r\sin\theta\;\mathrm{d}\phi\cdot\mathrm{d}t$$

$$= \rho\mathrm{d}r\cdot r\mathrm{d}\theta\cdot r\sin\theta\;\mathrm{d}\phi\cdot c\cdot\frac{\partial T}{\partial\tau}\cdot\mathrm{d}t$$

Dividing both sides by $k\mathrm{d}r\cdot r\mathrm{d}\theta\cdot r\sin\theta\cdot\mathrm{d}\phi\cdot\mathrm{d}t$,

$$\frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)+\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{q_G}{k}$$

$$= \frac{\boldsymbol{\rho}\mathbf{c}}{\mathbf{k}}\frac{\partial\mathbf{T}}{\partial\boldsymbol{\tau}} = \frac{\mathbf{1}}{\boldsymbol{\alpha}}\frac{\partial\mathbf{T}}{\partial\boldsymbol{\tau}} \tag{2.30}$$

This is the general heat conduction equation in spherical coordinates. If there is no heat source present and the heat flow is steady and only in radial direction. Equation (2.30) reduces to

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{\mathrm{d}T}{\mathrm{d}r}\right) = 0$$

Equation (2.30) can also be derived by transformation of coordinates as follows:

$$x = r\sin\theta\sin\phi;\quad y = r\sin\theta\cos\phi;\quad z = r\cos\theta.$$

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## 2.2 STEADY HEAT CONDUCTION IN SIMPLE GEOMETRIES

We will now derive solutions to the conduction equations as obtained in the previous section for simple geometrical systems with and without heat generation.

### 2.2.1 Plane Wall

#### (a) Without Heat Generation

In the first chapter we saw that the temperature distribution for one-dimensional steady conduction through a wall is linear. We will verify this result by simplifying the more general equation [Eq. (2.10)]

$$\nabla^2 T + \frac{q_G}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

For steady state, $\partial T/\partial t = 0$. Since $T$ is only a function of $x$, $\partial T/\partial y = 0$ and $\partial T/\partial z = 0$. There is no internal heat generation, $q_G = 0$. Therefore, the above equation reduces to

$$\frac{d^2T}{dx^2} = 0 \tag{2.31}$$

Integrating the ordinary differential equation twice yields the linear temperature distribution

$$T(x) = C_1 x + C_2 \tag{2.32}$$

For a wall (Fig. 2.8), at $x = 0$, $T = T_1$ and at $x = b$, $T = T_2$

$$T = -\frac{T_1 - T_2}{b}x + T_1 \tag{2.33}$$

which agrees with the linear temperature distribution deduced by integrating Fourier's law, $Q_k = -kA\dfrac{dT}{dx}$.

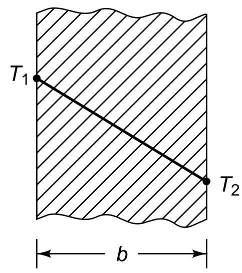


**Fig. 2.8** *Heat conduction through a wall*

#### (b) With Heat Generation

Let us now consider a heat source generating heat throughout the system. If the thermal conductivity is constant and the heat generation is uniform, Eq. (2.10) reduces to

$$\frac{d^2T}{dx^2} + \frac{q_G}{k} = 0 \tag{2.34}$$

On integration,

$$\frac{dT}{dx} = -\frac{q_G x}{k} + C_1$$

A second integration gives

$$T(x) = -\frac{q_G}{2k}x^2 + C_1 x + C_2 \tag{2.35}$$

where $C_1$ and $C_2$ are constants.

At $x = 0$, $T = T_1$ and at $x = b$, $T = T_2$ substituting in Eq. (2.35),

$$T_1 = C_2$$

$$T_2 = -\frac{q_G}{2k}b^2 + C_1 b + T_1$$

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$$C_1 = \frac{T_2 - T_1}{b} + \frac{q_G}{2k} b$$

Therefore, the temperature distribution is

$$T(x) = -\frac{q_G}{2k} x^2 + \frac{T_2 - T_1}{b} x + \frac{q_G b}{2k} x + T_1 \tag{2.36}$$

It may be seen that Eq. (2.32) is now modified by two terms containing the heat generation and that the temperature distribution is no longer linear but parabolic.

If the two surface temperatures are equal, $T_1 = T_2$, then the temperature distribution becomes

$$\mathbf{T(x) = -\frac{q_G}{2k} b^2 \left[\frac{x}{b} - \left(\frac{x}{b}\right)^2\right] + T_1} \tag{2.37}$$

which is parabolic and symmetric about the central plane with a maximum $T_{max}$ at $x = b/2$.

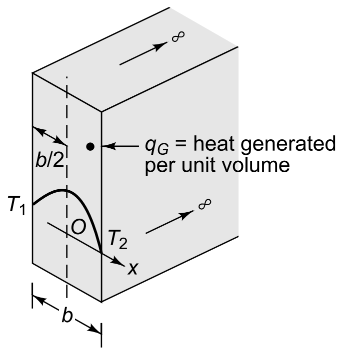


**Fig. 2.9** *Heat flow through a wall with heat generation*

$$\frac{dT}{dx} = -\frac{q_G}{2k} b^2 \left(\frac{1}{b} - \frac{2x}{b^2}\right) = 0$$

or $$\frac{2x}{b^2} = \frac{1}{b} \text{ or } x = \frac{b}{2}$$

and $$\mathbf{T_{max} = T_1 + \frac{q_G b^2}{8k}} \tag{2.38}$$

In dimensionless form, on dividing Eq. (2.37) by Eq. (2.38) and putting

$$\xi = x/b,$$

$$\frac{T(x) - T_1}{T_{max} - T_1} = 4(\xi - \xi^2) \tag{2.39}$$

Let us consider the case where heat is transferred from the two sides of the wall to the surrounding fluid at $T_\infty$ (Fig. 2.10). For simplicity, let us assume that both the wall surfaces are at $T_w$. At steady state and for one-dimensional heat flow,

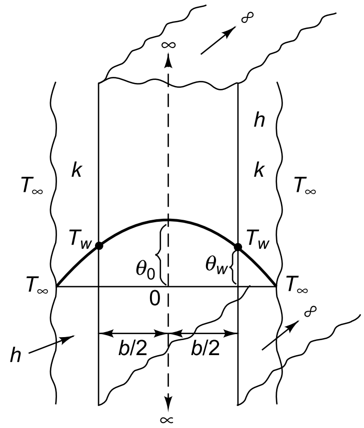


**Fig. 2.10** *Heat transfer from two sides of a wall having a heat source*

$$\frac{d^2T}{dx^2} + \frac{q_G}{k} = 0$$

Let the excess temperature be

$$\theta = T - T_\infty$$

so that, $$\frac{d^2\theta}{dx^2} = \frac{d^2T}{dx^2}$$

Therefore, $$\frac{d^2\theta}{dx^2} = -\frac{q_G}{k}$$

$$\frac{d\theta}{dx} = -\frac{q_G}{k} x + C_1$$

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and $$\theta = -\frac{q_G}{2k}x^2 + C_1 x + C_2 \tag{2.40}$$

which is parabolic. The central plane is the plane of symmetry where the solid temperature is the maximum and $d\theta/dx = 0$ and is taken as the reference plane with $x = 0$.

At $$x = 0, \frac{d\theta}{dx} = 0$$

$$C_1 = 0$$

At $$x = \frac{b}{2}, \left(\frac{d\theta}{dx}\right)_{b/2} = -\frac{q_G}{k}\frac{b}{2}$$

Again, at the wall

$$q_k = -k\left(\frac{\partial\theta}{\partial x}\right)_{b/2} = k\frac{q_G}{k}\frac{b}{2} = \frac{q_G b}{2}$$

$$= h(T_w - T_\infty) = h\theta_w$$

$$\theta_w = \frac{q_G b}{2h}$$

From Eq. (2.40)

$$\theta = -\frac{q_G}{2k}x^2 + C_2$$

when $$x = b/2, \theta = \theta_w$$

$$\theta_w = -\frac{q_G}{2k}\frac{b^2}{4} + C_2$$

$$C_2 = \theta_w + \frac{q_G}{2k}\frac{b^2}{4} = \frac{q_G b}{2h} + \frac{q_G}{8k}b^2$$

Therefore, the final temperature distribution is

$$\theta = -\frac{q_G}{2k}x^2 + \frac{q_G b}{2h} + \frac{q_G}{8k}b^2$$

$$\mathbf{T = \frac{q_G}{8k}(b^2 - 4x^2) + \frac{q_G b}{2h} + T_\infty} \tag{2.41}$$

This is the temperature distribution. At the mid-plane, $x = 0$ and $T = T_{max}$.

$$\mathbf{T_{max} = \frac{q_G}{8k}b^2 + \frac{q_G b}{2h} + T_\infty} \tag{2.42}$$

### 2.2.2 Cylindrical Wall

#### *(a) Without heat Generation*

Let us consider heat transfer through a hollow pipe with a fluid flowing inside (Fig. 2.11). Heat is assumed to flow only radially. We want to determine the temperature distribution and the heat transfer rate in a long hollow cylinder of length $L$ if the inside and outside surface temperatures are $T_i$ and $T_o$ respectively and there is no internal heat generation.

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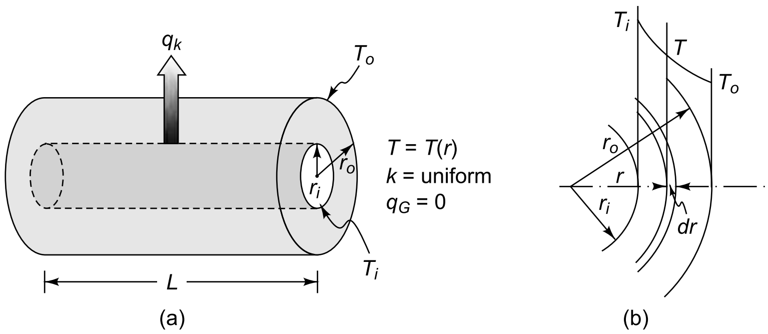


**Fig. 2.11** *Radial heat conduction through a hollow cylinder*

Since the temperatures at the two surfaces are constant, the temperature distribution in the wall is not a function of time, and the conduction equation is given by Eq. (2.23)

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}T}{\mathrm{d}r}\right) = 0$$

On integration

$$r\frac{\mathrm{d}T}{\mathrm{d}r} = C_1$$

or
$$\frac{\mathrm{d}T}{\mathrm{d}r} = \frac{C_1}{r}$$

A second integration yields

$$T = C_1 \ln r + C_2 \tag{2.43}$$

At $r = r_i,\ T = T_i$

$$T_i = C_1 \ln r_i + C_2$$

At $r = r_o,\ T = T_o$

$$T_o = C_1 \ln r_o + C_2$$
$$= C_1 \ln r_o + T_i - C_1 \ln r_i$$

$$C_1 = \frac{T_o - T_i}{\ln(r_o / r_i)} \text{ and } C_2 = T_i - \frac{T_o - T_i}{\ln(r_o / r_i)} \ln r_i$$

Substituting $C_1$ and $C_2$ in Eq. (2.43).

$$T = \frac{T_o - T_i}{\ln(r_o / r_i)} \ln r + T_i - \frac{T_o - T_i}{\ln(r_o / r_i)} \ln r_i$$

$$\frac{T(r) - T_i}{T_o - T_i} = \frac{\ln(r / r_i)}{\ln(r_o / r_i)} \tag{2.44}$$

The rate of heat transfer by conduction

$$Q_k = -kA\frac{\mathrm{d}T}{\mathrm{d}r} = -k2\pi rL\frac{C_1}{r} = -k2\pi L\frac{T_o - T_i}{\ln(r_o / r_i)}$$

$$\mathbf{Q_k = \frac{2\pi kL(T_i - T_o)}{\ln(r_o / r_i)}} \tag{2.45}$$

which is the same as given by Eq. (1.20).

The thermal resistance offered by the wall is

$$\mathbf{R_{th}} = \frac{\mathbf{T_i - T_o}}{\mathbf{Q_k}} = \frac{\mathbf{\ln (r_o / r_i)}}{\mathbf{2\pi kL}} \tag{2.46}$$

The rate of heat conduction through a composite cylindrical wall with convection at the inside and outside surfaces (Fig. 2.12) is given by

$$Q = \frac{\Delta T}{\Sigma R_{th}} = \frac{T_h - T_c}{\dfrac{1}{h_i\, 2\pi\, r_1 L} + \dfrac{\ln (r_2 / r_1)}{2\pi\, k_1 L} + \dfrac{\ln (r_3 / r_2)}{2\pi\, k_2 L} + \dfrac{1}{h_o 2\pi\, k_3 L}} \tag{2.47}$$

where $T_h$ and $T_c$ are the hot and cold fluid temperatures, $h_i$ and $h_o$ are the inside and outside heat transfer coefficients and $k_1$ and $k_2$ are the thermal conductivities of the two walls in series. Now,

$$Q = U_o A_o (T_h - T_c) \tag{2.48}$$

where $U_o$ is the overall heat transfer coefficient based on outside area $A_o = 2\pi\, r_o L$ given by

$$\frac{\mathbf{1}}{\mathbf{U_o A_o}} = \mathbf{\Sigma\, R_{th}} = \frac{\mathbf{1}}{\dfrac{\mathbf{1}}{\mathbf{h_i A_i}} + \dfrac{\mathbf{\ln (r_2 / r_1)}}{\mathbf{2\pi k_1 L}} + \dfrac{\mathbf{\ln (r_3 / r_2)}}{\mathbf{2\pi k_2 L}} + \dfrac{\mathbf{1}}{\mathbf{h_o A_o}}} \tag{2.49}$$

where $r_i = r_1$ and $r_o = r_3$.

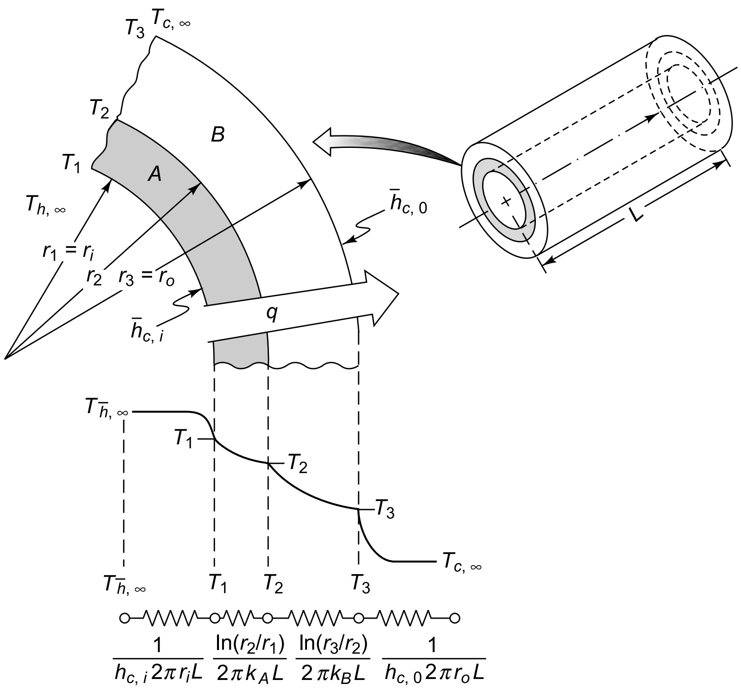


**Fig. 2.12** *Temperature distribution in a composite cylinder with convection at the interior and exterior surfaces*

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To find the wall temperatures $T_1$, $T_2$ and $T_3$, we can use

$$T_h - T_1 = QR_1 = \frac{Q}{h_i A_i}\,; \quad T_1 - T_2 = \frac{Q\, 2\pi\, k_1\, L}{\ln\,(r_2 / r_1)}$$

$$T_2 - T_3 = \frac{Q}{h_o\ A_o}$$

### (b) With heat Generation

Let us consider a long solid cylinder of radius $R$ with internal heat generation (Fig. 2.13), such as an electric coil in which heat is generated as a result of the electric current in the wire or a cylindrical nuclear fuel element in which heat is generated by nuclear fission. The one-dimensional heat conduction equation in cylindrical coordinates is

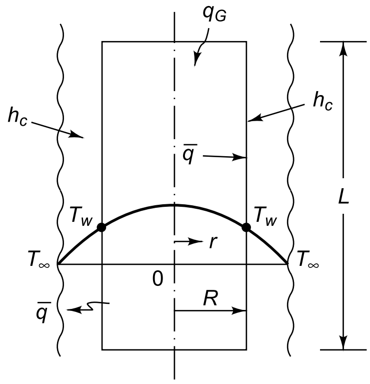


**Fig. 2.13** *Temperature distribution in a cylindrical rod with heat generation*

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}T}{\mathrm{d}r}\right) + \frac{q_G}{k} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}T}{\mathrm{d}r}\right) = -\frac{q_G r}{k}$$

On integration, $$r\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{q_G r^2}{2k} + C_1$$

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{q_G r}{2k} + \frac{C_1}{r} \tag{2.50}$$

By second integration,

$$T = -\frac{q_G r^2}{4k} + C_1 \ln r + C_2 \tag{2.51}$$

At $r = 0$, $\mathrm{d}T/\mathrm{d}r = 0$ (with origin at the centre line of the cylinder). But from Eq. (2.50), $\mathrm{d}T/\mathrm{d}r = \infty$, which is impossible.

At $$r = R,\ (\mathrm{d}T/\mathrm{d}r)_{r=R} = -\frac{q_G R}{2k} + \frac{C_1}{R} \tag{2.52}$$

Heat generated in the cylindrical rod $= q_G\, \pi R^2\, L$

This heat is conducted to the surface and then convected away.

$$q_G\, \pi R^2\, L = -k\, 2\pi\, LR\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_{r=R}$$

$$\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_{r=R} = -\frac{q_G R}{2k} \tag{2.53}$$

From Eqs (2.52) and (2.53), $C_1 = 0$

$$T = -\frac{q_G}{4k}\, r^2 + C_2 \tag{2.54}$$

At $$r = R,\ T = T_w$$

$$C_2 = T_w + \frac{q_G}{4k}\, R^2$$

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Substituting in Eq. (2.54),

$$T = -\frac{q_G}{4k} r^2 + T_w + \frac{q_G}{4k} R^2$$

or
$$T(r) = \frac{q_G}{4k} (R^2 - r^2) + T_w$$

$$= \frac{\mathbf{q_G R^2}}{\mathbf{4k}} \left( \mathbf{1} - \frac{\mathbf{r^2}}{\mathbf{R^2}} \right) + \mathbf{T_w} \tag{2.55}$$

This is the temperature variation along the wall radius. The maximum temperature occurs at $r = 0$,

$$T_{\max} = \frac{q_G R^2}{4k} + T_w \tag{2.56}$$

In dimensionless form Eq. (2.55) becomes

$$\frac{T(r) - T_w}{T_{\max} - T_w} = 1 - \left( \frac{r}{R} \right)^2 \tag{2.57}$$

For a hollow cylinder with uniformly distributed heat source and specified surface temperature $T = T_i$ at $r = r_i$ and $T = T_o$ at $r = r_o$, Eq. (2.51) gives

$$T_i = -\frac{q_G}{4k} r_i^2 + C_1 \ln r_i + C_2$$

$$T_o = -\frac{q_G}{4k} r_o^2 + C_1 \ln r_o + C_2$$

Evaluating $C_1$ and $C_2$ from the above two equations and substituting in Eq. (2.51) we can obtain the temperature distribution as

$$T(r) = \frac{q_G}{4k} (r_o^2 - r^2) + \frac{\ln (r / r_o)}{\ln (r_o / r_i)} \left[ \frac{q_G}{4k} (r_o^2 - r_i^2) + T_o - T_i \right] + T_o \tag{2.58}$$

If a solid cylinder is immersed in a fluid at $T_\infty$ (Fig. 2.13), and the convection heat transfer coefficient is $h_c$ and $T = T_w$ at $r = R$, the heat conduction from the cylinder is equal to the rate of convection at the surface, or

$$-k \left( \frac{dT}{dr} \right)_{r=R} = h_c (T_w - T_\infty) \tag{2.59}$$

From Eq. (2.53),

$$\left( \frac{dT}{dr} \right)_{r=R} = -\frac{h_c (T_w - T_\infty)}{k} = -\frac{q_G R}{2k}$$

$$T_w = T_\infty + \frac{q_G R}{2h_c} \tag{2.60}$$

From Eq. (2.55),

$$\mathbf{T(r)} = \frac{\mathbf{q_G R^2}}{\mathbf{4k}} \left( \mathbf{1} - \frac{\mathbf{r^2}}{\mathbf{R^2}} \right) + \mathbf{T_\infty} + \frac{\mathbf{q_G R}}{\mathbf{2h_c}} \tag{2.61}$$

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In dimensionless form,

$$\frac{T(r)-T_\infty}{T_\infty} = \frac{q_G R}{4h_c\, T_\infty}\left[2+\frac{h_c R}{k}\left\{1-\left(\frac{r}{R}\right)^2\right\}\right] \tag{2.62}$$

and the maximum temperature is

$$\frac{\mathbf{T_{max}}}{\mathbf{T_\infty}} = \mathbf{1} + \frac{\mathbf{q_G R}}{\mathbf{4h_c\, T_\infty}}\left(\mathbf{2}+\frac{\mathbf{h_c R}}{\mathbf{k}}\right) \tag{2.63}$$

There are two dimensionless parameters in the above equation which are important in conduction, viz. $q_G R/h_c T_\infty$, the heat generation number, and $h_c R/k$, the Biot number, which appears in problems with simultaneous conduction and convection.

The Biot number is the ratio of conduction resistance to convection resistance or

$$Bi = \frac{R_k}{R_c} = \frac{r_o / k}{1/h_c} = \frac{h_c r_o}{k}$$

The limits of Biot number are

$$Bi \to 0 \text{ when } R_k = \frac{r_o}{k} \to 0 \text{ or, } R_c = \frac{1}{h_c} \to \infty$$

$$Bi \to \infty \text{ when } R_c = \frac{1}{h_c} \to 0 \text{ or, } R_k = \frac{r_o}{k} \to \infty$$

The Biot number approaches zero when the conductivity of solid is very large ($k \to \infty$) or the convection coefficient of heat transfer is very low ($h_c \to 0$), i.e., when the solid is practically isothermal and the temperature change is mostly caused in the fluid by convection at the interface. On the contrary, the Biot number approaches infinity when the thermal resistance predominates ($k \to 0$) or the convection resistance is very low ($1/h_c \to 0$).

### 2.2.3 Spherical Wall

#### *(a) Without Heat Generation*

For a hollow sphere with uniform temperature at the inner and outer surfaces (Fig. 2.14), the temperature distribution without heat generation in the steady state can be obtained by simplifying Eq. (2.30). Under these conditions the temperature is only a function of the radius $r$, and the conduction equation in spherical coordinates is

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}T}{\mathrm{d}r}\right) = 0 \tag{2.64}$$

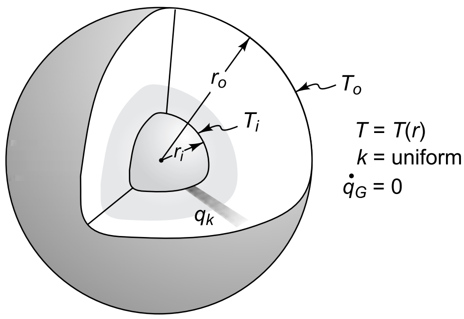


**Fig. 2.14** *Heat conduction in a hollow sphere without heat generation*

On integration, and putting at $r = r_i$, $T = T_i$ and at $r = r_o$, $T = T_o$

$$T(r) - T_i = (T_o - T_i)\,\frac{r_o}{r_o - r_i}\left(1-\frac{r_i}{r}\right) \tag{2.65}$$

The rate of heat transfer through the spherical wall is

$$Q_k = -k\,4\pi r^2\,\frac{\mathrm{d}T}{\mathrm{d}r}$$

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$$= \frac{\mathbf{T_i - T_o}}{\mathbf{(r_o - r_i)/4\pi\, k\, r_o r_i}} \tag{2.66}$$

which is the same as Eq. (1.24) derived in Chapter 1. The thermal resistance for a spherical wall is then

$$\mathbf{R_{th}} = \frac{\mathbf{r_o - r_i}}{\mathbf{4\pi\, k\, r_o r_i}} \tag{2.67}$$

### (b) With Heat Generation

The governing equation in spherical coordinates is given by

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}T}{\mathrm{d}r}\right) + \frac{q_G}{k} = 0$$

Upon integration,

$$r^2\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{q_G r^3}{3k} + C_1$$

or,

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{q_G r}{3k} + \frac{C_1}{r^2}$$

By integrating again,

$$T = -\frac{q_G r^2}{6k} - \frac{C_1}{r} + C_2 \tag{2.68}$$

**Case 1:** When the sphere is *solid*, the temperature gradient at the centre of the sphere ($r = 0$) will be finite and hence $C_1 = 0$.

If $T_w$ = temperature at the outer surface, $r = R$,

$$T_w = -\frac{q_G R^2}{6k} + C_2 \qquad \text{[from Eq. (2.68)]}$$

Therefore, the temperature distribution would be

$$T = T_w + \frac{q_G(R^2 - r^2)}{6k} \tag{2.69}$$

The temperature will be maximum at $r = 0$ and is given by

$$\mathbf{T_{max} = T_w + \frac{q_G R^2}{6k}} \tag{2.70}$$

**Case 2:** When the sphere is hollow with inner radius $r_1$ and outer radius $r_2$, the boundary conditions are: at $r = r_1$, $\frac{\mathrm{d}T}{\mathrm{d}r} = 0$, because the sphere is a closed surface, and at $r = r_2$, $T = T_2 = T_w$

$$0 = -\frac{q_G r_1}{3k} + \frac{C_1}{r_1^2}$$

$$\therefore \quad C_1 = \frac{q_G \cdot r_1^3}{3k}$$

$$\therefore \quad T = -\frac{q_G r^2}{6k} - \frac{q_G r_1^3}{3kr} + C_2$$

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At $\qquad r = r_2, \quad T = T_w$

$$T_w = -\frac{q_G r_2^2}{6k} - \frac{q_G r_1^3}{3k r_2} + C_2$$

$$\therefore \qquad C_2 = T_w + \frac{q_G}{3k}\left(\frac{r_2^2}{2} + \frac{r_1^3}{r_2}\right)$$

$\therefore$ The temperature distribution is given by

$$T = T_w + \frac{q_G}{6k}\,(r_2^2 - r^2) + \frac{q_G r_1^3}{3k}\left(\frac{1}{r_2} - \frac{1}{r}\right) \tag{2.71}$$

## 2.3 CRITICAL RADIUS OF INSULATION

Let us consider a small-diameter tube, cable or wire, the outside surface of which has a constant temperature and dissipates heat by convection into the surrounding air. Let the surface be covered with a layer of insulation. It is desired to examine the variation in heat loss from the tube surface as the thickness of insulation is changed. As insulation is added to the tube, the outer exposed surface temperature will decrease because of higher conduction resistance, but at the same time the surface area available for convective heat dissipation will increase causing more heat loss. These two opposing effects lead to an optimum insulation thickness.

Let the tube of radius $r_t$ and at temperature $T_t$ be covered with insulation. At the outside radius of the insulation $r_o$, a surface coefficient $h_a$ is assumed for heat transfer to the atmosphere at temperature $T_a$ (Fig. 2.15). As $r_o$ increases, $x_w$ $(= r_o - r_t)$ increases and $Q$ decreases. Again, as $r_o$ increases, $A_o$ increases and $Q$ increases. There is, thus, an optimum $r_o$ at which $Q$ is the maximum.

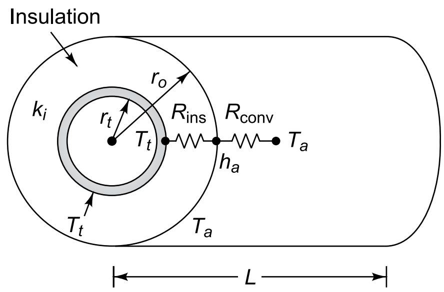


**Fig. 2.15** *Critical radius of insulation*

$$T_t - T_a = Q(R_1 + R_2) = Q\left[\frac{\ln (r_o / r_i)}{2\pi\, k_i\, L} + \frac{1}{2\pi\, r_o\, L\, h_a}\right] \tag{2.72}$$

$$Q = \frac{2\pi\, L\,(T_t - T_a)}{\dfrac{1}{k_i}\ln\dfrac{r_o}{r_t} + \dfrac{1}{h_a r_o}}$$

Since $T_t$, $T_a$, $k_i$, $r_t$ and $h_a$ are all constant, the heat loss $Q$ depends only on $r_o$. As $r_o$ increases, $\dfrac{1}{k_i}\ln (r_o/r_t)$ increases, but $1/(h_a r_o)$ decreases. Differentiating $Q$ with $r_o$,

$$\frac{\mathrm{d}Q}{\mathrm{d}r_o} = -2\pi L\,(T_t - T_a)\left(\frac{1}{k_i}\frac{r_t}{r_o}\frac{1}{r_t} - \frac{1}{h_a\, r_o^2}\right) = 0$$

$$\therefore \qquad \mathbf{(r_o)_{cr} = \frac{k_i}{h_a}} \tag{2.73}$$

If we take second derivative of $Q$ with respect to $r_o$,

$$\frac{\mathrm{d}^2Q}{\mathrm{d}r_o^2} = -2\pi L(T_t - T_a)\left(-\frac{1}{k_i r_o^2} + \frac{2}{h_a r_o^3}\right)$$

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Substituting $\quad \mathbf{r_o = r_{cr} = \dfrac{k_i}{h_a}}$,

$$\left.\frac{d^2Q}{dr_0^2}\right|_{r_o = r_c} = -2\pi L (T_t - T_a)\left(\frac{h_a^2}{k_i^3} + \frac{2h_a^2}{k_i^3}\right)$$

$$= -2\pi L (T_t - T_a)\,\frac{h_a^2}{k_i^3} < 0$$

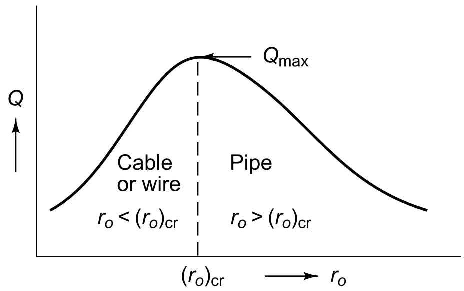


**Fig. 2.16** *Variation of insulation radius influencing heat loss to the outside*

Therefore, at $r_o = r_{cr}$, the heat loss will be maximum.

If $r_t < (r_o)_{cr}$, as $r_o$ increases, $Q$ increases till $r_o = (r_o)_{cr}$ (Fig. 2.16).

If $r_o > (r_o)_{cr}$, as $r_o$ increases, $Q$ decreases. If $r_t > (r_o)_{cr}$, any increase of insulation will decrease the rate of heat transfer. If $r_t < (r_o)_{cr}$, the increase of insulation will increase $Q$ till $Q = Q_{\max}$.

For pipes, $r_t$ is taken higher than $(r_o)_{cr}$, so that any insulation added will only decrease the heat loss from the pipe. For wires and cables, $r_t$ is kept lower than $(r_o)_{cr}$ so that insulation added increases the heat loss from the wire or cable. An insulated small diameter wire has a higher current carrying capacity than an uninsulated one. If the current flowing through an uninsulated wire increases, $I^2R$ increases, and if heat dissipation from the wire is not equal to $I^2R$, the temperature of the wire goes on increasing till it exceeds the melting point and the wire snaps. If the wire is insulated, it can dissipate more heat (till $r_t = (r_o)_{cr}$) and the wire temperature remains below the melting point.

In the case of a sphere, by following a similar procedure, the critical radius of insulation is given by

$$\mathbf{(r_o)_{cr} = \frac{2k_i}{h_a}} \tag{2.74}$$

The total thermal resistance (for spherical wall of outside radius $r$),

$$R_{th} = \frac{r_o - r_t}{4\pi\, k_i r_o r_t} + \frac{1}{4\pi\, r_o^2 h_a}$$

$$= \frac{1}{4\pi}\left[\left(\frac{1}{r_t} - \frac{1}{r_o}\right)\frac{1}{k_i} + \frac{1}{h_0 r_o^2}\right]$$

For resistance $R_{th}$ to be minimum and $Q$ to be maximum,

$$\frac{dR_{th}}{dr_a} = \frac{1}{4\pi}\left[\frac{1}{r_o^2 k_i} - \frac{2}{h_a r_o^3}\right] = 0$$

$$\therefore \qquad (r_o)_{cr} = 2k_i/h_o \tag{2.75}$$

The results given above for the critical radius do not include the effects of thermal radiation. The heat transfer coefficient $h_a$ at the outer surface of insulation is approximated by the sum of a convection componet $h_c$ and a radiation component $h_r$ in the form

$$h_a = h_c + h_r$$

Then the critical radius by Eqs (2.73) and (2.75) becomes respectively

$$\mathbf{(r_o)_{cr} = \frac{k}{h_c + h_r}} \quad \textbf{for a cylinder} \tag{2.73a}$$

$$\mathbf{(r_o)_{cr} = \frac{2k}{h_c + h_r}} \quad \textbf{for a cylinder} \tag{2.75a}$$

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The reduction of heat flow by insulation in practice always necessitates a compromise between the effectiveness of insulation and the minimization of capital cost. By employing a multilayer aluminium foil in vaccum, it is possible to create an almost perfect insulation, but the cost involved becomes prohibitive. Gases with their low values of thermal conductivity offer great resistance to heat flow by conduction, but allow appreciable heat transfer by convection and radiation. However, if the gas or air is trapped in a porous or fibrous material, its capacity to transfer heat by convection and radiation is very much reduced. The effective conductivity of a composite material may tend to reach the value of $k$ for air (0.026 W/mK) at 20°C. As insulation thickness increases, the capital cost increases, but the cost in terms of heat loss or energy cost decreases. The break-even point is called '*economic thickness*' of insulation. Various types of imsulation have been discussed in earlier section.

## 2.4 EXTENDED SURFACES

Convection heat transfer between a surface (at $T_w$) and the fluid surrounding it (at $T_\infty$) is given by

$$Q = hA(T_w - T_\infty)$$

where $h$ is the heat transfer coefficient and $A$ is the surface area of heat transfer. For gases $h(= k_f/\delta_t)$ is low, since the thermal conductivity $k_f$ of a gas film is low. For heat transfer from a hot gas to a liquid through a wall, $h_{\text{gas}} << h_{\text{liquid}}$. To compensate for low heat transfer coefficient, surface area $A$ on the gas side may be extended for a given $Q$. Such an extended surface is termed as *fin*.

Let us consider the plane wall of Fig. 2.17(a). If $T_w$ is fixed, there are three ways in which the heat transfer rate may be increased.

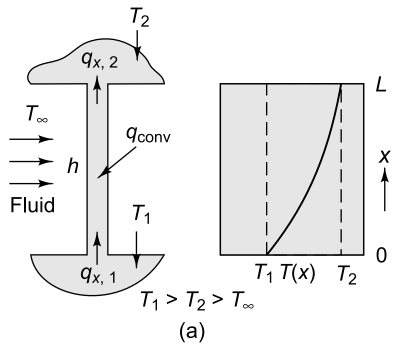


**Fig. 2.17(a)** *Combined conduction and convection in a bar*

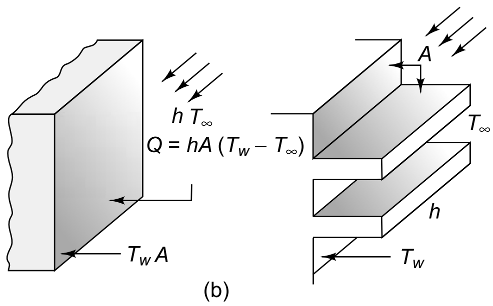


**Fig. 2.17(b)** *Use of fins to enhance heat transfer from a plane wall: Bare surface and Finned surface*

1. The convection coefficient $h$ could be increased by increasing the fluid velocity or/and the fluid temperature $T_\infty$ could be reduced. However, increasing $h$ even to the maximum possible value is often insufficient to obtain the desired heat transfer rate or the costs related to blower or pump power required to increase $h$ may be prohibitive.
2. The second option of reducing $T_\infty$ is often impractical.
3. The heat transfer rate may be increased by increasing the surface area across which convection occurs. This may be done by using fins that extend from the wall into the surrounding fluid [Fig. 2.17(b)]. The thermal conductivity of the fin material has a strong effect on the temperature distribution along the fin and thus the degree to which the heat transfer rate is enhanced. Figure 2.18 shows different fin configurations. A straight fin is any extended surface that is attached to a plane

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wall. It may be of uniform cross-sectional area, or its cross-sectional area may vary with the distance $x$ from the wall. An annular fin is one that is circumferentially attached to a cylinder. A pin fin or spine is an extended surface of circular cross-sections. Pin fins may also be of uniform or non-uniform cross-section.

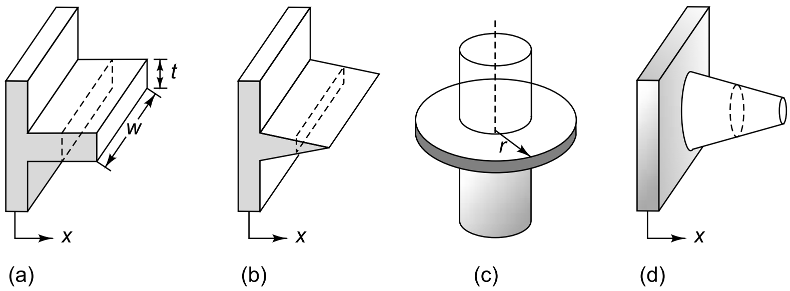


**Fig. 2.18** *Fin configurations (a) Straight fin of uniform cross-section, (b) straight fin of nonuniform cross-section, (c) annular fin and (d) pin fin*

### 2.4.1 Fins of Uniform Cross-sectional Area

Let us first consider the simplest case of straight and pin fins of uniform cross section (Fig. 2.19). Each fin is attached to the base surface of $T_o$ and extends into a fluid of temperature $T_\infty$. The perimeter of the fin, $P$, which is uniform is $2(L + b)$ or, $P \cong 2L$, if the thickness of the fin is small ($b << L$). If the fin is very thin and its length $l$ is long, it can be assumed that there is no radial temperature variation and heat gets conducted axially along the length. This heat is then dissipated to the surroundings by convection. The problem thus reduces to axial heat conduction along the fin with distributed heat sink from the sides. It is thus treated as one-dimensional heat conduction. Let us consider a small volume element at a distance $x$ from the base or root of the fin of thickness d$x$. The rate at which heat enters the element is $Q_x$ and the heat leaving the element is $Q_{x+\mathrm{d}x}$. In that small distance d$x$, let $Q_\Delta$ be the heat transferred by convection. Then by energy balance.

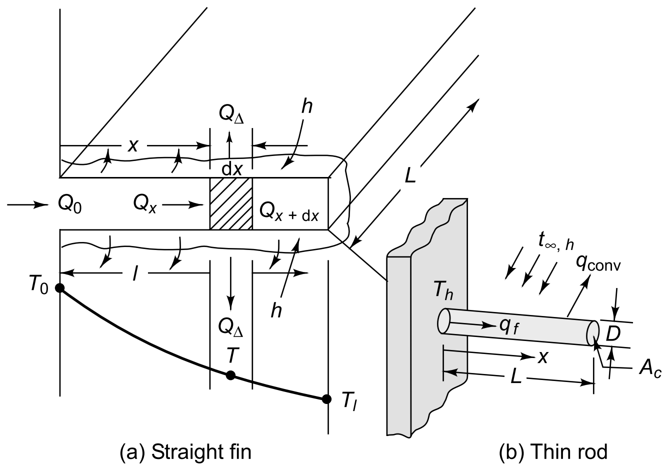


**Fig. 2.19** *Conduction and convection in a straight fin or a thin rod*

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$$Q_x = Q_{x+dx} + Q_\Delta \tag{2.76}$$

Now $$Q_x = -kA\frac{dT}{dx}$$

and $$Q_{x+dx} = Q_x + \frac{d}{dx}(Q_x)\,dx$$

$$Q_x - Q_{x+dx} = -\frac{d}{dx}(Q_x)\,dx = -\frac{d}{dx}\left(-kA\frac{dT}{dx}\right)dx$$

$$= kA\frac{d^2T}{dx^2}\,dx \tag{2.77}$$

Now $$Q_\Delta = hP\,dx\,(T - T_\infty) \tag{2.78}$$

Let excess temperature at any section $\theta = T - T_\infty$

$$\frac{d\theta}{dx} = \frac{dT}{dx}$$

$$\frac{d^2\theta}{dx^2} = \frac{d^2T}{dx^2} \tag{2.79}$$

From Eqs (2.76) – (2.79),

$$kA\frac{d^2\theta}{dx^2}\,dx = hP\,dx\,\theta$$

$$\frac{d^2\theta}{dx^2} = \frac{hP}{kA}\theta \tag{2.80}$$

Let $$m^2 = \frac{hP}{kA}\text{, so that}$$

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0 \tag{2.81}$$

By using the operator $D$,

$$(D^2 - m^2)\theta = 0$$

$$(D - m)(D + m)\theta = 0$$

Either $(D - m)\theta = 0$ or $(D + m)\theta = 0$

$$\frac{d\theta}{dx} = m\theta, \quad \frac{d\theta}{\theta} = m\,dx$$

$$\ln\theta = mx + A$$

or $$\boldsymbol{\theta = e^{mx}\,e^{A} = C_1\,e^{mx}} \tag{2.82}$$

where $C_1$ is a constant.

If $(D + m)\theta = 0$

$$\frac{d\theta}{dx} = -m\theta$$

$$\frac{d\theta}{\theta} = -m\,dx$$

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$$\ln\theta = -mx + B$$

or $$\theta = C_2 e^{-mx} \tag{2.83}$$

where $C_2$ is a constant.

Therefore, from Eqs (2.82) and (2.83), the general solution for temperature distribution is

$$\theta = C_1 e^{mx} + C_2 e^{-mx} \tag{2.84}$$

where $C_1$ and $C_2$ are constants to be evaluated from the boundary conditions given below:

(1) When $x = 0$, $T = T_0$, $\theta = \theta_0 = T_0 - T_\infty$

$$\theta_0 = C_1 + C_2 \tag{2.85}$$

(2) When $x = l$, $T = T_1$, $\theta = \theta_1 = T_1 - T_\infty$

$$\theta_1 = C_1 e^{ml} + C_2 e^{-ml} \tag{2.86}$$

**Case I** At the tip,

$$-k\left(\frac{d\theta}{dx}\right)_{x=l} = h\theta_1 \tag{2.87}$$

From Eq. (2.84) on differentiation and substitution,

$$-k\ (C_1 m\, e^{-ml} - C_2 m\, e^{-ml}) = h(C_1 e^{ml} + C_2\ e^{-ml}) \tag{2.88}$$

From Eqs (2.85) and (2.88) the constants $C_1$ and $C_2$ are given to be

$$C_1 = \frac{\theta_\theta}{1-\sigma e^{2ml}} \text{ and } C_2 = \frac{\theta}{\sigma - e^{-2ml}} \tag{2.89}$$

where $$\sigma = \frac{h+mk}{h-mk}$$

The temperature distribution, Eq. (2.84), becomes

$$\frac{\theta}{\theta_o} = \frac{\cosh m(l-x) + \dfrac{h}{mk}\sinh m(l-x)}{\cosh ml + \dfrac{h}{mk}\sinh ml} \tag{2.90}$$

At $$x = l,$$

$$\boldsymbol{\frac{\theta_1}{\theta_o} = \frac{1}{\cosh ml + \dfrac{h}{mk}\sinh ml}} \tag{2.91}$$

Heat transfer from the fin base

$$Q_o = -kA\left(\frac{d\theta}{dx}\right)_{x=0}$$

$$\boldsymbol{= (hPkA)^{1/2}\,\theta_o\,\frac{\dfrac{h}{mk} + \tanh ml}{1 + \dfrac{h}{mk}\tanh ml}} \tag{2.92}$$

It may be noted that conservation of energy demands that the rate at which heat is transferred by convection from the fin must be equal to the rate at which heat is conducted through the base of the fin. Accordingly,

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the alternative formation for $Q_o$ is

$$Q_o = \int_{A_f} h(T - T_\infty)\,\mathrm{d}A_s$$

$$= \int_{A_f} h\theta \mathrm{d}A_s \tag{2.93}$$

where $\mathrm{d}A_s$ = elemental surface area = $P$ d$x$ and $A_f$ is the *total fin surface area* (including the tip).

**Case 2** The fin is thin and long enough so that the heat loss from the tip is negligible. All the heat $Q_o$ is convected out along the length and no heat is dissipated from the tip surface.

$$Q_1 = -kA\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)_{x=l} = hA\ \theta_l = 0 \text{ (i.e. } T_l = T_\infty)$$

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)_{x=l} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(C_1 e^{mx} + C_2\, m\, e^{-mx})_{\ x=l} = 0$$

or,

$$(C_1\, m\, e^{mx} - C_2 m\, e^{-mx})_{x=1} = 0$$

$$C_1 e^{ml} = C_2\, e^{-ml}$$

$$\frac{C_2}{C_1} = e^{2ml} \tag{2.94}$$

Since

$$C_1 + C_2 = \theta_o$$

$$C_1 + C_1\, e^{2ml} = \theta_o$$

$$C_1 = \theta_o \frac{1}{1 + e^{ml}/e^{-ml}} = \theta_o \frac{e^{-ml}}{e^{ml} + e^{-ml}}$$

and

$$C_2 = C_1\, e^{2ml} = \theta_o \frac{e^{ml}}{e^{ml} + e^{-ml}}$$

Substituting in Eq. (2.84),

$$\theta = C_1\, e^{mx} + C_2\, e^{-mx}$$

$$= \theta_o \frac{e^{-ml}}{e^{ml} + e^{-ml}}\, e^{mx} + \frac{\theta_o\, e^{ml}}{e^{ml} + e^{-ml}}\, e^{-mx}$$

$$= \frac{\theta_o}{e^{ml} + e^{-ml}}\, [e^{-m(l-x)} + e^{m(l-x)}]$$

$$= \theta_o \frac{\cosh\, m\,(l - x)}{\cosh\, ml}$$

$$\boldsymbol{\frac{\theta}{\theta_o} = \frac{\cosh m\,(l - x)}{\cosh ml}} \tag{2.95}$$

This is the temperature distribution along the fin.

At the tip, $x = l$,

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$$\theta_l = T_1 - T_\infty = \frac{\theta_o}{\cosh ml}$$

$$= \text{Temperature difference at the tip}$$

$$\frac{\theta_1}{\theta_0} = \frac{1}{\cosh ml} \tag{2.96}$$

It gives the temperature of the fin at its tip ($T_1$).

The rate of heat transfer.

$$Q_o = -kA\left(\frac{d\theta}{dx}\right)_{x=0} = -kA\frac{d}{dx}\left[\theta_o \frac{\cosh m(l-x)}{\cosh ml}\right]_{x=0}$$

$$= -kA\theta_o\left[\frac{(-m)\sinh m(l-x)}{\cosh ml}\right]_{x=0}$$

$$\mathbf{Q_o = mk\,A\theta_o \tanh ml} \tag{2.97}$$

where $m = \left(\dfrac{hP}{kA}\right)^{1/2}$

It is found from Eqs (2.96) and (2.97), that as $l$ increases, $Q$ (i.e. $\tanh ml$) increases rapidly at first and then the rate slowly decreases and becomes asymptotic at $ml = 3$, which indicates that any further increase in length will not substantially increase the rate of heat transfer. Also, as $l$ increases, $\theta_l$ decreases (Fig. 2.20).

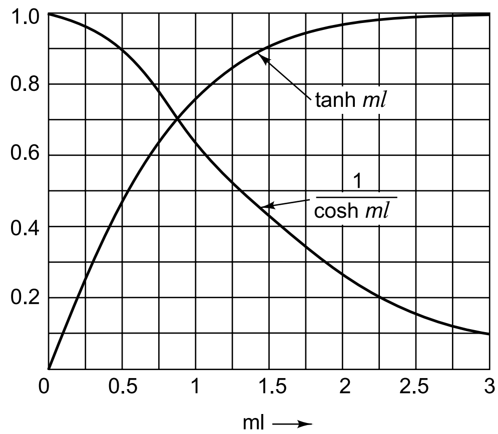


**Fig. 2.20** *Functions for determination of heat flow and temperature distribution in a thin rod*

**Case 3** If the temperature is given at the fin tip $T_1$, then $\theta_1 = T_1 - T_\infty$, and the resulting expressions for temperature distribution and heat transfer are

$$\frac{\theta}{\theta_o} = \frac{(\theta_l/\theta_o)\sinh mx + \sinh m(l-x)}{\sinh ml} \tag{2.98}$$

and $$\mathbf{Q_o = (hPkA)^{1/2}\,\theta_o \frac{\cosh ml - (\theta_l/\theta_o)}{\sinh ml}} \tag{2.99}$$

**Case 4** For very long fin, $l \to \infty$, $\theta_l \to 0$

$$\theta_o = C_1 + C_2$$

$$0 = C_1 e^{\infty} + C_2 e^{-\infty}$$

$$C_1 = 0,\ C_2 = \theta_1$$

$$\theta = \theta_1 e^{-mx}$$

$$\frac{\theta}{\theta_l} = e^{-mx} \tag{2.100}$$

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**Table 2.1** *Equations for temperature distribution and rate of heat transfer for fins of uniform cross section***

| S. No. | Tip condition ($x = L$) case | Temperature distribution, $\theta/\theta_o$ | Fin Heat Transfer Rate, $Q_{fin}$ |
|---|---|---|---|
| 1. | Convecting tip $h\theta(l) = -k \left.\dfrac{d\theta}{dx}\right\rvert_{x=l}$ | $\dfrac{[\cosh m(l-x) + (h/mk)\sinh m(l-x)]}{[\cosh ml + (h/mk)\sinh ml]}$ | $M\dfrac{[\sinh ml + (\bar{h}/mk)\cosh ml]}{[\cosh ml + (\bar{h}/mk)\sinh ml]}$ |
| 2. | Adiabatic tip: $\left.\dfrac{d\theta}{dx}\right\rvert_{x=1} = 0$ | $\dfrac{\cosh m(l-x)}{\cosh ml}$ | $M \tanh ml$ |
| 3. | Fixed tip temperature: $\theta(l) = \theta_l$ | $\dfrac{[(\theta_l/\theta_o)\sinh mx + \sinh m(l-x)]}{\sinh ml}$ | $M\dfrac{[\cosh ml\,(\theta_l/\theta_o)]}{\sinh ml}$ |
| 4. | Infinite fin ($l \to \infty$): $\theta_l(l) = 0$ | $e^{-mx}$ | $M$ |

** $\theta = T - T_\infty$; $\theta_o = \theta(0) = T_o - T_\infty$;

$$m^2 = \frac{hP}{kA};\ M = [hP\,kA\theta)^{1/2}\,\theta_o;\ \frac{\theta_{tip}}{\theta_o} = \frac{T_{tip} - T_\infty}{T_o - T_\infty}$$

$$\%\ \text{error} = \frac{100}{\cosh ml}$$

$$Q = -kA\left(\frac{d\theta}{dx}\right)_{x=0} = -kA\,[C_2(-m)\,e^o]$$

$$= \mathbf{mK\,A\theta_o} \tag{2.101}$$

The foregoing results are summarised in Table 2.1 and Fig. 2.21. A table of hyperbolic functions is given in Appendix B.1.

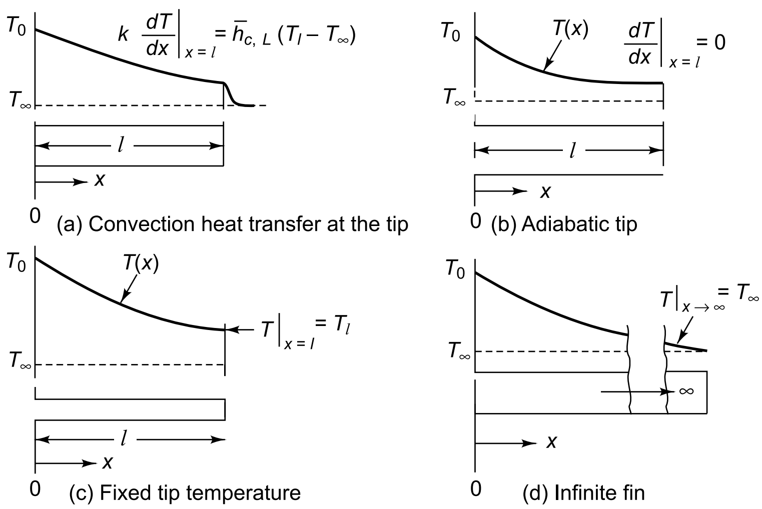


**Fig. 2.21** *Schematic representation of four boundary conditions at the tip of a fin*

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For nonuniform cross-sections of fins, the solutions are quite complex and the interested student is referred to Schneider [1] and Arpaci [2].

The temperature of a fluid flowing in a tube is often measured by a thermometer or thermocouple put into a well which is welded into the tube wall as shown in Fig. 2.22(a). If the fluid temperature differs greatly from the outside temperature, then the tube wall has a lower temperature than the gas and heat flows by conduction from the well to the tube wall. The end of the well where the thermometer bulb or thermocouple junction is placed may become colder than the fluid, and the indicated temperature will not be the true fluid temperature. The error can be calculated by Eq. (2.91) or (2.96), whichever is deemed appropriate. Figure 2.20 and Table 2.2 give the necessary length of the tube when the error must be confined within a certain limit.

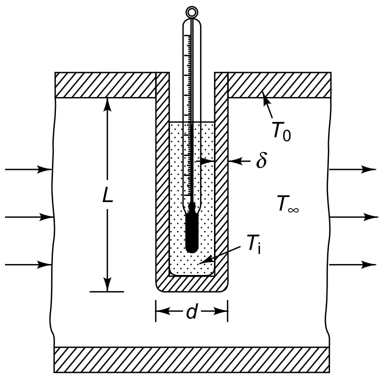


**Fig. 2.22(a)** *Thermometer well*

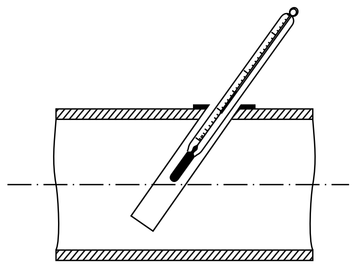

**Fig. 2.22(b)** *Temperature measurement in flow in a tube*

**Table 2.2** *Calculation functions for heat conduction in a rod*

| *ml* | *0* | *0.5* | *1* | *1.5* | *2* | *3* | *4* | *5* | *6* |
|---|---|---|---|---|---|---|---|---|---|
| cosh *ml* | 1 | 1.1276 | 1.543 | 2.352 | 3.762 | 10.07 | 27.31 | 74.21 | 201.7 |
| tanh *ml* | 0 | 0.4621 | 0.7616 | 0.9052 | 0.9640 | 0.9951 | 0.9993 | 0.9999 | 1 |

If the length of well is found to be greater than the tube diameter, it is necessary to locate the well obliquely in the tube [Fig. 2.22(b)]. Heat radiation between the end of the well and the tube wall may cause an additional error in temperature measurement. This will be dealt with in Chapter 7.

### 2.4.2 Fin Performance

We may recall that fins are used to increase the heat transfer from a surface by increasing the effective surface area, and not by increasing the heat transfer coefficient. However, the fin itself represents a conductive resistance to heat transfer from the original surface. Therefore, one cannot be sure that by the use of fins the heat transfer rate will be increased. It can be assessed by evaluating the fin effectiveness, $\varepsilon_f$. It is defined as a ratio given as follows:

$$\varepsilon_f = \frac{\text{Heat transfer rate with fin}}{\text{Heat transfer rate without fin}} = \frac{Q_o}{hA\,\theta_o} \tag{2.102}$$

where $A$ is the cross-sectional area of the fin. In any design $\varepsilon_f$ should be as large as possible, and in general, the use of fins is rarely justified unless $\varepsilon_f \geq 2$.

For any one of the four tip conditions given in Table 2.1, the effectiveness for a fin of uniform cross-section may be obtained by dividing the appropriate expression for $Q_o$ by $hA\theta_o$. For the infinite fin (Case 4) the result is

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$$\varepsilon_f = \frac{(hP\,kA)^{1/2}\theta_o}{hA\,\theta_o} = \left(\frac{kP}{hA}\right)^{1/2} \tag{2.103}$$

For Case 2 with negligible tip loss

$$\boldsymbol{\varepsilon_f = \frac{(hP\,kA)^{1/2}\theta_o \tanh ml}{hA\,\theta_o} = \left(\frac{kP}{hA}\right)^{1/2} \tanh ml} \tag{2.104}$$

It is observed that fin effectiveness is enhanced by

1. The choice of a material of high thermal conductivity like copper and aluminium. Although copper has a higher thermal conductivity, aluminium alloys are more common because they are of low cost and density.
2. Increasing the ratio of the perimeter to the cross-sectional area of the fin, $P/A$. The use of thin, but closely spaced, fins is preferred to that of thick fins.
3. The low value of heat transfer coefficient $h$. The fins are required when the fluid is a gas rather than a liquid, particularly when the heat transfer from the surface is by natural convection. If fins are to be used on a surface separating a gas and a liquid, they are generally placed on the gas side, which is the side of lower heat transfer coefficient ($h_{\text{liq}} >> h_{\text{gas}}$, $h_{\text{natural convection}} << h_{\text{forced convection}}$).

Fin performance may also be quantified in terms of a thermal resistance. Treating the difference between the base and fluid temperatures as the driving potential, a *fin resistance* may be defined as

$$R_{t,f} = \frac{\theta_0}{Q_0} \tag{2.105}$$

This result is very useful in the sense that a finned surface can be represented by a thermal circuit. An appropriate expression for $Q_0$ depending on the fin tip condition may be used from Table 2.1.

Dividing Eq. (2.105) into the expression for thermal resistance due to convection at the exposed base

$$R_{t,b} = \frac{1}{hA}$$

and substituting from Eq. (2.102), it follows that

$$\varepsilon_f = \frac{R_{t,b}}{R_{t,f}} \tag{2.106}$$

Hence, the fin effectiveness may be interpreted as a ratio of thermal resistances, and to increase $\varepsilon_f$ it is necessary to reduce the conductivity/convection resistance of the fin $R_{t,f}$. If the fin is to enhance heat transfer, its resistance must not exceed that of the exposed surface, $R_{t,b}$.

The thermal performance of a fin is also measured by a parameter called *fin efficiency*, $\eta_f$. The maximum driving potential for convection is the temperature difference between the base ($x = 0$) and the fluid, $\theta_o = T_o - T_\infty$. Hence, the maximum rate at which a fin could dissipate energy is the rate that would exist if the entire fin surface were at the base temperature, i.e. if the thermal conductivity of the fin is infinity.

Therefore, $$\eta_f = \frac{\text{Actual heat transfer from fin}}{\text{Maximum heat transfer from fin if entire fin surface were at fin base temperature}}$$

$$= \frac{Q_o}{Q_{\max}} = \frac{Q_o}{hA_f\,\theta_o} \tag{2.107}$$

where $A_f$ is the total surface area of the fin.

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Actual heat transfer from a fin.

$$(Q_o)_{act} = \eta_f (Q_o)_{max} = \eta_f h A_f \theta_o \tag{2.108}$$

For a long thin fin with insulated tip

$$Q_o = (hP\,kA)^{1/2}\,\theta_o \tanh ml$$

where $$m = \left(\frac{hP}{kA}\right)^{1/2}$$

The fin efficiency is then

$$\eta_f = \frac{(hP\,kA)^{1/2}\,\theta_o \tanh ml}{hPl\,\theta_o} = \left(\frac{kA}{hP}\right)^{1/2}\frac{\tanh ml}{l}$$

$$= \frac{\tanh ml}{ml} \tag{2.109}$$

For a rectangular fin (Fig. 2.19),

$$P = 2L + 2b \cong 2L,\ A = Lb$$

$$A_f = 2L1 + 2lb = 2l(L + b) \cong 2Ll$$

$$m = \left(\frac{h\,2L}{kLb}\right)^{1/2} = \left(\frac{2h}{kb}\right)^{1/2} \tag{2.110}$$

$$\eta_f = \frac{(h\,P\,k\,A)^{1/2}}{h\,A_f\,\theta_o}\theta_o \tanh ml$$

$$= \frac{(h\,2L\,k\,L\,b)^{1/2}\tanh ml}{h\,2L\,l} = \left(\frac{kb}{2h}\right)^{1/2}\frac{\tanh ml}{l}$$

$$= \frac{\tanh ml}{ml} \tag{2.110a}$$

It is the same expression as in Eq. (2.109). For a pin rod, $P = \pi d$ and $A = \frac{\pi}{4}d^2$, where $d$ is the pin diameter.

$$m = \left(\frac{h\pi d}{k\frac{\pi}{4}d^2}\right)^{1/2} = 2\left(\frac{h}{kd}\right)^{1/2} \tag{2.111}$$

By evaluating $\eta_f$, the actual heat transfer from a fin can be computed from Eq. (2.108).

Fins come in many shapes and forms, some of which are shown in Fig. 2.23. A compromise of the cost, the weight, the available space, the pressure drop of the fluid and heat transfer characteristics determines the suitability of fin geometry.

If a rectangular fin is long, wide and thin.

$$P/A = 2L/Lb = 2/b, \text{ and } m = (hP/kA)^{1/2}$$

$$\eta_f = \frac{\tanh\,(hPl^2/kA)^{1/2}}{(hPl^2/kA)^{1/2}}$$

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The heat loss from the fin tip can be taken into account approximately by increasing $l$ by $b/2$ and assuming that the tip is insulated. This approximation keeps the surface area from which heat is lost the same as in the real case, and fin efficiency then becomes

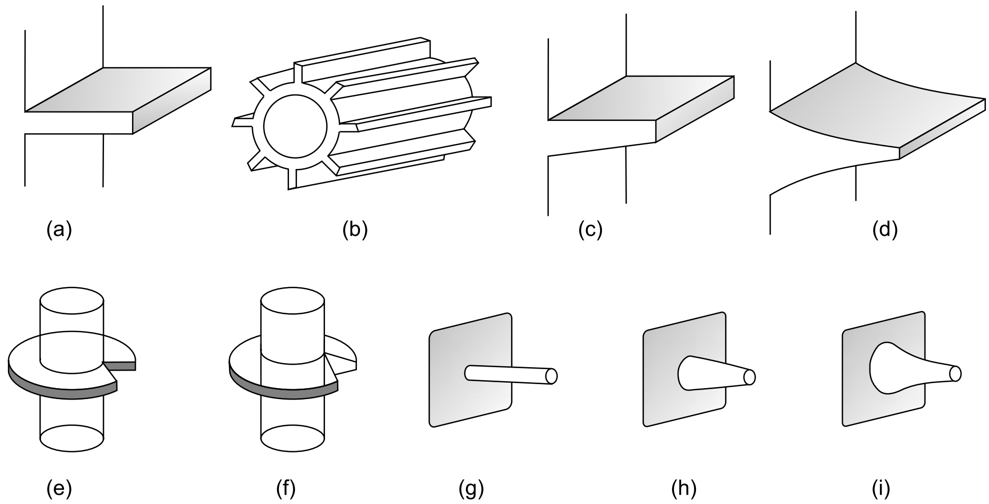


**Fig. 2.23** *Schematic diagrams of different types of fins. (a) Longitudinal fin of rectangular profile; (b) cylindrical tube with fins of rectangular profile; (c) longitudinal fin of trapezoidal profile; (d) longitudinal fin of parabolic profile; (e) cylindrical tube with radial fin of rectangular profile; (f) cylindrical tube with radial fin of truncated conical profile; (g) cylindrical pin fin; (h) truncated conical spine; (i) parabolic spine*

$$\eta_f = \frac{\tanh\,(2hl_c^2/kb)^{1/2}}{(2hl_c^2\,/\,kb)^{1/2}} \tag{2.112}$$

where the corrected fin length $l_c = l + b/2$. Kreith and Bohn [2] have, however, recommended $l_c = l + A/P$.

The error of this approximation is less than 8% [2] when

$$\frac{hb}{2k} \le \frac{1}{4}$$

For fins of non-uniform cross-section, the analysis is complex and such solutions are conveniently presented in graphical form [3]. In Fig. 2.24 the fin efficiency is plotted for three common profiles: the rectangular fin of uniform cross-sectional area and the triangular and parbolic fins of non-uniform cross-sectional area. Figure 2.25 presents results for annular fins of rectangular profile. These results are presented in terms of the corrected length $l_c$ $(= l + b/2)$. To use these results with Eq. (2.107), it is necessary to calculate the maximum heat transfer rate. For rectangular, triangular and parabolic fins,

$$Q_{\max} = hPlc\ \theta_o \tag{2.113a}$$

and for the annular fin,

$$Q_{\max} = 2\pi h\,(r_2^2 - r_1^2)\theta_o \tag{2.113b}$$

In practice, a finned heat transfer surface is composed of the fin surface and the unfinned surface. The total heat transfer will be the sum of the heat transfer from the two portions.

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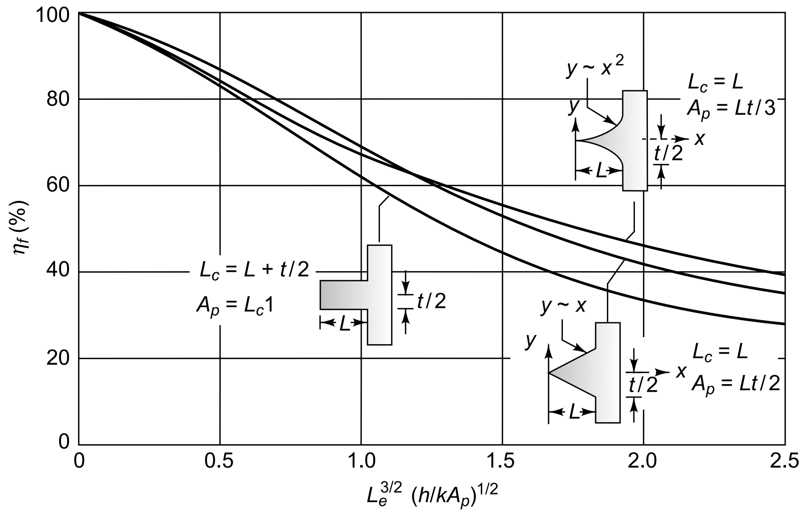


**Fig. 2.24** *Efficiency of straight fins (rectangular, triangular and parabolic profiles)*

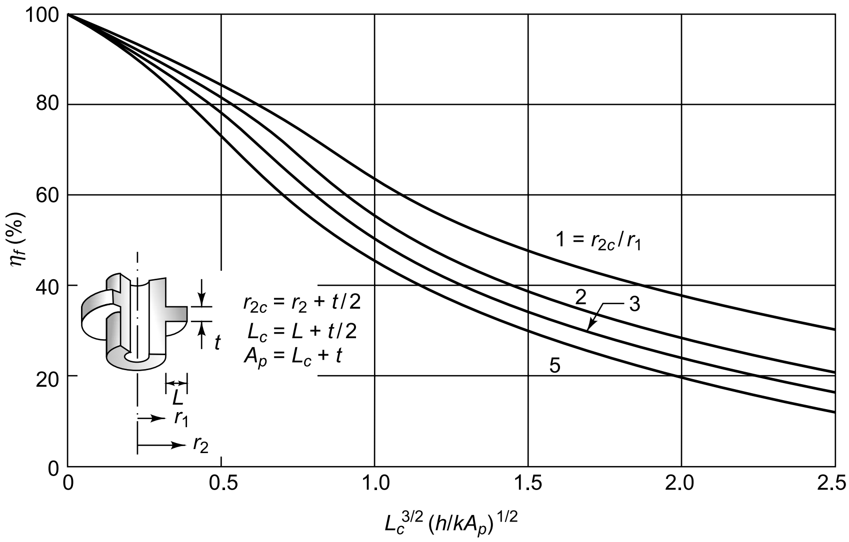


**Fig. 2.25** *Efficiency of annular fins of rectangular profile*

$$
\begin{aligned}
Q_{\text{total}} &= Q_{\text{fin}} + Q_{\text{unfinned}} \\
&= \eta_f A_f h\,\theta_o + (A - A_f)h\,\theta_o \\
&= h\theta_o\,(\eta_f A_f + A - A_f) \\
&= \mathbf{h\theta_o\,[A - (1 - \eta_f)A_f]}
\end{aligned}
\tag{2.114}
$$

where $A$ is the total area of fin and unfinned surfaces.

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### 2.4.3 Limitation of an Extended Surface

The installation of fins on a heat transferring surface increases the heat transfer area but it is not necessary that the rate of heat transfer would increase. For long fins the rate of heat loss from the fin is given by $\sqrt{h\,pkA}\ \theta_o = kA\sqrt{\dfrac{hp}{kA}}\ \theta_o = kAm\theta_o$.

$$\text{When} \qquad h/mk = 1, \quad \text{or} \quad h = mk,$$
$$Q = hA\theta_o$$

which is equal to the heat loss from the primary surface with *no* extended surface. Thus when $h = mk$, an extended surface will not increase the heat transfer rate from the primary surface whatever be the length of the extended surface.

For $h/mk > 1$, $Q < hA\theta_o$ and hence adding a secondary surface reduces the heat transfer and the added surface will act as an insulation. For $h/mk < 1$, $Q > hA\theta_o$, and the extended surface will increase the heat transfer. The heat transfer would be more effective when $h/k$ is low for a given geometry.

### 2.4.4 Rectangular Fin of Minimum Weight

For the design of cooling devices on vehicles, especially aircraft, the problem of exchanging the greatest amount of heat with the least amount of weight in the heat exchanger is of paramount importance. For a given weight, the maximum heat transfer is required.

Weight of one fin = $b \times l \times L \times \rho$

where $\rho$ is the density of fin material.

Let $A_1 = b \times l$ = area of fin cross-section normal to $L$.

The length $L$ is fixed at a given dimension, whereas the two dimensions $b$ and $l$ are to be changed so as to give maximum heat flow for a given area $A_1$ [Fig. 2.28(b)].

$$\text{We have} \qquad m = \left(\frac{2h}{kb}\right)^{1/2}, A = b \times L \text{ and } A_1 = b \times l.$$

If the tip loss is neglected,

$$Q_o = (hP\,kA)^{1/2}\,\theta_o \tanh ml = mkA\theta_o \tanh ml$$
$$= \left(\frac{2h}{kb}\right)^{1/2} kbL\ \theta_o \tanh\left[\left(\frac{2h}{kb}\right)^{1/2}\frac{A_1}{b}\right]$$
$$= (2hk)^{1/2}\,b^{1/2}\,L\ \theta_o \tanh\left(\frac{2h}{k}\right)^{1/2}\left(\frac{A_1}{b^{3/2}}\right) \tag{2.115}$$

For a given area $A_1$, $Q_1$ will be maximum when

$$\frac{dQ_1}{db} = (2hk)^{1/2}\,L\theta_o \tanh\left[\left(\frac{2h}{k}\right)^{1/2}\frac{A_1}{b^{3/2}}\right]\frac{1}{2}b^{\frac{1}{2}-1}$$
$$+ (2hk)^{1/2}\,L\theta_o b^{1/2}\,\frac{1}{\cosh^2\left[\left(\frac{2h}{k}\right)^{1/2}(A_1/b^{3/2})\right]} \times \left(\frac{2h}{k}\right)^{1/2} A_1\left(-\frac{3}{2}\right)b^{-5/2} = 0$$

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or $$(2hk)^{1/2} L\theta_o \frac{\tan h\left[\left(\frac{2h}{k}\right)^{1/2}(A_1/b^{3/2})\right]}{2(b)^{1/2}} - \frac{3}{2}\frac{A_1}{b^2}\left(\frac{2h}{k}\right)^{1/2} - b\frac{1}{\cos h^2\left[\left(\frac{2h}{k}\right)^{1/2}(A_1/b^{3/2})\right]} = 0$$

Putting $$u = \left(\frac{2h}{kb}\right)^{1/2}\frac{A_1}{b} = ml \tag{2.116a}$$

$$\frac{\tanh u}{2(b)^{1/2}} - \frac{3}{2}\frac{A_1}{b^2}\left(\frac{2h}{k}\right)^{1/2}\frac{1}{\cosh^2 u} = 0$$

or $$\tanh u - 3\frac{A_1}{b}\left(\frac{2h}{kb}\right)^{1/2}\frac{1}{\cosh^2 u} = 0$$

or $$\frac{\sinh u}{\cosh u} - \frac{3u}{\cosh^2 u} = 0$$

or $$\cosh u \sinh u - 3u = 0$$

or $$\frac{\sinh 2u}{2} = 3u$$

or $$\frac{e^{2u} - e^{-2u}}{2} = 6u$$

or $$\left(1 + 2u + \frac{4u^2}{2} + \frac{8u^3}{6} + \frac{16u^4}{24} + \frac{32u^5}{120} + \cdots +\right)$$
$$-\left(1 - 2u + \frac{4u^2}{2} - \frac{8u^3}{6} + \frac{16u^4}{24} - \frac{32u^5}{120} + \cdots +\right) = 12u$$

or $$4u + \frac{16u^3}{6} + \frac{64u^5}{120} = 12u$$

or $$u^4 + 5u^2 - 15 = 0$$

or $$u^2 = \frac{-5 + (25 + 60)^{1/2}}{2} = 2.1$$

or $$u = ml = 1.419$$

$$\left(\frac{2h}{kb}\right)^{1/2} l = 1.419$$

or $$l = 1.419\left(\frac{kb}{2h}\right)^{1/2}$$

$$\frac{l}{b/2} = 1.419\left(\frac{2k}{hb}\right)^{1/2} \tag{2.116b}$$

This is the condition for the maximum heat flow for a given weight of fin, giving the optimum ratio of fin height to half the fin thickness.

$$\theta_l = \frac{\theta_o}{\cosh ml} = \frac{\theta_o}{\cosh u} = \frac{\theta_o}{1.419}$$

or $$\theta_l = 0.457\theta_o \tag{2.117}$$

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$$\varepsilon_f = \text{Fin effectiveness} = \frac{Q_o\,(\text{with fin})}{Q_o\,(\text{without fin})}$$

$$= \frac{(hP\,kA)^{1/2}\theta_o}{hb\,L\theta_o}\tanh ml$$

$$= \left(\frac{\mathbf{2k}}{\mathbf{hb}}\right)^{\mathbf{1/2}} \mathbf{\tanh 1.419 = 0.889} \left(\frac{\mathbf{2k}}{\mathbf{hb}}\right)^{\mathbf{1/2}} \tag{2.118}$$

This equation makes it possible to determine the heat flow increase through the wall as a result of the addition of fins.

### 2.4.5 Generalized Equation for Fins

It is our prime interest to know the extent to which a particular extended surface or fin arrangement could improve heat transfer from a surface to the surrounding. To determine the heat transfer rate associated with a fin, we must first obtain the temperature distribution along the fin. Let us consider an extended surface of arbitrary shape (Fig. 2.26).

Assuming heat to be flowing only in the longitudinal direction ($x$), an energy balance for an elemental area of thickness d$x$ at a distance $x$ from the base is given as

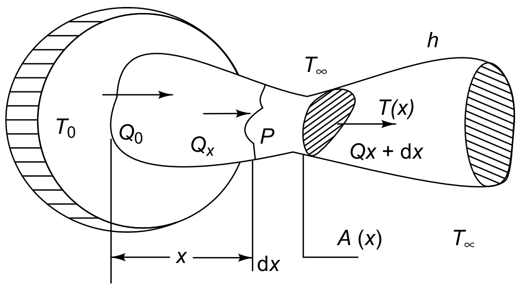


**Fig. 2.26** *General one dimensional fin*

$$Q_x = Q_{x+\mathrm{d}x} + Q_{\mathrm{conv}}.$$

$$-kA\frac{\mathrm{d}T(x)}{\mathrm{d}x} = -kA\frac{\mathrm{d}T(x)}{\mathrm{d}x} + \frac{\mathrm{d}}{\mathrm{d}x}\left[-kA\frac{\mathrm{d}T(x)}{\mathrm{d}x}\right] + h\cdot P\mathrm{d}x\,[T(x) - T_\infty]$$

Since both $A$ and P are functions of $x$,

$$kA(x)\frac{\mathrm{d}^2T}{\mathrm{d}x^2} + \frac{\mathrm{d}A(x)}{\mathrm{d}x}\cdot k\frac{\mathrm{d}T(x)}{\mathrm{d}x} = hP\,[T(x) - T_\infty]$$

Dividing throughout by $kA$ and putting $T(x) - T_\infty = \theta(x)$,

$$\frac{\mathrm{d}^2\theta(x)}{\mathrm{d}x^2} + \frac{\mathrm{d}A/\mathrm{d}x}{A}\cdot\frac{\mathrm{d}\theta(x)}{\mathrm{d}x} - \frac{hP}{kA}\theta(x) = 0 \tag{2.119}$$

This result provides a general form of the energy equation for an extended surface. Now let us consider its application to the case of a *triangular fin* (Fig. 2.27).

The perimeter $P \cong 2L$. The cross-sectional area $A = \dfrac{Lbx}{l}$. Substituting these values in Eq. (2.119), we obtain

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} + \frac{1}{x}\frac{\mathrm{d}\theta}{\mathrm{d}x} - \frac{2}{k}\frac{hl}{bx}\theta = 0 \tag{2.120}$$

It is a modified form of the Bessel equation, which in its general form for any value of $n$ is

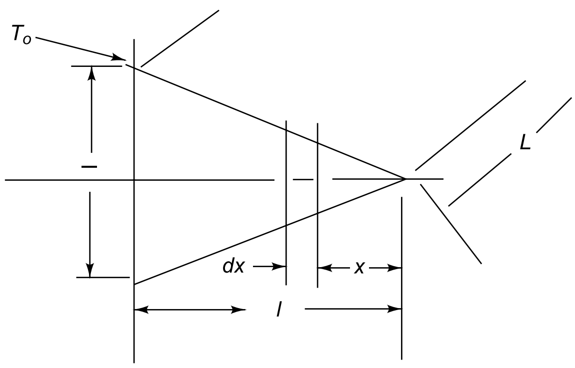


**Fig. 2.27** *Application of energy equation for a triangular fin*

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$$z^2 \frac{d^2 y}{dz^2} + z\frac{dy}{dz} - (z^2 + n^2)y = 0 \tag{2.121}$$

whose general solution is

$$y = C_1 I_n(z) + C_2 K_n(z) \tag{2.122}$$

where $C_1$ and $C_2$ are constants of integration and $I_n$ and $K_n$ are $n^{th}$ order Bessel functions of the first and second kind respectively.

In order to make use of the solutions available for the standard form of Eq. (2.121), Eq. (2.120) is converted into the same form of Eq. (2.121) as follows:

Putting $B^2 = \frac{2lh}{bk}$ and multiplying Eq. (2.120) by $x^2$ to get

$$x^2 \frac{d^2\theta}{dx^2} + x\frac{d\theta}{dx} - B^2 x\theta = 0 \tag{2.123}$$

Again, putting $z = 2B\sqrt{x}$, or $x = \frac{z^2}{4B^2}$

$$\therefore \quad \frac{dz}{dx} = Bx^{-\frac{1}{2}}, \quad \frac{d\theta}{dx} = \frac{d\theta}{dz}\cdot\frac{dz}{dx} = Bx^{-\frac{1}{2}}\cdot\frac{d\theta}{dz}$$

$$\frac{d^2\theta}{dx^2} = \frac{d}{dx}\left(\frac{d\theta}{dz}Bx^{-\frac{1}{2}}\right) = \frac{d\theta}{dz}B\left(-\frac{1}{2}x^{-\frac{3}{2}}\right) + \frac{d^2\theta}{dz^2}\cdot\frac{dz}{dx}\cdot Bx^{-\frac{1}{2}}$$

Substituting these relations in Eq. (2.123),

$$z^2\frac{d^2\theta}{dz^2} + z\frac{d\theta}{dz} - z^2\theta = 0$$

The above equation is identical to the modified Bessel equation of zero order ($n = 0$) and its general solution is

$$\theta = C_1 I_o\,(2B\sqrt{x}) + C_2 K_o\,(2B\sqrt{x}) \tag{2.124}$$

where $I_0$ and $K_0$ are modified zero order Bessel functions of the first and second kind respectively. Some typical values of $I_o(z)$ and $K_o(z)$ are tabulated in Table 2.3. It is seen that $I_o(0) = 1$ while $K_o(0) = \infty$.

The constants of integration $C_1$ and $C_2$ are evaluated by using the boundary conditions:

At the root, $\theta = \theta_o$ at $x = l$

At the tip, $\theta$ = finite at $x = 0$

Since at $x = 0$, $K_o(0)$ approaches infinity, $C_2 = 0$

$$\therefore \quad \theta = C_1 I_o\,(2B\sqrt{x})$$

From the first boundary condition,

$$C_1 = \frac{\theta_o}{I_o(2B\sqrt{l})}$$

$$\therefore \quad \frac{\theta}{\theta_o} = \frac{I_o(2B\sqrt{x})}{I_o(2B\sqrt{l})} \tag{2.125}$$

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Table 2.3 Typical values of Bessel Functions

| $z$ | $I_o(z)$ | $I_1(z)$ | $\frac{2}{\pi}K_o(z)$ | $\frac{2}{\pi}K_1(z)$ |
|---|---|---|---|---|
| 0.0 | 1.0000 | 0.0000 | ∞ | ∞ |
| 0.2 | 1.0100 | 0.1005 | 1.116 | 3.040 |
| 0.4 | 1.0404 | 0.2040 | 0.7095 | 1.391 |
| 0.6 | 1.0920 | 0.3137 | 0.4950 | 0.8294 |
| 0.8 | 1.1665 | 0.4329 | 0.3599 | 0.5486 |
| 1.0 | 2.2661 | 0.5652 | 0.2680 | 0.3832 |
| 2.0 | 2.2796 | 1.5906 | 0.07251 | 0.08904 |
| 3.0 | 4.8808 | 3.9534 | 0.02212 | 0.02556 |
| 4.0 | 11.3019 | 9.7595 | 0.007105 | 0.007947 |
| 5.0 | 27.2399 | 24.3356 | 0.00235 | 0.002575 |
| 6.0 | 67.2344 | 61.3419 | 0.000638 | 0.0006879 |
| 7.0 | 168.6 | 156.04 | 0.0002704 | 0.0002891 |
| 8.0 | 427.6 | 399.9 | 0.000093 | 0.0000989 |
| 9.0 | 1093.6 | 1030.9 | 0.000022 | 0.000034 |
| 10.0 | — | — | 0.000011 | 0.000011 |

The heat flow rate $Q$ from the fin is given by

$$Q = kA\left(\frac{d\theta}{dx}\right)_{x=L}$$

We know from the properties of Bessel functions that

$$\frac{d\,I_o(z)}{dx} = I_{n+1}(z)\,\frac{dz}{dx}$$

so that for $n = 0$,

$$\frac{d\,I_0(2B\sqrt{x})}{dx} = I_1(2B\sqrt{x})\,B\cdot x^{-\frac{1}{2}}$$

$$\therefore \qquad Q = kLb\,\theta_o\,\frac{I_1(2B\sqrt{l})}{I_o(2B\sqrt{l})}\,Bl^{-1/2}$$

$$= L\sqrt{2hkb}\cdot\theta_o\cdot\frac{I_1(2B\sqrt{l})}{I_o(2B\sqrt{l})} \tag{2.126}$$

If the heat flow into the triangular fin [Fig. 2.28(a)] is optimised in order to determine the best ratio of height $l$ to base $b$, the following expression as shown by Eckert and Drake [4] results.

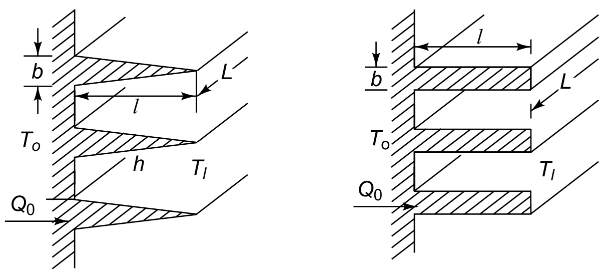


**Fig. 2.28** *Fins of (a) triangular cross section, (b) rectangular cross section*

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$$\frac{l}{b/2} = 1.309 \left(\frac{2k}{hb}\right)^{1/2} \tag{2.127}$$

The temperature excess at the tip of the fin is

$$\theta_l = 0.277\ \theta_o \tag{2.128}$$

The ratio of the thickness of the triangular fin to the thickness of the rectangular fin with equal heat flow is 1.31, and the ratio of the cross-sectional areas is 1:1.44. Therefore, the weight saved by using the triangular fin is 44%.

### 2.4.6 Fin of Minimum Weight

It is of further interest to determine the optimum shape of a fin having the minimum weight for a given heat flow. In such a fin, every part should be utilised to the same degree, and the specific rate of heat flow, $q$, should be constant throughout the fin [4]. The heat flow lines are equally spaced and parallel to the fin axis (Fig. 2.29).

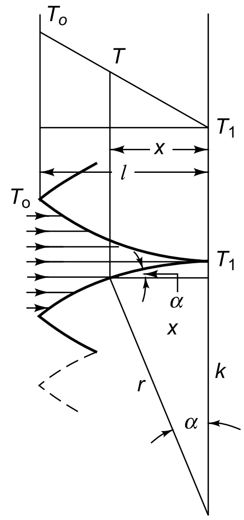


**Fig. 2.29** *Fin with smallest weight*

Since $$\bar{q} = -k\frac{d\theta}{dx} = \text{constant}$$

or $$\frac{d\theta}{dx} = \text{constant}$$

Temperature decreases linearly along any flow line from the value $T_o$ at the root of the fin to that at the tip which approaches $T_\infty$ of the surrounding fluid when $\alpha$ becomes zero. For a finite value of $\alpha$ there will be a temperature discontinuity between the fin tip and the surrounding fluid. From Fig. 2.29

$$\frac{x}{l} = \frac{T - T_\infty}{T_o - T_\infty} = \frac{\theta}{\theta_o}$$

$$\theta = \frac{x}{l}\ \theta_o$$

Let us consider a surface element of the fin at a distance $x$. The element is inclined to the fin axis by the angle $\alpha$. The specific rate of heat flow from this element is

$$q \sin\alpha = h(T - T_\infty) = h\theta$$

$$q \sin\alpha = h\frac{x}{l}\ \theta_o$$

or, $$\sin\alpha = \frac{h\theta_o x}{ql} = \frac{x}{ql/h\theta_o} = \frac{x}{r}$$

where $$\mathbf{r} = \frac{\bar{\mathbf{q}}\mathbf{l}}{\mathbf{h}\boldsymbol{\theta}_\mathbf{o}} = \textbf{constant} \tag{2.129}$$

Thus the contour lines of the fin are circles which meet tangentially at the tip for the smallest weight of a given heat flow. The difference in weight between a fin in the shape of a circular arc and the fin of triangular shape is very small. As the triangular shape is much easier to manufacture, it may, for practical purposes, be regarded as the best shape.

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### 2.4.7 Fin Arrangement

The cross-sectional area $A_1$ (= $bl$) necessary for a given heat flow in the example of a rectangular fin is derived by combining Eqs (2.115) and (2.116a), and solving for $A_1$.

$$A_1 = \left(\frac{Q_o}{\theta_o}\right)^3 \frac{l}{L^3} \frac{u}{\tanh^3 u} \frac{1}{4h^2 k} = \frac{2.109}{4L^3\, h^2\, k}\left(\frac{Q_o}{\theta_o}\right)^3 \tag{2.130}$$

This equation shows that it is advantageous to make the fins as thin ($b$) or as small ($A_1 = bL$) as possible. To double the heat flow, the area of one fin ($A_1$) or thickness $b$ must be increased eight times, whereas it is sufficient to use two fins of the original size.

Equation (2.130) shows that $A_1 \infty l/k$. The mass of the fin is, therefore, proportional to $\rho/k$. It may be seen that by using aluminium, instead of copper, a weight saving of 50% can be achieved. Iron fins have a ten-fold weight, and stainless steel about 50-fold weight as given in Table 2.4.

**Table 2.4** *Comparison of fin material*

| *Material* | *Thermal Conductivity, k(W/MK)* | *Density, (kg/m$^3$)* | $\rho/k$ | $\dfrac{\rho/k}{(\rho/k)_{Al}}$ |
|---|---|---|---|---|
| Copper | 380 | 8970 | 23.6 | 1.95 |
| Aluminium, pure | 225 | 2723 | 12.1 | 1.00 |
| Aluminium, alloy | 156 | 2659 | 17.0 | 1.40 |
| Magnesium, pure | 173 | 1762 | 10.2 | 0.84 |
| Steel | 55 | 7850 | 142.7 | 11.8 |
| Stainless steel | 14 | 7850 | 560.7 | 46.3 |

Extended surfaces are used in many engineering devices. For comprehensive discussions, the reader may consult Schneider [5], Arpaci [6], Kakac and Yener [7] and Kern and Kraus [8].

### 2.4.8 Cylindrical Fins

Fins which are arranged around tubes are called cylindrical fins and are quite important from an engineering point of view. Such a fin system is shown in Fig. 2.30. Here again the treatment is substantially the same as for rectangular fin except that the area must be allowed to vary with the radius. The area normal to the heat flux vector can be written as

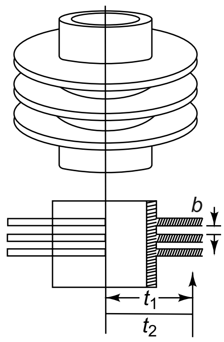


**Fig. 2.30** *Surface with circumferential fins*

$$A = 2\pi rb$$

and the periphery can be expressed as

$$P = 4\pi r$$

We choose an annular element of radius $r$ and thickness $dr$ (Fig. 2.31). By making an energy balance

$$-k\,2\pi rb\,\frac{dT}{dr} = -k2\pi(r + dr)\,b\left(\frac{dT}{dr} + \frac{d^2T}{dr^2}\cdot dr\right) + h\,4\pi r dr\cdot(T - T_\infty)$$

or,

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} - \frac{2h}{kb}\,(T - T_\infty) = 0$$

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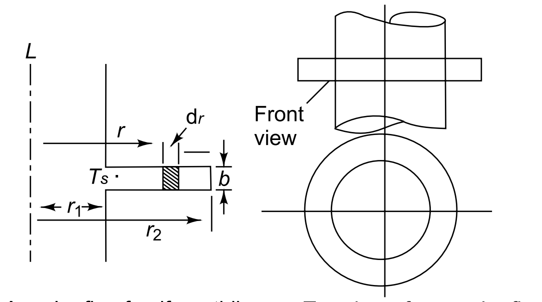
Let $\theta = T - T_\infty$, then the above equation reduces to

$$\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} - \left(\frac{2h}{kb}\right)\theta = 0 \tag{2.131}$$




Annular fin of uniform thikness   Top view of a annular fin

**Fig. 2.31** *Element of an annular fin*

The equation is recognised as the Bessel's equation of zero order and its solution is

$$\theta = C_1\, I_o(mr) + C_2\, K_o(mr), \text{ where } m = \sqrt{\frac{2h}{kb}},$$

$I_o$ = modified Bessel function, 1st kind, and $K_o$ = modified Bessel function, 2nd kind, zero order. The constants $C_1$ and $C_2$ are evaluated by applying the two boundary conditions:

at $\quad r = r_1,\ T = T_w$ and $\theta = T_w - T_\infty$

at $\quad r = r_2,\ \dfrac{dT}{dr} = 0$, or $\dfrac{d\theta}{dr} = 0$ since $b << (r_2 - r_1)$

By using the above boundary conditions the temperature distribution is given by

$$\frac{\theta}{\theta_o} = \frac{I_o(mr)\,K_1(mr_2) + K_o(mr)\,I_1(mr_2)}{I_o(mr_1)\,K_1(mr_2) + K_o(mr_1)\,I_l(mr_2)} \tag{2.132}$$

where $I_1\,(mr)$ and $K_1(mr)$ are Bessel functions of order one and $m = \sqrt{\dfrac{2h}{kb}}$.

The rate of heat transfer is given by

$$\theta_o = 2\pi k\, mb\, \theta_o\, r_1 \frac{K_1(mr_1)\,I_1(mr_2) - I_1(mr_1)\,K_1(mr_2)}{K_o(mr_1)\,I_1(mr_2) + I_o(mr_1)\,K_1(mr_2)} \tag{2.133}$$

Table B.5 in the Appendix gives the selected values of Modified Bessel Functions of the 1st and 2nd kind, of order zero and one (see C.R. Wylie, Jr., Adv. Engg. Mathematics, McGraw-Hill, N.Y.).

The fin efficiency for a convective tip then becomes

$$\eta_c = \frac{Q_o}{2\pi h(r_2^2 - r_1^2)\theta_o} = \frac{2r_1}{m(r_2^2 - r_1^2)} \cdot \frac{K_1(mr_1)\,I_1(mr_2) - I_1(mr_1)\,K_1(mr_2)}{K_o(mr_1)\,I_1(mr_2) + I_o(mr_1)\,K_1(mr_2)} \tag{2.133a}$$

where the tip radius $r_2$ is replaced by a corrected radius $r_{2c} = r_2 + \dfrac{b}{2}$. Results are represented graphically in Fig. 2.25, where $b$ has been shown as $t$.

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## 2.5 TWO- AND THREE-DIMENSIONAL STEADY-STATE HEAT CONDUCTION

In the preceding part of this chapter we dealt with problems in which the temperature and the heat flow can be treated as functions of a single variable. But when the boundaries of a system are irregular or the temperature along a boundary is non-uniform, a one-dimensional treatment may not be satisfactory. Temperature here may be a function of two or even three coordinates. We will first study two-dimensional problems because they are less cumbersome to solve, and then extend it to analyse three-dimensional systems.

For steady-state two-dimensional heat conduction in the absence of any heat source, and uniform thermal conductivity, Laplace equation applies

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{2.134}$$

The solution of this equation can be obtained by (a) analytical, (b) graphical, (c) analogue and (d) numerical methods.

### 2.5.1 Analytical Solution

It is applicable only to simple geometries and boundary conditions. The solutions of Eq. (2.134) will give $T(x, y)$, the temperature as a function of two space coordinates $x$ and $y$. The components of the heat flow per unit area (heat flux) $q$ in the $x$ and $y$-direction are obtained from Fourier's law:

$$q_x = -k\,\frac{\partial T}{\partial x} \quad \text{and} \quad q_y = -k\,\frac{\partial T}{\partial y}$$

The heat flux depends on the temperature gradient and is, therefore, a vector, while temperature is a scalar.

The heat flux $q$ at a given point $(x, y)$ is the resultant of the components $q_x$ and $q_y$ at that point and is directed perpendicular to the isotherm (Fig. 2.32). If the temperature distribution in a system is known, the rate of heat flow can easily be calculated. Therefore, heat flow analyses usually concentrate on determining the temperature field.

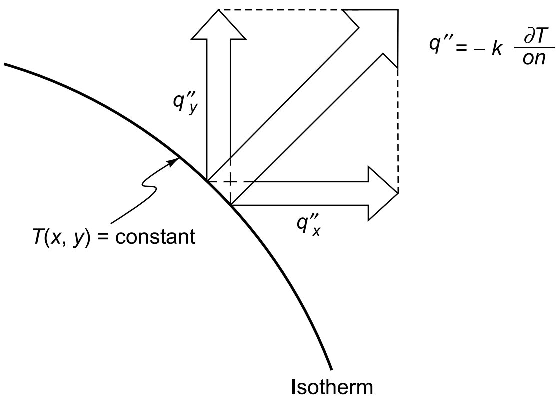


**Fig. 2.32** *Heat flow in two dimensions*

Let us consider a simple case of a thin rectangular plate (Fig. 2.33), free of heat sources and insulated at the top and bottom surfaces. Since $\partial T/\partial z \cong 0$, the temperature is a function of $x$ and $y$ only. If $k$ is uniform, the temperature distribution must satisfy Eq. (2.134), a linear and homogeneous partial differential equation that can be integrated by assuming a product solution for $T(x, y)$ of the form

$$T = X(x)\, Y(y) \tag{2.135}$$

where $X = X(x)$, a function of $x$ only, and $Y = Y(y)$, a function of $y$ alone. Differentiating Eq. (2.135) twice, first with respect to $x$ and then with respect to $y$.

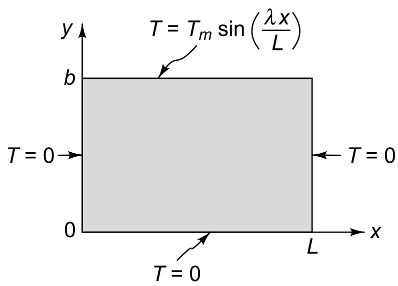


**Fig. 2.33** *Rectangular adiabatic plate with sinusoidal temperature distribution on one edge*

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$$\frac{\partial^2 T}{\partial x^2} = Y\frac{\partial^2 X}{\partial x^2} \text{ and } \frac{\partial^2 T}{\partial y^2} = X\frac{\partial^2 Y}{\partial y^2}$$

Substituting in Eq. (2.134)

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} = 0$$

or,

$$-\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = \lambda^2 \text{ (say)} \tag{2.136}$$

The variables are now separated. The LHS is a function of $x$ only, while the RHS is a function of $y$ alone. Since neither side can change as $x$ and $y$ vary, both must be equal to a constant, say $\lambda^2$. We have, therefore, two ordinary differential equations

$$\frac{d^2 X}{dx^2} + \lambda^2 X = 0 \tag{2.137}$$

$$\frac{d^2 Y}{dy^2} - \lambda^2 Y = 0 \tag{2.138}$$

The general solution to Eq. (2.137) is

$$X = C_1 \cos \lambda x + C_2 \sin \lambda x$$

and the general solution to Eq. (2.138) is

$$Y = C_3 e^{-\lambda y} + C_4 e^{\lambda y}$$

Therefore, from Eq. (2.135),

$$T = X \cdot Y = (C_1 \cos \lambda x + C_2 \sin \lambda x)(C_3 e^{-\lambda y} + C_4 e^{\lambda y}) \tag{2.139}$$

where $C_1$, $C_2$, $C_3$ and $C_4$ are constants to be evaluated from the boundary conditions. As shown in Fig. 2.33, the boundary conditions to be satisfied are

$$T = 0 \text{ at } x = 0$$

$$T = 0 \text{ at } x = L$$

$$T = 0 \text{ at } y = 0$$

$$T = T_m \sin\left(\frac{\lambda x}{L}\right) \text{ at } y = b$$

Substituting these conditions in Eq. (2.139), from the 3rd condition (at $y = 0$)

$$(C_1 \cos \lambda x + C_2 \sin \lambda x)(C_3 + C_4) = 0 \tag{2.140}$$

from the first condition (at $x = 0$)

$$C_1 (C_3 e^{-\lambda y} + C_4 e^{\lambda y}) = 0 \tag{2.141}$$

and from the second condition

$$(C_1 \cos \lambda L + C_2 \sin \lambda L)(C_3 e^{-\lambda y} + C_4 e^{\lambda y}) = 0 \tag{2.142}$$

Equation (2.140) gives $C_3 = -C_4$ and Eq. (2.141) gives $C_1 = 0$. Using these results in Eq. (2.138)

$$2C_2C_3 \sin \lambda L \sinh \lambda y = 0 \tag{2.143}$$

To satisfy this condition, $\sin \lambda L = 0$ or, $\lambda = \frac{n\pi}{L}$, where $n = 1, 2, 3, \ldots$. There exists, therefore, a different

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solution for each integer $n$, and each solution has a separate integration constant $C_n$. Summing these solutions, we get from Eq. (2.139),

$$T = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi y}{L} \tag{2.144}$$

The last boundary condition needs that at $y = b$

$$T_m \sin \frac{\pi x}{L} = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi b}{L}$$

The first term in the series solution gives

$$T_m \sin \frac{\pi x}{L} = C_1 \frac{\pi x}{L} \sinh \frac{\pi b}{L}$$

$$C_1 = \frac{T_m}{\sinh (\pi b / L)}$$

The solution therefore becomes [from Eq. (2.144)]

$$T(x, y) = T_m \frac{\sinh (\pi y / L)}{\sinh (\pi b / L)} \sin \frac{\pi x}{L} \tag{2.145}$$

The corresponding temperature field is shown in Fig. 2.34. The solid lines are isotherms, and the dashed lines are heat flow lines, which are orthogonal.

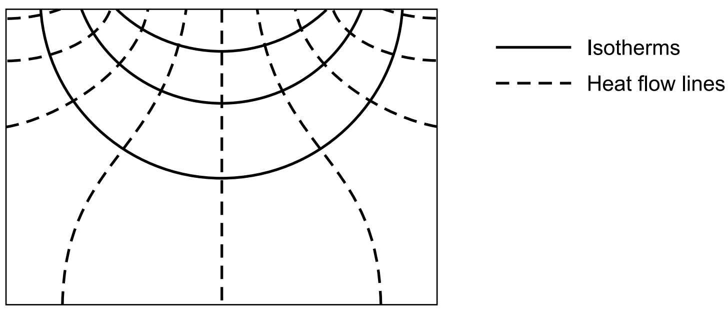


**Fig. 2.34** *Isotherms and heat flow lines for the plate in Fig. 2.33*

The separation-of-variables method can be extended to three-dimensional problems, by assuming that $T = X \cdot Y \cdot Z$. Substituting this expression for $T$ in equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0$$

separating the variables, integrating the total differential equation and using the given boundary conditions, the solution for temperature distribution can be obtained [1, 5, 7].

### 2.5.2 Graphical Method

This method can rapidly provide an approximate estimate of the temperature distribution and heat flow in geometrically complex two-dimensional systems, but its application is limited to cases with isothermal and adiabatic boundaries. The object of the graphic solution is to construct a network of isotherms and adiabatics. The flux lines are analogous to streamlines in a potential fluid flow, i.e., tangent to the direction of heat flow at any point. Therefore, no heat can flow across the constant-flux lines. The isotherms are

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analogous to constant-potential lines and heat flows perpendicular to them. Thus, isotherms and adiabatics intersect at right angles.

To obtain the temperature distribution, one first prepares a scale model, and then draws isotherms and adiabatics freehand, by trial and error, until they form a network of curvilinear squares. Then a constant amount of heat flows between any two flux lines.

In Fig. 2.35(a), a corner section of unit depth ($\Delta z$ = 1) has been considered with $ABC$ at temperature $T_1$, faces $FED$ at temperature $T_2$ and faces $CD$ and $AF$ insulated. The curvilinear network of isotherms and adiabatics are shown in Fig. 2.35(b). Flux lines leading to or from a corner of an isothermal boundary bisect the angle between the surfaces forming the corner.

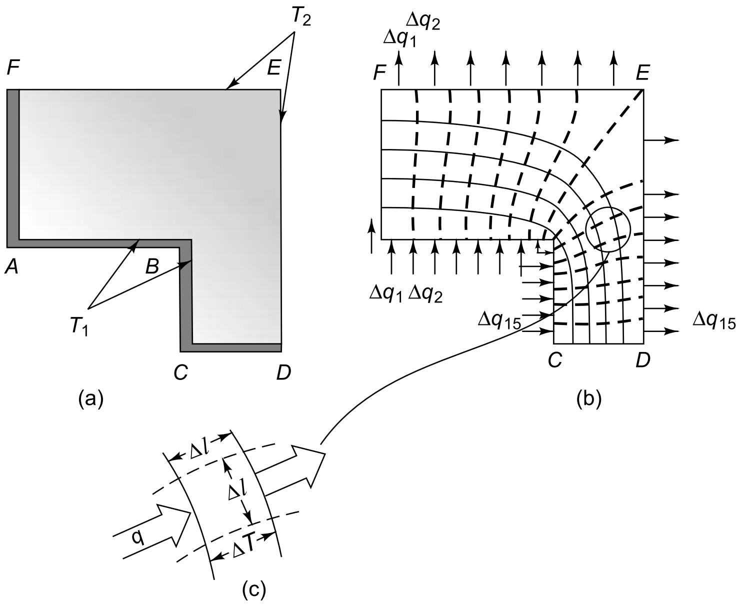


**Fig. 2.35** *Construction of a network of curvilinear squares for a corner section: (a) scale model, (b) flux plot and (c) typical curvilinear square*

A graphic solution, like an analytic solution of a heat conduction problem described by the Laplace equation and the associated boundary condition, is unique. Any curvilinear network, irrespective of its size, that satisfies the boundary conditions, represents the correct solution. For any curvilinear square [Fig. 2.35(c)], the rate of heat flow is given by Fourier law:

$$Q = -k\,(\Delta l \times l)\,\frac{\Delta T}{\Delta l} = -\,k\Delta T$$

This heat flow will remain the same across any square within any one heat flow lane from the boundary at $T_1$ to the boundary at $T_2$. The temperature difference $\Delta T$ across any one element in the heat flow lane is therefore

$$\Delta T = \frac{T_2 - T_1}{N}$$

where $N$ is the number of temperature increments between the two boundaries at $T_1$ and $T_2$. The total heat flow from $T_1$ to $T_2$ is equal to the sum of the heat flows of all the lanes.

$$Q = \sum_{n=1}^{n=M} \Delta Q_n = \frac{M}{n}\;k\,(T_1 - T_2) = \frac{M}{n}\;k\,(\Delta T)_{\text{overall}} \tag{2.146}$$

where $\Delta Q_n$ is the rate of heat flow through the $n$th lane, and $M$ is the number of heat flow lanes.

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Only a network of the curvilinear squares need be constructed. By counting the number of temperature increments ($N$) and heat flow lanes ($M$), one can estimate the rate of heat transfer. The accuracy of the method depends on the skill and patience of the person sketching the network. Even a crude sketch can give a fairly good estimate of the heat transfer, which can be refined by the numerical method described later.

The ratio $M/N$ depends on the shape of the system and is called the *shape factor S*. The rate of heat transfer is then

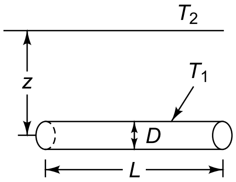
$$Q = kS\,(\Delta T)_{\text{overall}} \tag{2.147}$$

Values of $S$ for several shapes are summarized in Table 2.5.

**Table 2.5** *Conduction shape factors S for several configurations to determine steady heat transfer rate $Q = kS\,(T_1 - T_2)$ between surfaces at temperatures $T_1$ and $T_2$*

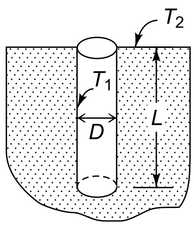
1. Isothermal cylinder of length $L$ buried in a semi-infinite medium ($L >> D$ and $z > 1.5D$)

$$S = \frac{2\pi L}{\ln(4z/D)}$$



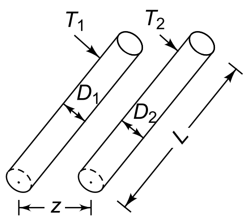

2. Vertical isothermal cylinder of length $L$ buried in a semi-infinite medium ($L >> D$)

$$S = \frac{2\pi L}{\ln(4L/D)}$$



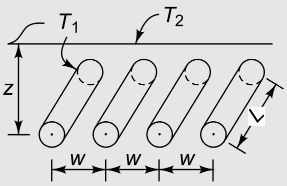

3. Two parallel isothermal cylinders placed in an infinite medium ($L >> D_1, D_2, z$)

$$S = \frac{2\pi L}{\cosh^{-1}\left(\dfrac{4z^2 - D_1^2 - D_2^2}{2D_1D_2}\right)}$$



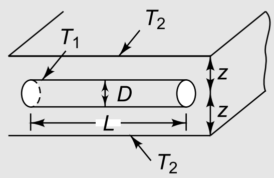

4. A row of equally spaced parallel isothermal cylinders buried in a semi-infinite medium ($L >> D$, $z$ and $w > 1.5\,D$)

$$S = \frac{2\pi L}{\ln\left(\dfrac{2w}{\pi D}\sinh\dfrac{2\pi z}{w}\right)} \quad \text{(per cylinder)}$$



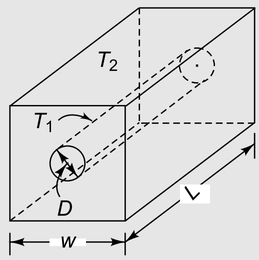

5. Circular isothermal cylinder of length $L$ in the midplane of an infinite wall ($z > 0.5D$)

$S = (8z/\pi D)$




6. Circular isothermal cylinder of length $L$ at the centre of a square solid bar of the same length

$$S = \frac{2\pi L}{\ln(1.08w/D)}$$

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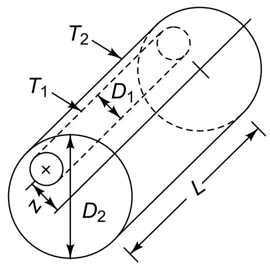
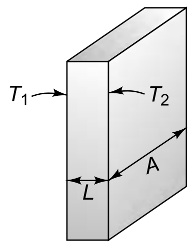
7. Eccentric circular isothermal cylinder length $L$ in a cylinder of the same length ($L > D_2$)

$$S = \frac{2\pi L}{\cosh^{-1}\left(\dfrac{D_1^2 + D_2^2 - 4z^2}{2D_1 D_2}\right)}$$



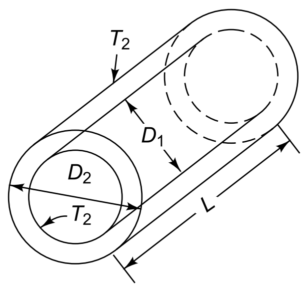

8. Large plain wall

$$S = \frac{A}{L}$$




9. A long cylindrical layer

$$S = \frac{2\pi L}{\ln (D_2 / D_1)}$$



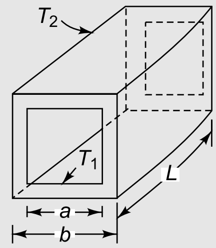

10. A square flow passage

(a) For $a/b > 1.4$.

$$S = \frac{2\pi L}{0.93 \ln (0.948a / b)}$$

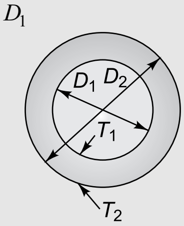
(b) For $a/b < 1.41$.

$$S = \frac{2\pi L}{0.785 \ln (a / b)}$$




11. A spherical layer

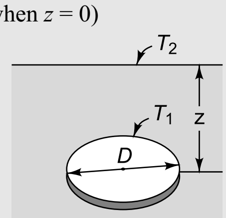
$$S = \frac{2\pi D_1 D_2}{D_2 - D_1}$$




12. Disk buried parallel to the surface in a semi-infinite medium ($z >> D$)

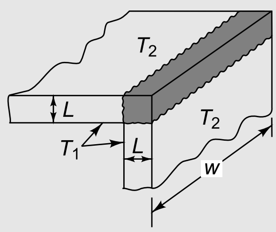
$S = 4D$

($S = 2D$ when $z = 0$)



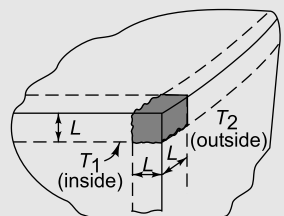

13. The edge of two adjoining walls of equal thickness

$S = 0.54\ w$




14. Corner of three walls of equal thickness

$S = 0.15\ L$

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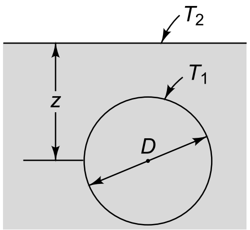
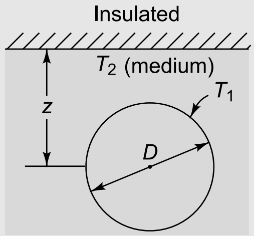
15. Isothermal sphere buried in a semi-infinite medium

$$S = \frac{2\pi D}{1 - 0.25D/z}$$




16. Isothermal sphere buried in a semi-infinite medium at $T_2$ whose surface is insulated

$$S = \frac{2\pi D}{1 + 0.25D/z}$$




### 2.5.3 Numerical Methods

Analytical solutions are usually possible only for relatively simple problems. Many practical problems, however, involve complex geometries, complex boundary conditions, or variable properties, for which analytical solutions are very difficult and sometimes, impossible. These problems can be solved with relative ease by methods of numerical analysis. The numerical analysis not only saves on computer time, vis-a-vis analytical method, but also allows changes in problem parameters to determine the behaviour of a thermal system and also to optimise it.

In the last two decades, the amount of computing power per unit cost available in personal computers increased very significantly. Also available at reasonable cost is powerful software such as finite element codes, equation solvers and compilers, that can be employed in solving complex heat transfer problems.

Analytical method solves the governing differential equations and provides a solution at every point in space and time within the problem boundaries. In contrast, numerical methods provide the solution only at discrete points within the problem boundaries and yield only an approximate solution. By solving only for a finite number of discrete points, we simplify the solution method to one of solving a system of simultaneous algebraic equations as opposed to solving the differential equation. The solution of a system of simultaneous equations is a task best suited to digital computers.

In addition to replacing the differential equation with a system of algebraic equations, a process called *discretisation*, there are several other important considerations for a complete numerical solution. First, the boundary conditions or initial conditions specified for the problem must also be discretised. Second, we need to be aware that as an approximation to the exact solution, the numerical method introduces errors into the solution. We need to know how to minimise these errors. Finally, the numerical method may give a solution that oscillates in time or space. We need to know how to avoid these stability problems.

Several methods are available for discretising the differential equations of heat conduction, like the finite difference, finite element and control volume approaches.

We would first illustrate the finite difference method. A two-dimensional body is divided into equal segments of $\Delta x$ and $\Delta y$. The nodes are designated by points $(m, n)$, $(m + 1, n)$, $(m, n + 1)$, ... as shown, $m$ location indicating $x$ increment and $n$ location indicating $y$ increment. It is desired to find the temperatures at the nodes. The temperature at any node represents the temperature in the region $\pm x/2$ and $\pm y/2$ around the node.

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The temperature gradients at $A$, $B$, $C$ and $D$ (Fig. 2.36) are respectively

$$\left(\frac{\partial T}{\partial x}\right)_{m+\frac{1}{2},n} \cong \frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$

$$\left(\frac{\partial T}{\partial x}\right)_{m-\frac{1}{2},n} \cong \frac{T_{m,n} - T_{m-1,n}}{\Delta x}$$

$$\left(\frac{\partial T}{\partial y}\right)_{m,n+\frac{1}{2}} \cong \frac{T_{m,n+1} - T_{m,n}}{\Delta y}$$

$$\left(\frac{\partial T}{\partial z}\right)_{m,n-\frac{1}{2}} = \frac{T_{m,n} - T_{m,n-1}}{\Delta y}$$

$$\left(\frac{\partial^2 T}{\partial x^2}\right)_{m,n} = \frac{\left(\frac{\partial T}{\partial x}\right)_{m+\frac{1}{2},n} - \left(\frac{\partial T}{\partial x}\right)_{m-\frac{1}{2},n}}{\Delta x}$$

$$= \frac{T_{m+1,n} - T_{m,n} - T_{m,n} + T_{m-1,n}}{(\Delta x)^2}$$

$$= \frac{T_{m+1,n} + T_{m-1,n} - 2T_{m,n}}{(\Delta x)^2}$$

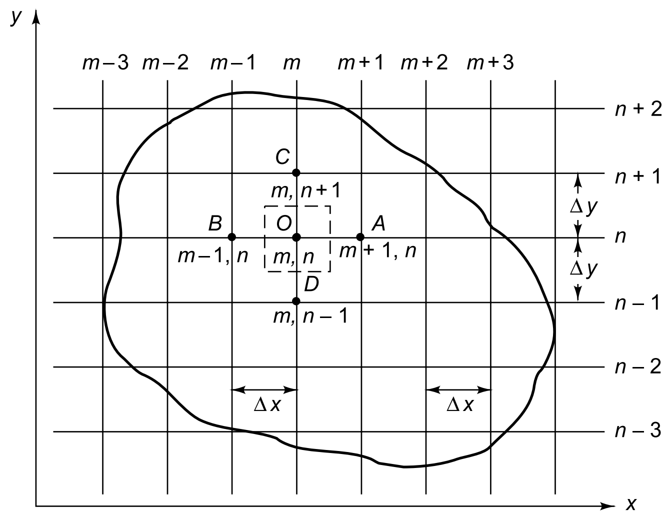


**Fig. 2.36** *Two-dimensional body is discretised into segments, $\Delta x \times \Delta y$*

Similarly,

$$\left(\frac{\partial^2 T}{\partial y^2}\right)_{m,n} = \frac{T_{m,n+1} + T_{m,n-1} - 2T_{m,n}}{(\Delta y)^2}$$

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Substituting in Laplace equation

$$\frac{T_{m+1,n} + T_{m-1,n} - 2T_{m,n}}{(\Delta x)^2} + \frac{T_{m,n+1} + T_{m,n-1} - 2T_{m,n}}{(\Delta y)^2} = 0$$

If $\Delta x = \Delta y$, we have

$$T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1} - 4T_{m,n} = 0 \quad (2.148)$$

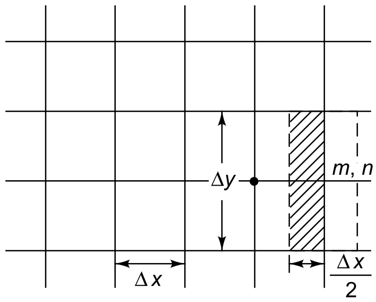


**Fig. 2.37** *A nodal point at the boundary*

For nodal points at the convection boundary (Fig. 2.37), separate energy balance would be needed,

$$-k\Delta y \frac{T_{m,n} - T_{m-1,n}}{\Delta x} - k\frac{\Delta x}{2}\frac{T_{m,n} - T_{m,n+1}}{\Delta y}$$

$$-k\frac{\Delta x}{2}\frac{T_{m,n} - T_{m,n-1}}{\Delta y} = h\Delta y\,(T_{m,n} - T_\infty)$$

For $\Delta x = \Delta y$,

$$T_{m,n}\left[\frac{h\Delta x}{k} + 2\right] - \frac{h\Delta x}{k}\,T_\infty - \frac{1}{2}$$

$$(2T_{m-1,n} + T_{m,n+1} - T_{m,n-1}) = 0 \quad (2.149)$$

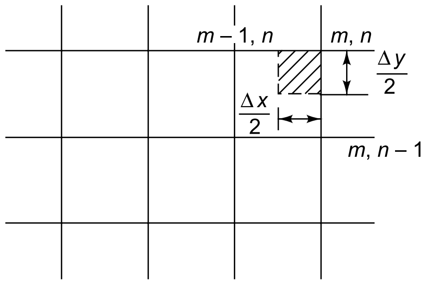


**Fig. 2.38** *A nodal point at a corner*

Similarly, for a nodal point at a corner (Fig. 2.38),

$$-k\frac{\Delta y}{2}\frac{T_{m,n} - T_{m-1,n}}{\Delta x} - k\frac{\Delta x}{2}\frac{T_{m,n} - T_{m,n+1}}{\Delta y}$$

$$= h\frac{\Delta x}{2}(T_{m,n} - T_\infty) + h\frac{\Delta y}{2}(T_{m,n} - T_\infty)$$

For $\Delta x = \Delta y$,

$$2T_{m,n}\left(1 + \frac{h\Delta x}{k}\right) - \frac{h\Delta x}{k}\,T_\infty - (T_{m-1,n} + T_{m,n-1}) = 0 \quad (2.150)$$

Other boundary conditions may be treated in a similar manner. In this way, equations are written for all nodal points and arranged in a matrix form.

$$\begin{aligned}
&a_{11}T_1 + a_{12}T_2 + \ldots + a_{\mathrm{ln}}T_n = C_1\\
&a_{21}T_1 + a_{22}T_2 + \ldots + a_{2n}T_n = C_2\\
&A_{31}T_1 + a_{32}T_2 + \ldots + a_{3n}T_n = C_3\\
&\vdots\\
&a_{n1}T_1 + a_{n2}T_2 + \ldots + a_{nn}T_n = C_n
\end{aligned} \quad (2.151)$$

where $a_{ij}$'s and $C_j$'s are known constants and $T_j$'s are unknown temperatures.

Equation (2.151) can be condensed and written in matrix notation as

$$AT = C$$

where $A$ is a $n \times n$ coefficient defined by

$$\begin{Bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{Bmatrix} \quad (2.152)$$

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where $T$ and $C$ are column matrices consisting of $n$ elements each:

$$T = \begin{Bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{Bmatrix} \text{ and } C = \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \\ C_n \end{Bmatrix} \tag{2.153}$$

To calculate the unknown temperatures we must determine the inverse of the matrix, $A^{-1}$, which satisfies the equation

$$T = A^{-1}C \tag{2.154}$$

If the elements of the inverse of matrix $A$ are given by

$$B = A^{-1} = \begin{Bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & & & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{Bmatrix} \tag{2.155}$$

the unknown nodal temperatures are given by the equations

$$\begin{Bmatrix} b_{11}c_1 + b_{12}c_2 + b_{13}c_3 + \cdots + b_{1n}c_n = T_1 \\ b_{21}c_1 + b_{22}c_2 + b_{23}c_3 + \cdots + b_{2n}c_n = T_2 \\ \vdots \\ b_{n1}c_1 + b_{n2}c_2 + b_{n3}c_3 + \cdots + b_{nn}c_n = T_n \end{Bmatrix} \tag{2.156}$$

Since the values of all the $c_i$'s are known, the problem of calculating the temperatures depends upon determining the inverse of matrix $A$, which can be calculated by using standard mathematical techniques if the number of nodes is small. When the number of nodes is large, elements in the inverted matrix can be determined with a computer by using standard subroutines.

#### 2.5.3.1 Gauss-Siedel Iteration Technique

A numerical method that is particularly well-suited for a computer solution is an *iteration method* based on solving each nodal equation explicitly for the temperature of that node. For an interior node in a two-dimensional solid, the energy balance equation gives

$$T_1 + T_2 + T_3 + T_4 - 4T_o = 0$$

If the node is located on a boundary transferring heat to a fluid by convection

$$T_o = \frac{1/2(T_2 + T_3) + T_1 + Bi\, T_\infty}{2 + Bi}$$

An equation for the temperature of each node may be written in terms of the temperatures of the neighbouring nodes. The number of equations equals the number of nodes with unknown temperatures.

First assume a set of values for all the nodal temperatures. Next calculate new values for the temperatures using the nodal equations. Replace the old temperatures by the new ones till a tolerance level of the difference between the old and new values is reached. This is *Gauss-Siedel iteration method.*

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### Relaxation Method

It was first introduced by Southwell [9]. It is a numerical method to solve a set of algebraic equations. Let us take an example

$$-4x + y + 56 = 0$$
$$x - 2y + 34 = 0$$

If we write $F_1 = -4x + y + 56$ and $F_2 = x - 2y + 34$, then $F_1$ and $F_2$ are called *residuals*. The object is to reduce the values of residuals to zero, as illustrated in Table 2.6. If we stop at $x = 21$ and $y = 27$, the residuals are $F_1 = -1$ and $F_2 = 1$. On further relaxation, we find $x = 20.8$, $y = 27.4$, when the residual $F_2$ is reduced to zero and $F_1 = 0.2$.

**Table 2.6**

| | | $F_1$ | $F_2$ |
|---|---|---|---|
| $x = 0$ | $y = 0$ | 56 | 34 |
| $\Delta x = 14$ | – | 0 | 48 |
| | $\Delta y = 24$ | 24 | 0 |
| $\Delta x = 6$ | | 0 | 6 |
| | $\Delta y = 3$ | 3 | 0 |
| $\Delta x = 1$ | | –1 | 1 |
| $x = 21$ | $y = 27$ | –1 | 1 |
| $\Delta x = -0.3$ | | 0.2 | 0.7 |
| | $\Delta y = 0.4$ | 0.6 | –0.1 |
| $\Delta x = 0.1$ | | 0.2 | 0.0 |
| $x = 20.8$ | $y = 27.4$ | 0.2 | 0.0 |

To apply the relaxation method to a two-dimensional steady-state heat conduction problem, we have to convert the partial differential equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

to a set of algebraic equations. Let us consider a solid of thickness $b$ where no variation in temperature exists along the axis representing thickness (Fig. 2.39).

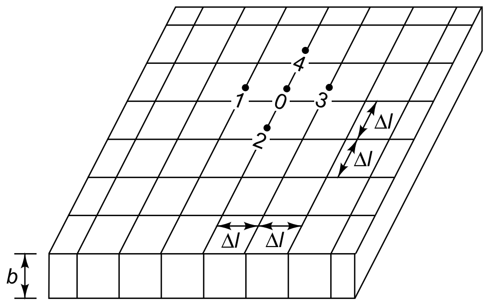


**Fig. 2.39** *Solid divided into sub-volumes, $\Delta l \times \Delta l \times b$*

Let us divide the whole solid into sub-volumes $\Delta l \times \Delta l \times b$. The centre of each sub-volume is a nodal point. Material between nodal points may be replaced by fictitious rods having the same $k$ as the solid material (Fig. 2.40). Steady-state energy balance gives

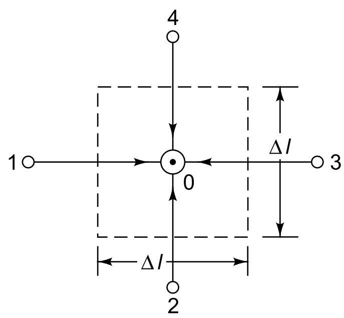


**Fig. 2.40** *Each sub-volume replaced by a nodal point*

$$Q_{1-0} + Q_{2-0} + Q_{3-0} + Q_{4-0} = 0$$

$$\frac{k\Delta l \times b}{\Delta l}(T_1 - T_o) + \frac{k\Delta l \times b}{\Delta l}(T_2 - T_o)$$
$$+ kb(T_3 - T_o) + kb\,(T_4 - T_o) = 0$$

or $$kb\;[(T_1 - T_o) + (T_2 - T_o) + (T_3 - T_o) + (T_4 - T_o)] = 0$$

or $$T_1 + T_2 + T_3 + T_4 - 4T_o = 0$$

Residual equation for an interior nodal point is

$$Q_o' = q^* = T_1 + T_2 + T_3 + T_4 - 4T_o = \frac{Q_o}{kb} \tag{2.157}$$

The object is to reduce the residuals to zero for each nodal point in the body. Any unit change in temperature in any of the neighbouring points can change the residual $q^*$ by $\pm 1$, whereas a unit change in temperature of the node itself changes the residual by $\pm 4$. This is the key to the procedure (Fig. 2.41).

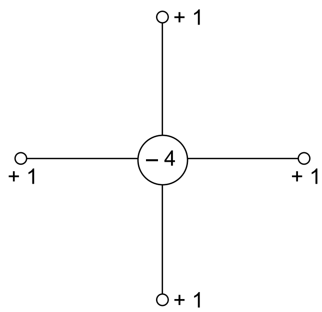


**Fig. 2.41** *Key to relaxation method*

For a nodal point at an isothermal boundary (Fig. 2.42), $T_4 = T_2 = T_o$.

$$\bar{q}\,(\Delta lb) + \frac{k(\Delta lb)}{\Delta l}\,(T_3 - T_o) = 0$$

or

$$\bar{q}\,\frac{\Delta l}{k} + T_3 - T_o = q^*$$

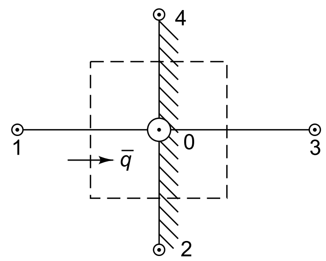


**Fig. 2.42** *Nodal point at an isothermal boundary*

For a nodal point at a surface in contact with a fluid (Fig. 2.43), $T_3 = T_\infty$.

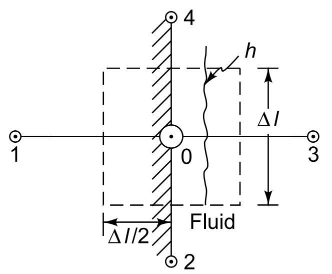


**Fig. 2.43** *Node in contact with a fluid*

$$h\Delta lb\,(T_\infty - T_o) + \frac{k\Delta lb}{\Delta l}\,(T_1 - T_o) + \frac{k\Delta lb}{2\Delta l}\,(T_4 - T_o) + \frac{kb}{2}\,(T_2 - T_o) = 0$$

The residual equation is

$$q^* = \frac{h\Delta l}{k} T - \frac{h\Delta l}{k}\,T_o + T_1 - T_o + \frac{T_4 - T_o}{2} + \frac{T_2 - T_o}{2}$$

$$q^* = \frac{T_2 + T_4}{2} + T_1 + T_\infty\,\frac{h\Delta l}{k} - T_o\left(2 + \frac{h\Delta l}{k}\right)$$

and so on.

The residual equations for a two-dimensional case may thus be set up for different nodal points. The resulting simultaneous equations may be solved by relaxation method and the steady-state temperature distribution can be obtained.

### 2.5.4 Electrical Analogy

The governing equations of electrostatics and heat conduction are similar. The potential distribution $E$ in an electrostatic field and the temperature distribution $T$ in conduction are both governed by the Laplace equation

$$\nabla^2 E = 0,\ \nabla^2 T = 0$$

In an analogy between electrical flow and heat conduction, the temperature can be replaced by

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the voltage and vice versa. This analogy between the two fields is utilised in practice by plotting the temperature distribution in complex problems of heat conduction. A special paper which has the capacity to conduct electricity is cut to the shape and size of the body for which temperature distribution is desired. Isothermal boundaries (equipotentials) are obtained by attaching wires to the paper or by painting the paper with a silver paint, which is a very good conductor. The unpainted edges of the paper will be insulated boundaries. The model thus prepared is connected to an e.m.f. source and the equipotential or isothermal lines are progressively plotted using a stylus probe and a null detector connected to a suitable bridge circuit. The heat flow lines are then plotted either by hand, drawing lines orthogonal to the isotherms or by resisting the boundaries and repeating as desired [10]. The analogue field plotter is faster and gives more accurate results than free hand flux plotting.

## 2.6 THREE-DIMENSIONAL HEAT CONDUCTION

Analytical and numerical methods described above for two-dimensional problems can be extended to three-dimensional (3D) problems. The general equation for the temperature distribution in steady state is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{2.158}$$

subject to boundary conditions of the problem. Since $T$ is a function of $(x, y, z)$, the solution would be of the form

$$T = X(x)\ Y(y)\ Z(z) \tag{2.159}$$

The separation-of-variables technique is applied, which leads Eq. (2.158) to the form

$$\frac{1}{X}\frac{d^2X}{dX^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = -\frac{1}{Z}\frac{d^2Z}{dz^2}$$

which can be solved for $X$, $Y$, $Z$ subject to given boundary conditions.

The numerical method of solving 3D heat conduction problems is exactly the same as that for 2D cases except that a 3D grid should be made now. As shown in Fig. 2.44, in a 3D problem there will be

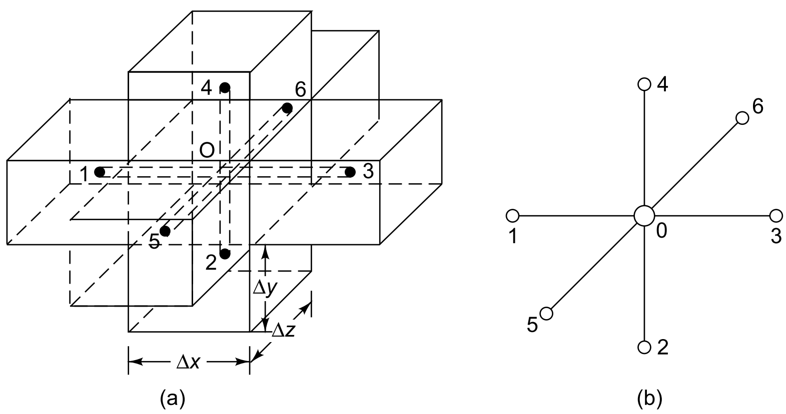


**Fig. 2.44** *Relaxation grid for an internal node in a three-dimensional system*

six neighbouring nodes for any internal node. Heat flow by conduction into the nodal point 0 from all the neighbouring six nodes is given by

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$$Q_{1-0} + Q_{2-0} + Q_{3-0} + Q_{4-0} + Q_{5-0} + Q_{6-0} = 0$$

$$k(\Delta y\, \Delta z)\, \frac{(T_1 - T_o)}{\Delta x} + k(\Delta z\, \Delta x)\, \frac{(T_2 - T_o)}{\Delta y} + k\Delta\, y\Delta z\, \frac{(T_3 - T_o)}{\Delta x} + k\Delta z\, \Delta x\, \frac{(T_4 - T_o)}{\Delta y}$$

$$+ k\Delta\, x\Delta y\, \frac{(T_5 - T_o)}{z} + k\Delta x\, \Delta y\, \frac{(T_6 - T_o)}{\Delta z} = 0$$

If $\Delta x = \Delta y = \Delta z = b$,

$$bk\,(T_1 - T_o + T_2 - T_o + T_3 - T_o + T_4 - T_o + T_5 - T_o + T_6 - T_o) = 0$$

$$T_1 + T_2 + T_3 + T_4 + T_5 + T_6 - 6T_o = 0 \tag{2.160}$$

The residual equations for the boundary nodes can be written by performing energy balances following the approach described earlier for two-dimensional systems.

Three-dimensional conduction is often encountered as in the walls of a furnace if they are very thick. Langmuir solved the problem by means of experimental studies on electrical analogues [9]. For this purpose he determined the conductance of a copper sulphate solution in a container of the same shape as the wall and compared it with the conductance of an equivalent plane wall with constant cross-section. He thus determined a mean area $A_m$ to give the same heat flow as through the given wall, so that

$$Q = \frac{kA_m (T_1 - T_2)}{\Delta x}$$

where $\Delta x$ is the thickness of the wall. Sometimes this is expressed in the form

$$Q = kS\,(T_1 - T_2)$$

where $S = A_m/\Delta x$ is the shape factor, which is given as

(a) for one edge of length $y$, $S = 0.54\, y$

(b) for a corner, $S = 0.15\, \Delta x$

(c) for a cylindrical pipe, $S = 2\pi L \ln \dfrac{r_2}{r_1}$

(d) for a spherical wall, $S = \dfrac{4\pi\, r_1\, r_2}{r_2 - r_1}$

**Case 1** Let us suppose that all inside dimensions exceed 1/5th of the wall thickness, i.e. $y > \Delta\, x/5$. In this case $A_m$ is obtained by adding to the inside surface area $A_1$ the quantity $0.54\, \Delta x\, \Sigma y$ to account for the 12 edges and the quantity $0.15\, (\Delta x)^2$ for each of the 8 corners.

$$A_m = A_1 + 0.54\, \Delta x\, (\Sigma y) + 8 \times 0.15\, (\Delta x)^2 \tag{2.161}$$

**Case 2** Let one dimension be less than $\Delta x/5$. The lengths of the four inside edges less than $\Delta x/5$ are neglected to give

$$A_m = A_1 + 0.465\, \Delta x\, \Sigma y + 0.35\, (\Delta x)^2 \tag{2.162}$$

**Case 3** If two inside dimensions are each less than $\Delta x/5$, then

$$A_m = \frac{2.78\, y_{\max}\, \Delta x}{\log_{10} (A_2 / A_1)} \tag{2.163}$$

where $y_{\max}$ is the largest interior dimensions.

**Case 4** If all the there inside dimensions are less than $\Delta x/5$, then

$$A_m = 0.79\, (A_1\, A_2)^{1/2} \tag{2.164}$$

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## Solved Problems

**Example 2.1** *Given*: The composite wall having unit length normal to the plane of paper and the equivalent thermal circuit are shown in Fig. Ex. 2.1

$H_A = H_D = 3$ m, $H_B = H_C = 1.5$ m
$L_1 = L_3 = 0.05$ m, $L_2 = 0.1$ m
$k_A = k_D = 50$ W/mK, $k_B = 10$ W/mK, $k_C = 1$ W/mK
$T_1 = 200°$C, $h_1 = 50$ W/m$^2$ K, $T_2 = 25°$C, $h_2 = 10$ W/m$^2$K.

*To find:* The rate of heat transfer through the wall.

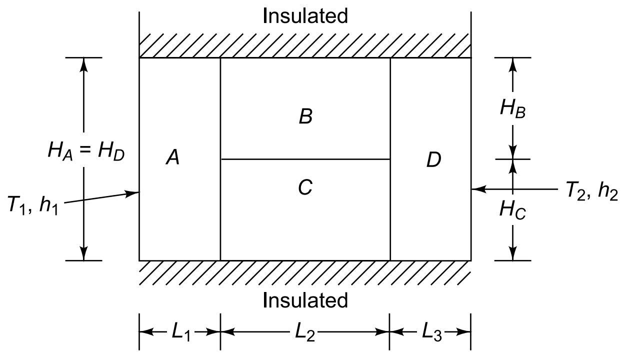


**Fig.** Ex 2.1

**Solution** The total thermal resistance

$$R_{tot} = \frac{1}{h_1 A_A} + \frac{L_1}{k_A A_A} + \frac{L_2}{k_B A_B + k_C A_C} + \frac{L_3}{k_D A_D} + \frac{1}{h_2 A_D}$$

$$= \frac{1}{50 \times 3} + \frac{0.05}{50 \times 3} + \frac{0.1}{10 \times 1.5 + 1 \times 1.5} + \frac{0.05}{50 \times 3} + \frac{1}{10 \times 3}$$

$$= 0.0467 \text{ mK/W}$$

$$\therefore \quad Q = \frac{T_1 - T_2}{R_{tot}} = \frac{200 - 25}{0.0467} = 3745 \text{ W/m} = 3.745 \text{ kW/m} \quad \textit{Ans.}$$

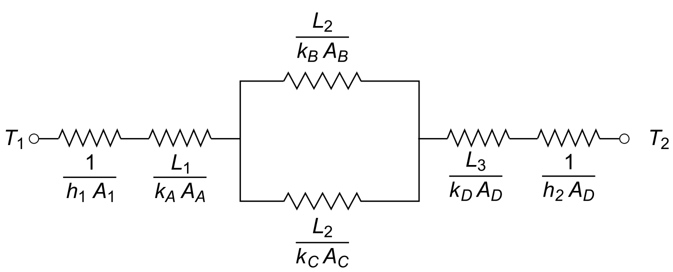


**Fig.** Ex 2.1(a)

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**Example 2.2** *Given:* A hollow cylinder with $r_1$ = 30 mm and $r_2$ = 50 mm, $k$ = 15 W/mK, is heated on the inner surface at a rate of $10^5$ W/m$^2$ and dissipates heat by convection from the outer surface to a fluid at 100°C with $h$ = 400 W/m$^2$K.
*To find:* The temperatures of insde and outside surfaces of the cylinder.

**Solution** Thermal resistances to heat flow are shown in Fig. 2.11. The rate of heat transfer is given by

$$Q = (2\pi r_1 L)q \qquad (1)$$

Also,
$$Q = \frac{(T_1 - T_2)2\pi kL}{\ln(r_2/r_1)} = h2\pi r_2 L\,(T_2 - T_\infty)$$

$$= \frac{T_1 - T_\infty}{\dfrac{\ln(r_2/r_1)}{2\pi kL} + \dfrac{1}{2\pi r_2 Lh}} \qquad (2)$$

Equating Eqs (1) and (2),

$$2\pi r_1 Lq = \frac{(T_1 - T_\infty)2\pi L}{\dfrac{1}{k}\ln\left(\dfrac{r_2}{r_1}\right) + \dfrac{1}{r_2 h}}$$

$$\therefore \qquad T_1 = \left[\frac{r_1}{k}\ln\left(\frac{r_2}{r_1}\right) + \frac{r_1}{r_2 h}\right]q + T_\infty$$

$$= \left[\frac{0.03}{15}\ln\frac{5}{3} + \frac{0.03}{0.05 \times 400}\right] \times 10^5 + 100$$

$$= 352.2\text{°C} \quad \textit{Ans.}$$

$$2\pi r_1 Lq = (T_2 - T_\infty)\,h2\pi r_2 L$$

$$\therefore \qquad T_2 = \frac{r_1}{r_2 h}q + T_\infty = \frac{0.03}{0.05 \times 400} \times 10^5 + 100$$

$$= 250\text{°C} \quad \textit{Ans.}$$

**Example 2.3** *Given:* A hollow sphere, $r_1$ = 4 cm, $r_2$ = 6 cm, $k$ = 20 W/mK, is electrically heated at the inner surface at a rate of $10^5$ W/m$^2$. Heat is dissipated at the outer surface by convection to a fluid at 100°C with $h$ = 450 W/m$^2$K.
*To find:* The inner and outer surface temperatures of the sphere.

**Solution** The thermal resistance network for the hollow sphere is given in Fig. Ex. 2.3. The heat transfer is

$$Q = 4\pi r_1^2 q = \frac{T_1 - T_2}{(r_2 - r_1)/4\pi k r_1 r_2} = \frac{T_2 - T_\infty}{1/(4\pi r_2^2 h)}$$

$$= \frac{T_1 - T_\infty}{(r_2 - r_1)/(4\pi k r_1 r_2) + 1/(4\pi r_2^2 h)}$$

$$4\pi r_1^2 q = \frac{4\pi r_1^2 (T_1 - T_\infty)}{r_1(r_2 - r_1)/kr_2 + (r_1/r_2)^2 \cdot \dfrac{1}{h}}$$

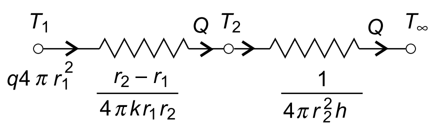


**Fig.** Ex. 2.3

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$$\therefore \qquad T_1 = \left[\frac{r_1(r_2-r_1)}{kr_2} + \left(\frac{r_1}{r_2}\right)^2 \frac{1}{h}\right] q + T_\infty$$

$$= \left[\frac{0.04\times 0.02}{20\times 0.06} + \left(\frac{4}{6}\right)^2 \frac{1}{450}\right] \times 10^5 + 100$$

$$= [6.67\times 10^{-4} + 9.88\times 10^{-4}]\times 10^5 + 100$$

$= 265.5$ºC *Ans.*

Again, $\quad 4\pi r_1^2 . q = 4\pi r_2^2 h\,(T_2 - T_\infty)$

$$\therefore \qquad T_2 = \left(\frac{r_1}{r_2}\right)^2 \frac{1}{h}\cdot q + T_\infty$$

$$= \left(\frac{0.04}{0.06}\right)^2 \frac{1}{450}\times 10^5 + 100$$

$= 198.8$ºC *Ans.*

**Example 2.4** *Given:* A tube 2 cm o.d. maintained at uniform temperature of $T_i$ is covered with insulation ($k$ = 0.20 W/mK) to reduce heat loss to the ambient air at $T_\infty$ with $h_a$ = 15 W/m$^2$K.
*To find:* (*i*) The critical thickness $r_c$ of insulation (*ii*) the ratio of heat loss from the tube with insulation to that without insulation, (*a*) if the thickness of insulation is equal to $r_c$, (*b*) if the thickness of insulation is ($r_c$ + 2) cm.

**Solution** Critical radius of insulation $r_c = k/h_a$ or, $r_c = \dfrac{0.20\times 100}{15} = 1.33$ cm.

Critical thickness of insulation = 1.33 – 1 = 0.33 cm *Ans.*

Heat loss from the tube with insulation is

$$Q_{\text{with}} = \frac{T_1 - T_\infty}{\dfrac{1}{2\pi kL}\ln\dfrac{r_o}{r_i} + \dfrac{1}{2\pi r_o L h_a}} = \frac{2\pi r_o L h (T_1 - T_\infty)}{1 + \dfrac{r_o h_a}{k}\ln\dfrac{r_o}{r_i}}$$

Heat loss from the tube without insulation

$$Q_{\text{without}} = 2\pi r_i L h_a\,(T_1 - T_\infty)$$

$\therefore$ The ratio

$$\frac{Q_{\text{with}}}{Q_{\text{without}}} = \frac{r_o}{r_i}\left[1 + \frac{r_o h_a}{k}\ln\frac{r_o}{r_i}\right]^{-1} \qquad \text{(a)}$$

(a) If $\qquad r_o = r_c = \dfrac{k}{h_a}$

$$\frac{Q_{\text{with}}}{Q_{\text{without}}} = \frac{r_o}{r_i}\left[1 + \ln\frac{r_o}{r_i}\right]^{-1} \qquad \text{(b)}$$

$$= \frac{1.33}{1.0}[1 + \ln 1.33]^{-1} = 1.035 \quad \textit{Ans.}$$

Thus, heat loss is increased by 3.5% in spite of the fact that there is an insulation of thickness 0.33 cm.

(b) If another layer of 2 cm thickness of insulation is added, $r_o$ = 1.33 + 2 = 3.33 cm. Substituting in Eq. (a),

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$$\frac{Q_{\text{with}}}{Q_{\text{without}}} = \frac{3.33}{1.00}\left[1+\frac{0.0333\times 15}{0.20}\ln\frac{3.33}{1.0}\right]^{-1} = 0.83$$

Insulation of 2.33 cm thickness reduces the heat loss by 17%. *Ans.*

**Example 2.5** *Given*: A 1 mm dia electric wire is covered with 2 mm thick layer of insulation ($k$ = 0.5 W/mK). Air surrounding the wire is at 25°C and $h$ = 25 W/m$^2$K. The wire temperature is 100°C.
*To find:* (*i*) The rate of heat dissipation from the wire per unit length with and without insulation, (*ii*) The critical radius of insulation, (*iii*) The maximum value of heat dissipation.

**Solution** Heat transfer rate per unit length with the insulation.

$$\frac{Q}{l} = \frac{T_w - T_\infty}{\dfrac{1}{2\pi k}\ln\dfrac{r_o}{r_i} + \dfrac{1}{2\pi r_o h_a}}$$

$$= \frac{100-25}{\dfrac{\ln(2.5/1.5)}{2\pi\times 0.5} + \dfrac{1}{2\pi\times 0.025\times 10}} = 10.90 \text{ W/m} \quad \textit{Ans.}$$

Heat transfer rate without insulation

$$\frac{Q}{l} = 2\pi r_0 h_a (T_w - T_\infty)$$

$$= 2\pi \times 0.5 \times 10^{-3} \times 10(100 - 25)$$

$$= 2.36 \text{ W/m} \quad \textit{Ans.}$$

The addition of insulation increases the rate of heat transfer from the wire by a factor of 10.9/2.36 or 4.62 *Ans.* (i)

Critical radius of insulation, $r_c = k/h_a$

$$\therefore \qquad r_c = \frac{0.5}{10} = 0.05 \text{ m} = 50 \text{ mm} \quad \textit{Ans.} \text{ (ii)}$$

Peak heat flux per unit length,

$$\frac{Q}{l} = \frac{100-25}{\dfrac{1}{2\pi\times 0.5}\ln(50/0.5) + \dfrac{1}{2\pi(50\times 10^{-3})\times 10}}$$

$$= 42.07 \text{ W/m} \quad \textit{Ans.}$$

**Example 2.6** *Given*: A 1 mm dia aluminium ($k$ = 204 W/mK) wire can carry a current till wire temperature does not exceed $T_{\max}$ = 200°C when suspended in air at 25°C where $h$ = 10 W/m$^2$K. For the wire, the specific resistance is 0.037 ohm/m.
*To find:* The maximum current the wire can carry.

**Solution** Heat conducted to the outside surface of the wire = Heat convected to the air,

$$-k\frac{dT}{dr} = h(T_w - T_\infty) \tag{1}$$

where $T_w$ is the outside surface temperature of the wire. At the centre, $T$ is maximum and $\left(\dfrac{dT}{dr}\right) = 0$.

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From the conduction equation,

$$\frac{l}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{q_G}{k} = 0$$

$$\therefore \quad \frac{dT}{dr} = -\frac{q_G r}{2k}+\frac{C_1}{r} \tag{2}$$

$$T = -\frac{q_G r^2}{4k}+C_1 \ln r + C_2 \tag{3}$$

At $r = r_0$, $\frac{dT}{dr} = \infty$, which is impossible

$$\text{At} \quad r = r_o, \left(\frac{dT}{dr}\right)_{r=r_o} = \frac{-q_G r_o}{2K}+\frac{C_1}{r_o} \tag{4}$$

Also, $$q_G \pi r_o^2 L = -k2\pi r_o L\left(\frac{dT}{dr}\right)_{r=r_o}$$

$$\therefore \quad \left(\frac{dT}{dr}\right)_{r=r_o} = -\frac{q_G r_o}{2k} \tag{5}$$

From Eqs (4) and (5), $C_1 = 0$

Equation (3) reduces to

$$T = -\frac{q_G r^2}{4k}+C_2 \tag{6}$$

At $r = r_o$, $T = T_w$.

Substituting $C_2$ in Eq. (6)

$$T = -\frac{q_G r^2}{4k}+\frac{q_G r_o^2}{4k}+T_w$$

$$\therefore \quad T = \frac{q_G r_o^2}{4k}\left(1-\frac{r^2}{r_o^2}\right)+T_w \tag{7}$$

From Eq. (1), $$\left(\frac{dT}{dr}\right)_{r=r_o} = -\frac{h(T_w-T_\infty)}{k} = -\frac{q_G r_o}{2k}$$

$$\therefore \quad T_w = \frac{q_G r_o}{2k}+T_\infty$$

Substituting $T_w$ in Eq. (7), we get the temperature distribution,

$$T = \frac{q_{G r_o^2}}{4k}\left(1-\frac{r^2}{r_o^2}\right)+\frac{q_G r_o}{2h}+T_\infty \tag{8}$$

$$\therefore \quad \frac{T-T_\infty}{T_\infty} = \frac{q_G r_o}{2hT_\infty}\left(1+\frac{hr_o}{2k}+\frac{hr^2}{2r_o^2 k}\right)$$

Putting $r = 0$, $T = T_{max}$

$$\therefore \quad T_{max} = T_\infty+\frac{q_G r_o}{2h}\left(1+\frac{hr_o}{2k}\right)$$

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Now,
$$q_G = \frac{I^2 R}{V} = \frac{I^2 R}{\pi r_o^2 L}$$

$\therefore$
$$T_{\max} = T_\infty + \frac{I^2}{2\pi r_o h}\frac{R}{L}\left(1 + \frac{h r_o}{2k}\right)$$

$$200 = 25 + \frac{I^2}{2\pi(0.5\times10^{-3})\times10}\times0.037\times\left(1+\frac{10\times0.5\times10^{-3}}{2\times204}\right)$$

$\therefore$
$$I_{\max} = 12.19 \text{ amperes} \quad \textit{Ans.}$$

**Example 2.7** The air inside a chamber at $T_{\infty, i}$ = 50°C is heated convectively with $h_i$ = 20 W/m$^2$ K by a 200-mm thick wall having a thermal conductivity of 4 W/m K and a uniform heat generation of 1000 W/m$^3$. To prevent any heat, generated within the wall, from being lost to the outside of the chamber at $T_{\infty,0}$ = 25°C with $h_o$ = 5 W/m$^2$ K, a very thin electrical strip heater is placed on the outer wall to provide a uniform heat flux $q_o''$.

(a) Sketch the temperature distribution in the wall on $T$–$x$ coordinates for the condition where no heat generated within the wall is lost to the outside chamber. Identify $T_{\infty,i}$ and $T_{\infty,0}$ on the plot. (b) What are the temperatures at the wall boundaries $T_o$ and $T_L$ (at $x$ = 0 and $x$ = $L$) for the conditions of Part (a).

(c) Determine the value of $q_o''$ that must be supplied by the strip heater so that all heat generated within the wall is transferred to the inside of the chamber for the temperature $T_o$ as computed in Part (b). (d) If he heat generated in the wall were switched off while the heat flux to the strip heater remains constant, what would be the steady-state temperature $T_o$ of the outer wall surface?

**Solution**
$$\frac{d^2T}{dx^2} + \frac{q_G}{k} = 0$$

$$\frac{dT}{dx} = -\frac{q_G}{k}x + C_1 \tag{1}$$

$$T = -\frac{q_G}{2k}x^2 + C_1 x + C_2 \tag{2}$$

At $x = 0$, $T = T_o$, $dT/dx = 0$

from Eq. (1), $C_1 = 0$

$$T = -\frac{q_G}{2k}x^2 + C_2$$

At $x = L$, $T = T_L$

$$T_L = -\frac{q_G}{2k}L^2 + C_2 \quad\text{or}\quad C_2 = T_L + \frac{q_G L^2}{2k}$$

The temperature distribution in the wall is thus

$$T = \frac{q_G}{2k}(L^2 - x^2) + T_L \tag{3}$$

The temperature profile is shown in Fig. Ex. 2.7(a).

At $x = 0$, from Eq. (3),

$$T = T_o = T_{\max} = \frac{q_G}{2k}L^2 + T_L$$

$$= \frac{1000\times(0.2)^2}{2\times4} + T_L \tag{4}$$

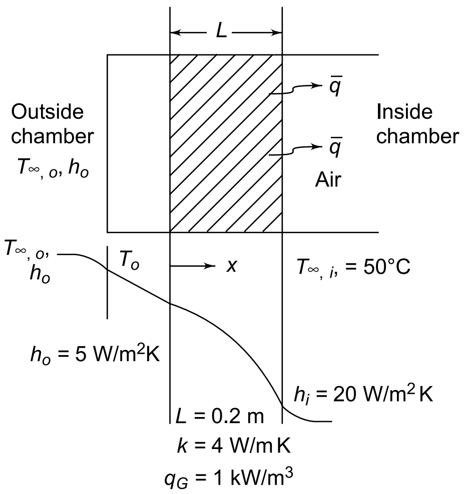


**Fig.** Ex. 2.7(a)

Again, at the inside wall surface, from Eq. (1),

$$h_i\,(T_L - T_{\infty,\,i}) = -k\left(\frac{dT}{dx}\right)_{x=L} = \frac{q_G L}{k}\,k$$

$$T_L - T_{\infty,\,i} = \frac{q_G L}{h_i} = \frac{1000 \times 0.2}{20} = 10°\text{C}$$

$$T_L = 50 + 10 = 60°\text{C} \quad \textit{Ans.}\ \text{(a)}$$

From Eq. (4), $T_o = T_{max} = 5 + 60 = 65°\text{C}$ *Ans.* (b)

(c) $q_o = h_o\,(T_o - T_{\infty,\,o}) = 5(65 - 25)$

$= 200\ \text{W/m}^2$ *Ans.* (c)

The heat flow path is shown in Fig. Ex. 2.7(b).

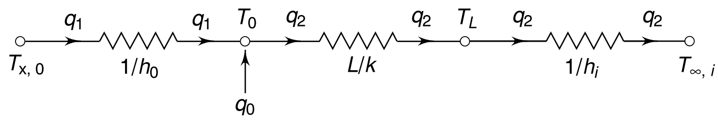


**Fig.** Ex. 2.7(b)

Heat transferred by the strip heater

$$q_o = q_1 + q_2 = 200\ \text{W/m}^2 = \frac{T_o - T_{\infty,\,o}}{1/h_o} + \frac{T_o - T_{\infty,\,i}}{(L/k + 1/h_i)}$$

$$200 = \frac{T_o - 25}{1/5} + \frac{T_o - 50}{(0.2/4 + 1/20)}$$

$$= 5T_o - 125 + \frac{20T_o - 1000}{1+1} = 15T_o - 625$$

or $T_o = 55°\text{C}$ *Ans.* (d)

**Example 2.8** In a cylindrical fuel rod of a nuclear reactor heat is generated internally according to the equation

$$q_G = q_o\left[1 - \left(\frac{r}{r_o}\right)^2\right]$$

where $q_G$ is the local rate of heat generation per unit volume at radius $r$, $r_o$ is the outside radius, and $q_o$ is the rate of heat generation per unit volume at the centre line. Calculate the temperature drop from the centre line to the surface for a 2.5 cm outer diameter rod having $k$ = 25 W/m K, if the rate of heat removal from the surface is 1650 kW/m$^2$.

**Solution** In cylindrical coordinates, the radial variation of temperature at steady state when there is heat generation is given by (Fig. Ex. 2.8)

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = -\frac{q_G}{k}$$

or
$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = -\frac{q_G r}{k} = -q_o\left(1 - \frac{r^2}{r_o^2}\right)\frac{r}{k}$$

or
$$r\frac{dT}{dr} = -\frac{q_o}{k}\left(\frac{r^2}{2} - \frac{r^4}{4r_o^2}\right) + C_1$$

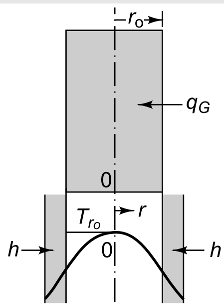


**Fig.** Ex. 2.8

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or
$$k\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{q_o r}{2}\left(1-\frac{r^2}{2r_o^2}\right)+\frac{C_1}{r} \quad (1)$$

At $r = 0, \left(k\frac{\mathrm{d}T}{\mathrm{d}r}\right)$ cannot be infinite, $\therefore C_1 = 0$

$$k\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{q_o r}{2}\left(1-\frac{r^2}{2r_o^2}\right)$$

Heat transferred from the rod

$$q_F = -k\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_{r=r_o} = \frac{q_o r_o}{2}\left(1-\frac{1}{2}\right) = \frac{q_o r_o}{4}$$

Given: $q_F = 1650 \text{ kW/m}^2$, $r_o = 1.25$ cm

$$1650 = \frac{q_o \times 1.25\times 10^{-2}}{4}$$

$$q_o = \frac{1650\times 100\times 16}{5} = 528\times 10^3 \text{ kW/m}^3$$

This is the volumetric heat generation at the centre line of the rod. Now,

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{q_o r}{2k}\left(1-\frac{r^2}{2r_o^2}\right)$$

$$T = -\frac{q_o}{2k}\left(\frac{r^2}{2}-\frac{r^4}{8r_o^2}\right)+C_2$$

At, $r = 0$, $T = T_c$, the centre line temperature.

$$C_2 = T_c$$

$$T = -\frac{q_o}{2k}\left(\frac{r^2}{2}-\frac{r^4}{8r_o^2}\right)+T_c$$

At, $r = r_o$, the temperature drop,

$$T_c - T_{r_o} = \frac{q_o r_o^2}{2k}\left(\frac{1}{2}-\frac{1}{8}\right) = \frac{3}{16}\frac{q_o r_o^2}{k}$$

$$= \frac{3}{16}\times\frac{528\times 10^6(\text{W/m}^3)\times(1.25)^2\times 10^{-4}\text{ m}^2}{25 \text{ W/mK}}$$

$$= 618.7°\text{C} \quad \textit{Ans.}$$

**Example 2.9** *Given:* Three 10 mm dia rods *A*, *B* and *C* protrude from a steam bath at 100°C to a length of 25 cm into an atmosphere at 20°C. The temperatures at the other ends are $T_A$ = 26.76°C, $T_B$ = 32°C and $T_C$ = 36.93°C. *To find:* The thermal conductivities of the rods *A*, *B* and *C*, if $h$ = 23 W/m²K in each case.

**Solution** Neglecting the heat loss from the tips, since the rods are thin and long,

$$A/P = \frac{\pi}{4}d^2/\pi d = \frac{d}{4} = \frac{10\text{ mm}}{4} = 2.5 \text{ mm.}$$

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As recommended by Kreith and Bohn [2], the corrected length $L_c = L + A/P = 250 + 2.5 = 250$ mm. The temperature difference at the tip is given by Eq. (2.96),

$$\theta_l = T_l - T_\infty = \frac{\theta_0}{\cosh ml} = \frac{T_o - T_\infty}{\cosh ml}$$

For rod $A$,
$$26.76 - 20 = \frac{100-20}{\cosh ml}$$

$$\therefore \quad \cosh ml = \frac{80}{6.76} = 11.8$$

$$ml = 3.16$$

$$m = \frac{3.16}{0.25} = 12.64 \text{ m}^{-1} = \left(\frac{hp}{kA}\right)^{1/2}$$

$$\frac{h \times \pi d}{k \frac{\pi}{4} d^2} = (12.64)^2$$

$$\therefore \quad k_A = \frac{23 \times 4}{12.64 \times 12.64 \times 0.01} = 57.7 \text{ W/mK} \quad \textit{Ans.}$$

For rod $B$,

$$32 - 20 = \frac{100-20}{\cosh ml}$$

$$\cosh ml = \frac{80}{12} = 6.67$$

$$ml = 2.6$$

$$m = 10.4 = \left(\frac{hP}{kA}\right)^{1/2}$$

$$k_B = \frac{23 \times 4}{(10.4)^2 \times 0.04} = 88.6 \text{ W/mK} \quad \textit{Ans.}$$

For rod $C$,

$$36.93 - 20 = \frac{100-20}{\cosh ml}$$

$$\cosh ml = \frac{80}{16.93} = 4.73$$

$$ml = 2.23$$

$$m = 8.92 = \left(\frac{hP}{kA}\right)^{1/2}$$

$$k_c = \frac{23 \times 4}{(8.92)^2 \times 0.01} = 115.6 \text{ W/m}^2\text{K} \quad \textit{Ans.}$$

**Example 2.10** A 50 mm × 50 mm iron bar 0.4 m long is connected to the walls of two heated reservoirs, each at 120°C. The ambient air temperature is 35°C and the convective heat transfer coefficient is 17.4 W/m$^2$ K. Calculate the rate of heat loss from the bar and the temperature of the bar midway between the reservoirs. The thermal conductivity of iron is 52 W/mK.

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**Solution** Because of symmetry, we would consider half-length of the bar (Fig. Ex. 2.10).

$P = 2(a + b) = 2(50 + 50) = 200 \text{ mm} = 0.2 \text{ m}$

$A = 50 \text{ mm} \times 50 \text{ mm} = 2500 \text{ mm}^2 \times 10^{-6}$

$= 25 \times 10^{-4} \text{ m}^2$

Here, $m = \left(\frac{hP}{kA}\right)^{1/2} = \left(\frac{17.4 \times 0.2}{52 \times 25 \times 10^{-4}}\right)^{1/2} = 5.174 \text{ m}^{-1}$

$ml = 5.174 \times 0.2 = 1.0348$

$\tanh ml = 0.776$

$$\frac{Q_o}{2} = mkA\theta_o \tanh ml$$

$$= 5.174 \times 52 \times 0.0025 \times (120 - 35) \times 0.776$$

$$= 44.37 \text{ W}$$

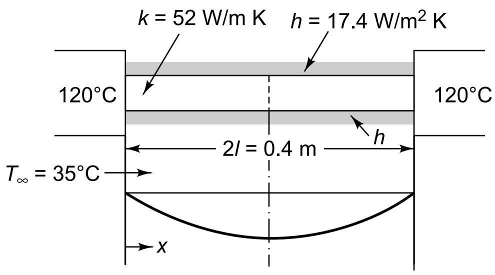


Fig. Ex. 2.10

Rate of heat loss from the bar is

$$Q_o = 88.74 \text{ W} \quad \textit{Ans.}$$

$$\theta_1 = \frac{\theta_o}{\cosh ml} = \frac{85}{1.58} = 53.8 = T_1 - T_\infty$$

Mid-temperature is 88.8°C. *Ans.*

**Example 2.11** Fins, 12 in number, having $k = 75$ W/m K and 0.75 mm thickness protrude 25 mm from a cylindrical surface of 50 mm diameter and 1 m length placed in an atmosphere of 40°C. If the cylindrical surface is maintained at 150°C and the heat transfer coefficient is 23 W/m$^2$ K, calculate (a) the rate of heat transfer, (b) the percentage increase in heat transfer due to fins, (c) the temperature at the centre of fins and (d) the fin efficiency and the fin effectiveness.

**Solution** The perimeter (Fig. Ex. 2.11) of one fin

$$P = 2(L + b) \cong 2 \text{ m}$$

$$A = bL = 0.75 \times 10^{-3} \times 1$$

$$= 0.00075 \text{ m}^2$$

$$m = \left(\frac{hP}{kA}\right)^{1/2} = \left(\frac{23 \times 2}{0.00075 \times 75}\right)^{1/2} = 28.6 \text{ m}^{-1}$$

$$ml = 28.6 \times 0.025 = 0.715$$

$$\tanh ml = \tanh 0.715 = 0.61$$

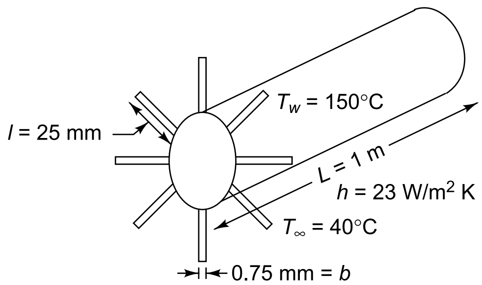


Fig. Ex. 2.11

Heat transfer from one fin,

$$Q_o = mkA\theta_o \tanh ml$$

$$= 28.6 \times 75 \times 0.00075 \times (150 - 40) \times 0.61$$

$$= 108 \text{ W}$$

Heat transfer from 12 fins

$$Q = 108 \times 12 = 1296 \text{ W}$$

Heat transfer from unfinned portion of the surface

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$$Q' = hA\theta_o$$
$$= 23[(\pi \times 0.05 \times l) - (12 \times 0.00075 \times l)] \times (150 - 40)$$
$$= 23 \times 0.148 \times 110 = 374.44 \text{ W}$$

Total heat transfer from the cylindrical surface

$$Q_t = 1296 + 374.4 = 1670.4 \text{ W} \quad \textit{Ans.} \text{ (a)}$$

If the cylinder were without fins, the heat transfer would have been

$$Q = 23 \times (\pi \times 0.05 \times l) \times 110$$
$$= 397.4 \text{ W}$$

Increase in heat transfer due to fins

$$= \frac{1670.4 - 397.4}{397.4} \times 100 = 320.3\% \quad \textit{Ans.} \text{ (b)}$$

$$\frac{\theta}{\theta_o} = \frac{\cosh m(l - x)}{\cosh ml}$$

$$\frac{T_c - 40}{150 - 40} = \frac{\cosh(28.6 \times 0.0125)}{\cosh 0.715} = \frac{1.062}{1.257} = 0.845$$

Centre temperature of fins, $T_c = 133°\text{C}$ *Ans.* (c)

If tip loss is neglected

$$\eta_{\text{fin}} = \frac{\tanh ml}{ml} = \frac{0.61}{0.715} = 0.853 \text{ or, } 85.3\% \quad \textit{Ans.}$$

Fin effectiveness $\varepsilon = \dfrac{1670.4}{397.4} = 4.2$ *Ans.* (d)

**Example 2.12** A stainless steel fin ($k$ = 20 W/m K) having a diameter of 20 mm and a length of 0.1 m is attached to a wall at 300°C. The ambient temperature is 50°C and the heat transfer coefficient is 10 W/m$^2$ K. The fin tip is insulated. Determine (a) the rate of heat dissipation from the fin, (b) the temperature at the fin tip, (c) the rate of heat transfer from the wall area covered by the fin if the fin was not used and (d) the heat transfer rate from the same fin geometry if the stainless steel fin is replaced by a fictitious fin with infinite thermal conductivity.

**Solution**

(a) The heat transfer rate from the fin

$$Q_o = (hPkA)^{1/2}\, \theta_o \tanh ml$$

where $$m = \left(\frac{hP}{kA}\right)^{1/2} = \left(\frac{10 \times \pi \times 0.02}{20 \times \pi \times (0.01)^2}\right)^{1/2} = 10 \text{ m}^{-1}$$

$$ml = 10 \times 0.1 = 1$$

$$Q_o = [10 \times \pi \times 0.02 \times 20 \times \pi \times (0.01)^2]^{1/2} (300 - 50) \tanh(1)$$
$$= 0.06283 \times 250 \times \tanh(1) = 11.96 \text{ W} \quad \textit{Ans.} \text{ (a)}$$

(b) The fin tip temperature is given by

$$\theta_1 = \frac{\theta_o}{\cosh ml} = \frac{300 - 50}{\cosh(1)} = \frac{250}{1.543} = 162°\text{C}$$

$$T_1 = 162 + 50 = 212°\text{C} \quad \textit{Ans.} \text{ (b)}$$

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(c) $$Q = hA(T_o - T_\infty) = 10 \times \pi \times (0.01)^2\,(300 - 50)$$
$$= 0.785 \text{ W} \quad \textit{Ans. } (c)$$

The presence of fin increases heat dissipation 11.96/0.785 or 15.24 times

(d) $$Q_{\text{ideal}} = h\,A_s(T_o - T_\infty)$$
$$= 10\pi\,(0.02)\,(0.1)\,(300 - 50)$$
$$= 15.71 \text{ W}$$

The stainless steel fin dissipates

$$\frac{15.71 - 11.96}{15.71} = 0.24$$

or 24% less heat than the ideal fin of infinite thermal conductivity.

**Example 2.13** One end of a long rod is inserted into a furnace while the other projects into ambient air. Under steady state the temperature of the rod is measured at two points 75 mm apart and found to be 125°C and 88.5°C, respectively, while the ambient temperature is 20°C. If the rod is 25 mm in diameter and *h* is 23.36 W/m$^2$ K, find the thermal conductivity of the rod material.

**Solution** The temperature distribution is given by

$$\theta = C_1\, e^{-mx} + C_2\, e^{mx}$$

where $$m = \left(\frac{hP}{kA}\right)^{1/2}$$

At $x = 0$, $\theta = \theta_o$, and at $x = \infty$, $\theta = 0$

$\therefore$ $$C_2 = 0$$
$$\theta = C_1\, e^{-mx}$$

Let *l* be the distance between the two points where the temperatures are measured. Then,

$$\theta_1 = \theta_o\, e^{-mx_1}$$

and $$\theta_2 = \theta_o\, e^{-m(x_1+1)}$$

$$\frac{\theta_1}{\theta_2} = \frac{e^{-mx_1}}{e^{-mx_1}\,e^{-ml}} = e^{ml}$$

or $$\frac{125-20}{88.5-20} = \frac{105}{68.5} = 1.533 = e^{ml}$$

$$ml = 0.427$$

or $$m = \frac{0.427}{0.075} = 5.696 = \left(\frac{hP}{kA}\right)^{1/2} = \left(\frac{h\pi d}{k\dfrac{\pi}{4}d^2}\right)^{1/2}$$

$$= 2\left(\frac{h}{kd}\right)^{1/2}$$

$$m^2 = 4 \times \frac{h}{kd} = 32.44$$

$$k = \frac{4 \times 23.36}{0.025 \times 32.44} = 115.2 \text{ W/mK} \quad \textit{Ans.}$$

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**Example 2.14** An electric current of 34,000 A flows along a flat steel plate 12.5 mm thick and 100 mm wide. The temperature of one surface of the plate is 82°C and that of the other is 95°C. Find the temperature distribution across the plate, and the value and position of the maximum temperature. Also calculate the total amount of heat generated per metre length of the plate and the flow of heat from each surface of the plate. The end effects along the short sides of the plate may be neglected. Given for steel: $\rho = 12 \times 10^{-6}\ \Omega$ cm and $k = 52.4$ W/m K.

**Solution** The temperature distribution in the plate (Fig. Ex. 2.14) is obtained from

$$\frac{d^2T}{dx^2} + \frac{q_G}{k} = 0$$

$$\frac{dT}{dx} = -\frac{q_G x}{k} + C_1 \qquad (1)$$

$$T = -\frac{q_G x^2}{2k} + C_1 x + C_2 \qquad (2)$$

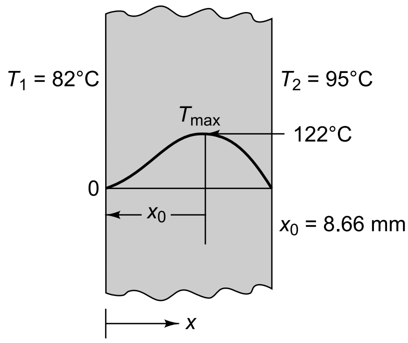


**Fig.** Ex. 2.14

Current density $\quad i = \frac{I}{A} = \frac{34,000}{(12.5 \times 100) \times 10^{-6}}$

$$= 2.72 \times 10^7 \text{ A/m}^2$$

$$q_G = i^2 \rho = (2.72)^2 \times 10^{14} \times 12 \times 10^{-8}$$

$$= 88.78 \times 10^6 \text{ W/m}^3$$

From Eq. (2), at $\quad x = 0, T = 82°C$

$$C_2 = 82°C$$

At $\quad x = 12.5 \times 10^{-3}$ m, $T = 95°C$

From Eq. (1),

$$95 = -\frac{88.78 \times 10^6 \times (12.5)^2 \times 10^{-6}}{2 \times 52.4} + C_1 (12.5 \times 10^{-3}) + 82$$

$$= -50.37 + C_1 (12.5 \times 10^{-3})$$

$$C_1 = 11629.2 \text{ (K/m)}$$

$$T = -\frac{88.78 \times 10^6}{2 \times 52.4} x^2 + 11629.2x + 82$$

$$= -0.847 \times 10^6 x^2 + 11629.2x + 82 \quad \textit{Ans.}$$

It is the temperature distribution across the plate. For the maximum temperature,

$$\frac{dT}{dx} = -1.694 \times 10^6 x + 11629.2 = 0$$

$$x = \frac{11629.2}{1.694 \times 10^6} = 0.0686 \text{ m} = 6.86 \text{ mm} \quad \textit{Ans.}$$

$$T_{max} = -0.847 \times 10^6 \times (0.00686)^2 + 11629.2 \times 0.00686 + 82$$

$$= -39.86 + 79.78 + 82 = 121.92°C \quad \textit{Ans.}$$

$Q_1$ = Heat transfer from the left side

$$= -kA \left(\frac{dT}{dx}\right)_{x=0}$$

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$$Q_1/A = 52.4 \times 11629.2 = 609370 \text{ W/m}^2 = 609.37 \text{ kW/m}^2 \quad \textit{Ans.}$$

$$Q_r/A = -k\left(\frac{dT}{dx}\right)_{x=1} = 88.78 \times 10^6 \times 12.5 \times 10^{-3} + 11629.2 \times 52.4$$

$$Q_r/A = -1109.75 \times 10^3 + 609370$$

$$= -500{,}380 \text{ W/m}^2 = -500.38 \text{ kW/m}^2 \quad \textit{Ans.}$$

**Example 2.15** Two large steel plates at temperatures of 120°C and 80°C are separated by a steel rod 300 mm long and 25 mm in diameter. The rod is welded to each plate. The space between the plates is filled with insulation, which also insulates the circumference of the rod. Because of a voltage difference between the two plates, current flows through the rod, dissipating electrical energy at a rate of 150 W. Find out the maximum temperature in the rod and the heat flux. Take $k$ for the rod as 47 W/m K.

**Solution**

$$q_G = \frac{150\ W}{(\pi/4) \times (0.025)^2 \times 0.3} = 1018.59 \text{ kW/m}^3$$

The temperature distribution is

$$\frac{dT}{dx} = -\frac{q_G}{k}x + C_1$$

$$T = -\frac{q_G}{2k}x^2 + C_1 x + C_2$$

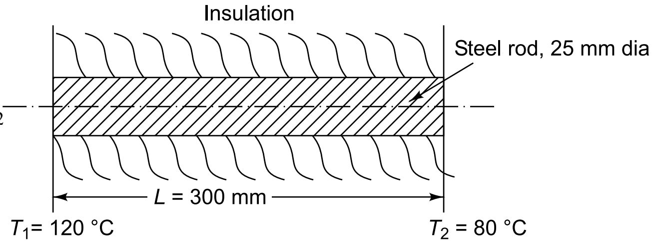


**Fig.** Ex. 2.15

With reference to Fig. Ex. 2.15,

At $x = 0, T = T_1$

At $x = l, T = T_2$

$$C_2 = T_1 = 120°\text{C}$$

$$80 = -\frac{1018590}{2 \times 47} \times (0.3)^2 + C_1 \times 0.3 + 120$$

$$C_1 = 3117.48 \text{ K/m}$$

$$T = -10836.06\, x^2 + 3117.48\, x + 120$$

For maximum temperature,

$$\frac{dT}{dx} = -21672.12x + 3117.48 = 0$$

$$x = 0.1438 \text{ m or, } 14.38 \text{ cm} \quad \textit{Ans.}$$

$$T_{max} = -224.07 + 448.29 + 120 = 344.22°\text{C} \quad \textit{Ans.}$$

$$(Q_1)_{x=0} = -47 \times \frac{\pi}{4} \times (0.025)^2 \times 3117.48$$

$$= -71.92 \text{ W} \quad \textit{Ans.}$$

$$(Q_2)_{x=1} = -47 \times \frac{\pi}{4} \times (0.025)^2 \times (-21672.12 \times 0.3 + 3117.48)$$

$$= 78.08 \text{ W} \quad \textit{Ans.}$$

$$Q_{total} = 78.08 + 71.92 = 150 \text{ W}$$

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**Example 2.16** A bar of square cross-section connects two metallic structures. One structure is maintained at a temperature 200°C and the other is maintained at 50°C. The bar, 20 mm × 20 mm, is 100 mm long and is made of mild steel ($k$ = 0.06 kW/m K). The surroundings are at 20°C and the heat transfer coefficient between the bar and the surroundings is 0.01 kW/m$^2$ K.

Derive an equation for the temperature distribution along the bar and hence calculate the total heat flow rate from the bar to the surroundings.

**Solution** From Fig. Ex. 2.16,

$$Q_1 = -kA\frac{dT}{dx}$$

$$Q_2 = Q_1 + \frac{dQ_1}{dx}\,dx$$

$$Q_1 - Q_2 = -\frac{d}{dx}\left(-kA\frac{dT}{dx}\right)dx = kA\frac{d^2T}{dx^2}\,dx$$

$$= hP\,dx\,(T - T_\infty)$$

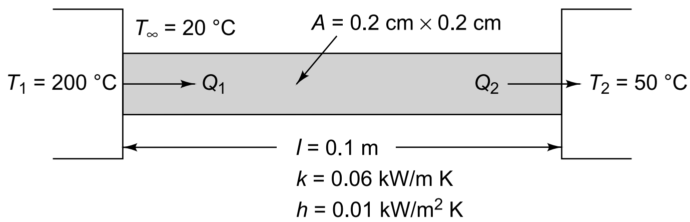


**Fig.** Ex. 2.16

Letting $\theta = T - T_\infty$,

$$\frac{d^2\theta}{dx^2} - \frac{hP}{kA}\theta = 0$$

or $(D^2 - m^2)\,\theta = 0$, where $m = \left(\frac{hP}{kA}\right)^{1/2}$

The general solution is

$$\theta = C_1 e^{mx} + C_2 e^{-mx} \tag{1}$$

At $x = 0$, $\theta = \theta_1 = C_1 + C_2 = 180°C$ (2)

At $x = 1$, $\theta = \theta_2 = C_1 e^{ml} + C_2 e^{-ml} = 30°C$ (3)

$$m = \left(\frac{hP}{kA}\right)^{1/2} = \left(\frac{0.01\times 4\times 0.02}{0.06\times 0.02\times 0.02}\right)^{1/2} = \left(\frac{4}{0.12}\right)^{1/2}$$

$$ml = 0.577,\ e^{ml} = 1.78, \qquad e^{-ml} = 0.561 = 5.77\ \text{m}^{-1}$$

$$30 = C_1 \times 1.78 + C_2 \times 0.561$$

$$3.173\,C_1 + C_2 = 53.476$$

From Eq. (2) $\quad C_1 + C_2 = 180.00$

On subtraction, $\quad 2.173\,C_1 = -126.524$

$$C_1 = -58.22$$

From Eq. (2), $C_2 = 238.22$

The temperature distribution is

$$\theta = -58.22\, e^{5.77x} + 238.22\, e^{-5.77x}$$

$$Q_1 = -kA\left(\frac{d\theta}{dx}\right)_{x=0} = -kA\,(-58.22 \times 5.77\, e^{5.77x} - 238.22 \times 5.77\, e^{-5.77x})_{x=0}$$

$$= 0.06 \times 4 \times 10^{-4}\,(335.93 + 1374.53)$$

$$= 0.0410 \text{ kW} = 41.0 \text{ W}$$

$$Q_2 = -kA\left(\frac{d\theta}{dx}\right)_{x=1} = -kA\,(-58.22 \times 5.77 \times 1.78 - 238.22 \times 5.77 \times 0.561)$$

$$= 0.06 \times 4 \times 10^{-4}\,(598 + 771.5)$$

$$= 0.0328 \text{ kW} = 32.8 \text{ W}$$

Heat flow rate from the bar to the surroundings

$$= Q_1 - Q_2 = 41.0 - 32.8 = 8.2 \text{ W} \quad \textit{Ans.}$$

**Example 2.17** The cooling system of an electronic package has to dissipate 0.153 kW from the surface of an aluminium plate 100 mm × 150 mm. It is proposed to use eight fins, each 150 mm long and 1 mm thick. The temperature difference between the plate and the surroundings is 50 K, the thermal conductivity of plate and fins is 0.15 kW/m K and the heat transfer coefficient is 0.04 kW/m² K. Calculate the height of fins required.

**Solution**

$$\theta_1 = T_1 - T_\infty = 50 \text{ K}$$

Unfinned surface area (Fig. Ex. 2.17)

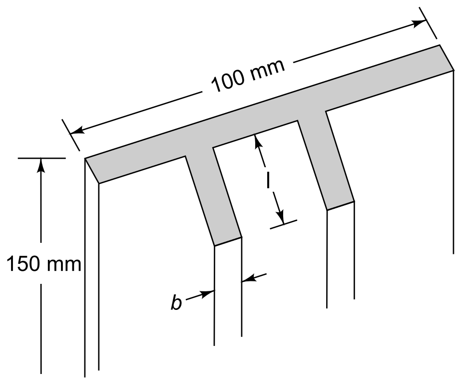


**Fig.** Ex. 2.17

$$A_{uf} = (0.1 \times 0.15) - (8 \times 1 \times 10^{-3} \times 0.15)$$

$$= 0.0138 \text{ m}^2$$

$$(Q_1)_{uf} = 0.04 \times 0.0138 \times 50 = 0.0276 \text{ kW}$$

$$Q_{total} = 0.153 \text{ kW (given)}$$

Heat transfer from finned surface

$$Q_f = 0.153 - 0.0276 = 0.1254 \text{ kW}$$

Heat transfer from one fin $= \dfrac{0.1254}{8} = 0.015675$ kW

$$m = \left(\frac{hP}{kA}\right)^{1/2} = \left(\frac{h2L}{kbL}\right)^{1/2} = \left(\frac{2h}{kb}\right)^{1/2}$$

$$= \left(\frac{2 \times 0.04}{0.15 \times 0.001}\right)^{1/2} = 23.094 \text{ m}^{-1}$$

$$A = 0.001 \times 0.15 = 0.00015 \text{ m}^2$$

$$Q_o = 0.015675 = mkA\theta \tanh ml$$

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$$= 23.094 \times 0.15 \times 0.00015 \times 50 \times \tanh ml$$

$$\therefore \quad \tanh ml = 0.6033$$

$$ml = 0.6981$$

$$l = \text{height of fins} = \frac{0.6981}{23.094} \times 10^3 = 30.2 \text{ mm} \quad Ans.$$

**Example 2.18** Determine the heat transfer rate from the rectangular fin of length 20 cm, width 40 cm and thickness 2 cm. The tip of the fin is not insulated and the fin has a thermal conductivity of 150 W/m K. The base temperature is 100°C and the fluid is at 20°C. The heat transfer coefficient between the fin and the fluid is 30 W/m$^2$ K.

**Solution** The extended length

$$L_C = l + \frac{A}{P} = 20 + \frac{40 \times 2}{84} = 20.95 \text{ cm} = 0.21 \text{ m}$$

$$A = 40 \times 2 = 80 \text{ cm}^2 = 0.008 \text{ m}^2$$

$$A_S = L_C P = 0.21 \times 0.84 = 0.1764 \text{ m}^2$$

$$m = \left(\frac{hP}{kA}\right)^{1/2} = \left(\frac{30 \times 0.84}{150 \times 0.008}\right)^{1/2} = 4.583 \text{ m}^{-1}$$

$$Q_o = mkA\theta_o \tanh ml$$

$$= 4.583 \times 150 \times 0.008 \times 80 \times \tanh (4.583 \times 0.21)$$

$$= 328.0 \text{ W} \quad Ans.$$

Also,
$$\eta_{\text{fin}} = \frac{\tanh ml}{ml} = \frac{\tanh (4.583 \times 0.21)}{(4.583 \times 0.21)} = 0.775$$

$$Q_o = \eta_{\text{fin}} A_s h_C (T_o - T_\infty)$$

$$= 0.775 \times 30 \times 0.1764 (100 - 20) = 328 \text{ W} \quad Ans.$$

**Example 2.19** The inside dimensions of a furnace are 3 m × 2.5 m × 2 m. The walls are 20 cm thick and have $k$ = 1.2 W/m K. If the inner and outer surface temperatures are 220°C and 80°C, calculate the rate of heat loss.

**Solution** Since all inside dimensions are greater than one-fifth wall thickness $\left(\frac{1}{5} \times 20 = 4 \text{ cm}\right)$, case 1 applies. Hence, from Eq. (2.161)

$$A_m = A_1 + 0.54 \, \Delta x \, \Sigma y + 1.2 \, (\Delta x)^2$$

where the inside surface area

$$A_1 = 2 \, (3 \times 2.5 + 2.5 \times 2 + 3 \times 2) = 37 \text{ m}^2$$

$$\Sigma y = 4(3 + 2.5 + 2) = 30 \text{ m}$$

$$\Delta x = 0.2 \text{ m}$$

$$A_m = 37 + 0.54 \times 0.2 \times 30 + 1.2 \, (0.2)^2 = 40.288 \text{ m}^2$$

$$Q = \frac{kA_m (T_1 - T_2)}{\Delta x} = \frac{1.2 \times 40.288 \, (220 - 80)}{0.2}$$

$$= 33842 \text{ W} = 33.84 \text{ kW} \quad Ans.$$

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**Example 2.20** The inside and outside temperatures of a hollow 75 cm × 75 cm rectangular duct are 500°C and 100°C respectively. It is 30 cm thick. Find the rate of heat loss per unit length of the duct. Take $k$ = 20 W/mK.

**Solution** Only one corner of the duct (Fig. Ex. 2.20) is considered. It is divided into a number of squares 15 cm × 15 cm. $A$, $B$, $C$, $D$ and $E$ are nodes. To start with, let $T_A = T_B = T_C = T_D = (500 + 100)/2 =$ 300°C. The residuals at the nodes are then found out. The largest residual is relaxed first. The nine steps of reducing the residuals at the four nodes $A$, $B$, $C$ and $D$ are given in the following Table. The steady-state temperatures are $T_A$ = 184°C, $T_B$ = 269°C, $T_C$ = 292°C, $T_D$ = 298°C and $T_E = T_B$ = 269°C (by symmetry).

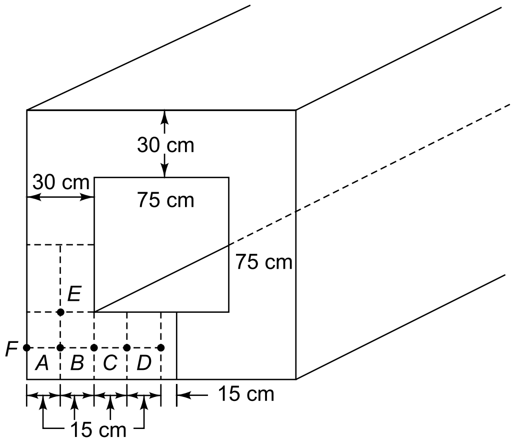


**Fig.** Ex. 2.20

Heat loss from the duct

$$Q = 8\,(Q_B + Q_C + Q_D) + 4Q_A$$

$$= 8\left[k\frac{0.15\times 1}{0.15}(269 - 100) + k\,(292 - 100) + k\,(293 - 100)\right] + 4k\,(184 - 100)$$

$$= 8k\,(169 + 192 + 198) + 4k\,84$$

$$= 4808\,k = 4808 \times 20$$

$$= 96160 \text{ W} = 96.16 \text{ kW/m} \quad \textit{Ans.}$$

| *Steps* | *A* | | *B* | | *C* | | *D* | |
|---|---|---|---|---|---|---|---|---|
| | $q^*$ | *T(°C)* | $q^*$ | *T(°C)* | $q^*$ | *T(°C)* | $q^*$ | *T(°C)* |
| 1 | – 400 | 300 | 0 | 300 | 0 | 300 | 0 | 300 |
| 2 | 0 | 200 | –100 | 300 | 0 | 300 | 0 | 300 |
| 3 | – 50 | 200 | 0 | 275 | –25 | 300 | 0 | 300 |
| 4 | 30 | 180 | –20 | 275 | –25 | 300 | 0 | 300 |
| 5 | 30 | 180 | –28 | 275 | 7 | 292 | –8 | 300 |
| 6 | 16 | 180 | 0 | 268 | 0 | 292 | –8 | 300 |
| 7 | 0 | 184 | 4 | 268 | 0 | 292 | –8 | 300 |
| 8 | 0 | 184 | 4 | 268 | –2 | 292 | 0 | 298 |
| 9 | 2 | 184 | 0 | 269 | – 1 | 292 | 0 | 298 |

**Example 2.21** An oil-filled thermometer well made of a steel tube ($k$ = 55.8 W/m K), 120 mm long and 1.5 mm thick is installed in a tube through which air is flowing. The temperature of the air stream is measured with the help of a thermometer placed in the well. The surface heat transfer coefficient from the air to the well is 23.3 W/m$^2$ K and the temperature recorded by the thermometer is 88°C. Estimate the measurement error and the percentage error if the temperature at the base of the well is 40°C.

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**Solution** When the heat loss from the tip is neglected,

$$\frac{\theta_l}{\theta_o} = \frac{1}{\cosh ml}$$

where $$m = \left(\frac{hP}{kA}\right)^{1/2}$$

From Fig. Ex. 2.21, $P = \pi d$ and $A = \pi d\delta$

$$m = \left(\frac{h}{k\delta}\right)^{1/2} = \left(\frac{23.3}{55.8 \times 1.5 \times 10^{-3}}\right)^{1/2}$$

$$= 16.68 \text{ m}^{-1}$$

$$ml = 16.68 \times 0.12 = 2.0$$

$$\cosh 2 = 3.76, \ T_1 = 88°\text{C}$$

$$\frac{T_1 - T_\infty}{T_0 - T_\infty} = \frac{1}{3.76}, \ T_0 = 40°\text{C}$$

$$3.76 \times 88 - 3.76\ T_\infty = 40 - T_\infty$$

$$T_\infty = 105.4°\text{C}$$

Measurement error = 105.4°C – 88°C = 17.4°C *Ans.*

$$\% \text{ error} = \frac{17.4}{105.4} \times 100 = 16.5\% \quad \textit{Ans.}$$

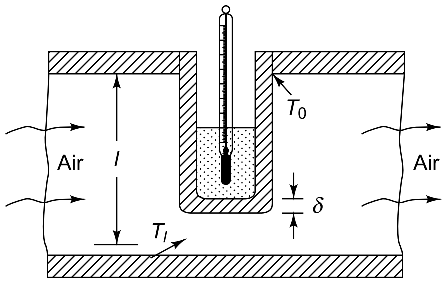


**Fig.** Ex. 2.21

**Example 2.22** *Given:* An aluminium alloy conductor of cross-section 6.25 cm$^2$ and length 2.5 cm has three of its sides insulated (Fig. Ex. 2.22) and carries a current of 800 amp. The exposed surface is in contact with air at 30°C and $h$ = 16.28 W/m$^2$K. $\rho_{Al}$ = 5 × 10$^{-4}$ Ω cm$^2$/m and $k$ = 201.5 W/mK.
*To find:* The maximum temperature of the conductor at steady state.

**Solution** Heat conduction with uniform heat generation $q_G$ (W/m$^3$) is given by

$$\frac{d^2T}{dx^2} + \frac{q_G}{k} = 0$$

$$\frac{dT}{dx} = -\frac{q_G x}{k} + C_1$$

$$\therefore \quad T(x) = -\frac{q_G x^2}{2k} + C_1 x + C_2$$

At $x = 0$, $$\frac{dT}{dx} = 0 \quad \therefore\ C_1 = 0$$

$$\therefore \quad T(x) = -\frac{q_G x^2}{2k} + C_2$$

At $x = l$, $T = T_1$

$$T_1 = -\frac{q_G l^2}{2k} + C_2$$

$$\therefore \quad C_2 = T_1 + \frac{q_G l^2}{2k}$$

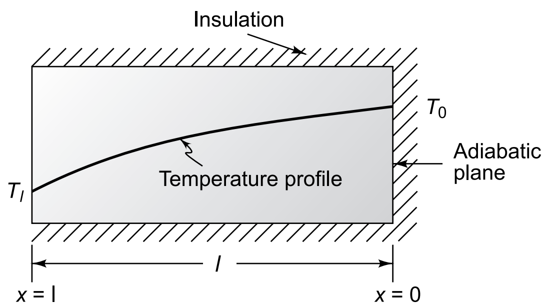


**Fig.** Ex. 2.22

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$$\therefore \qquad T(x) = -\frac{q_G x^2}{2k} + \frac{q_G l^2}{2k} + T_1$$

$$= \frac{q_G}{2k}(l^2 - x^2) + T_1 \qquad (1)$$

Also,

$$q_k = -k\left(\frac{dT}{dx}\right)_{x=l} = h(T_1 - T_\infty)$$

$$\left(\frac{dT}{dx}\right)_{x=l} = \frac{h(T_1 - T_\infty)}{k} = -\frac{q_G l}{k}$$

$$h(T_1 - T_\infty) = q_G l$$

$$T_1 = \frac{q_G l}{h} + T_\infty \qquad (2)$$

From Eqs (1) and (2),

$$T(x) = \frac{q_G}{2k}(l^2 - x^2) + \frac{q_G l}{h} + T_\infty$$

At $x = 0,\ T = T_{max}$

$$\therefore \qquad T_{max} = \frac{q_G}{2k} l^2 + \frac{q_G l}{h} + T_\infty$$

Here, $A = 6.25\ \text{cm}^2,\ l = 2.5\ \text{cm}$

$$i = \frac{I}{A} = \frac{800\ \text{A}}{6.25\ \text{cm}^2} = 128\ \text{A/cm}^2$$

$$q_G = \frac{I^2 R}{V} = \frac{I^2 Pl}{A \times l \times A} = i^2 \rho$$

$$= (128)^2 \frac{A^2}{\text{cm}^4} \times 5 \times 10^{-4} \frac{\Omega\ \text{cm}^2}{\text{m}} \times \frac{10^4\ \text{cm}^2}{1\ \text{m}^2}$$

$$= 81.92\ \text{kW/m}^2$$

$$\therefore \qquad T_{max} = q_G l\left(\frac{l}{2k} + \frac{1}{h}\right) + T_\infty$$

$$= 81.92 \times 2.5 \times 10^{-2}\left[\frac{2.5 \times 10^{-2}}{0.203 \times 2} + \frac{1}{0.01628}\right] + 30$$

$$= 2.05\ (0.0616 + 61.39) + 30$$

$$= 156^\circ\text{C} \quad \textit{Ans.}$$

**Example 2.23** *Given:* A spherical shell containing a 100 W heat source (Fig. Ex. 2.23)

$r_1 = 26$ cm, $r_2 = 34$ cm

$T_1 = 339$ K, $T_2 = 311$ K

*To find:* Thermal conductivity $k$.

**Solution** The rate of heat transfer through the spherical shell is given by (Eq. 2.66),

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$$Q = \frac{4\pi k r_1 r_2 (T_1 - T_2)}{r_2 - r_1}$$

$$100 = \frac{4\pi k \times 0.26 \times 0.34 \times (339 - 311)}{0.34 - 0.26}$$

$$= \frac{4\pi k \times 0.26 \times 0.34 \times 28}{0.08}$$

$\therefore$ $\quad k = 0.257$ W/mK $\quad$ *Ans.*

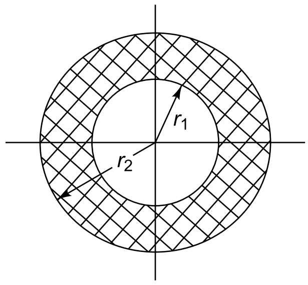


**Fig.** *Ex. 2.23*

**Example 2.24** A copper fin ($k$ = 396 W/mK) 0.25 cm in diameter protrudes from a wall at 95°C into ambient air at 25°C. The heat transfer coefficient by free convection is equal to 10 W/m$^2$K. Calculate the heat loss if (a) the fin is infinitely long, (b) the fin is 2.5 cm long and the coefficient at the end is the same as around the circumference.

**Solution**

$$m = \sqrt{\frac{hP}{kA}} = \sqrt{\frac{10 \times \pi \times d \times 4}{396 \times \pi \times d^2}} = \sqrt{\frac{1}{99 \times 0.0025}}$$

$$= 2.01 \text{ m}^{-1}$$

(a) For infinitely long fin,

$$Q_0 = mkA\theta_0$$

$$= \sqrt{hPkA}\ \theta_0$$

$$= \sqrt{10 \frac{W}{m^2 K} \times \pi \times 0.0025\, m \times 396 \frac{W}{mK} \times \frac{\pi}{4}(0.0025)^2 m^2} \cdot (95 - 25)k$$

$$= 0.865 \text{ W} \quad \textit{Ans.}$$

(b) $$Q_0 = \sqrt{hPkA}\ (T_0 - T_\infty) \frac{\sinh ml + \frac{h}{mk}\cosh ml}{\cosh ml + \frac{h}{mk}\sinh ml}$$

$$ml = 2.01 \times 0.025 = 0.05025$$

$$h/mk = \frac{10}{2.01 \times 396} = 0.01256$$

Substituting: $$Q_0 = 0.865 \times \frac{\sinh(0.05025) + 0.01256 \times \cosh(0.05025)}{\cosh(0.05025) + 0.01256 \times \sinh(0.05025)}$$

$\therefore$ $\quad Q_0 = 0.140$ W $\quad$ *Ans.*

**Example 2.25** Heat is generated uniformly in uranium ($k$ = 29.5 W/mK) rods of 5 cm diameter at the rate of $7.5 \times 10^7$ W/m$^3$. Cooling water at 120°C is circulated in the annulus around the rods with heat transfer coefficient of 55 kW/m$^2$K. Find the maximum temperature of the fuel rods.

**Solution** Rate of heat flow through the rod surface = Rate of internal heat generation.

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$$2\pi r_o L\left(-k\frac{\mathrm{d}T}{\mathrm{d}r}\right)_{r=r_0} = q_G \cdot \pi r_0^2 L$$

$$\therefore \qquad -k\left.\frac{\mathrm{d}T}{\mathrm{d}r}\right|_{r=r_0} = \frac{q_G \cdot r_0}{2} = \frac{7.5\times10^7\times0.025}{2}$$

$$= 9.375 \times 10^5 \text{ W/m}^2$$

Rate of heat flow by conduction at the outer surface = Rate of heat flow by convection to the water

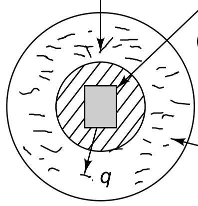


Fig. Ex. 2.25

$$2\pi r_0\left(-k\frac{\mathrm{d}T}{\mathrm{d}r}\right)_{r=r_0} = 2\pi r_0\, h_0(T_0 - T_\infty)$$

$$\therefore \qquad T_0 = \frac{-k\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_{r=r_0}}{h_0} + T_\infty = \frac{9.375\times10^5}{55\times10^3} + 120$$

$$= 137°\text{C} = T_w$$

From Eq. (2.56)

$$T_{\max} = T_w + \frac{q_G R^2}{4k}$$

$$= 137 + \frac{7.5\times10^7\times(0.025)^2}{4\times29.5}$$

$$= 534°\text{C} \quad \textit{Ans.}$$

Also, from Eq. (2.63),

$$\frac{T_{\max}}{T_\infty} = 1 + \frac{q_G R}{4h_c T_\infty}\left(2 + \frac{h_c R}{k}\right)$$

$$\therefore \qquad T_{\max} = T_\infty + \frac{q_G R}{4h_c}\left(2 + \frac{h_c R}{k}\right)$$

$$= 120 + \frac{7.5\times10^7\times0.025}{4\times55\times10^3}\left(2 + \frac{55\times10^3\times0.025}{29.5}\right)$$

$$= 534°\text{C} \quad \textit{Ans.}$$

**Example 2.26** An aluminium heat sink for electronic components has a base of width 50 mm and length 70 mm. There are eight aluminium ($k$ = 180 W/mK) fins of height 12 mm and thickness 3 mm. The fins are cooled by air at 25°C with a convective heat transfer coefficient of 10 W/m$^2$ K. Assuming that the heat transfer coefficient is uniform all along the fin and the tip, determine (a) the heat flow through the heat sink for a base temperature of 50°C, (b) the fin effectiveness, (c) the fin efficiency, (d) the length of the fin if the heat flow is 95% of the heat flow for an infinite fin.

**Solution** Total heat flow to the sink

$Q$ = Heat flow through the unfinned portion of the base + Heat flow through the fins

$$= Q_{uf} + Q_f$$

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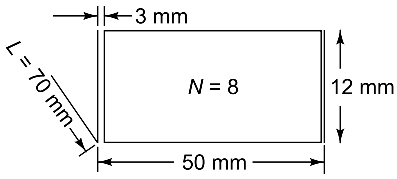


**Fig.** Ex. 2.26

$$Q_{uf} = hA_{uf}(T_0 - T_\infty)$$
$$= 10 \times 0.07(0.05 - 8 \times 0.003) \times (50 - 25)$$
$$= 0.456 \text{ W}$$

$$Q_f = N\sqrt{hPkA_c}\,\theta_0 \frac{\sinh ml + \dfrac{h}{mk}\cosh ml}{\cosh ml + \dfrac{h}{mk}\sinh ml}$$

where $P = 2(70 + 3) \times 10^{-3} = 0.146$ m

$$A_c = 2.1 \times 10^{-4} \text{ m}^2$$

$$m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{10 \times 0.146}{180 \times 2.1 \times 10^{-4}}} = 6.2148 \text{ m}^{-1}$$

$$ml = 6.2148 \times 0.012 = 0.0746$$

$$\cosh ml = 1.00278; \sinh ml = 0.07465$$

$$\sqrt{hPkA_c} = 0.235 \text{ W/K}$$

$$\frac{h}{mk} = 8.939 \times 10^{-3}$$

$$\therefore \quad Q_f = 8 \times 0.235 \times 25 \times \frac{0.07465 + 8.939 \times 10^{-3} \times 1.00278}{1.00278 + 8.939 \times 10^{-3} \times 0.07465}$$
$$= 3.915 \text{ W}$$

$$\therefore \quad Q = 0.456 + 3.915 = 4.371 \text{ W} \qquad \textit{Ans. (a)}$$

Fin effectiveness, $\varepsilon = \dfrac{Q_f}{h\,A_c\,\theta_0}$

$$= \frac{3 \cdot 915/8}{10 \times 0.003 \times 0.07 \times 25}$$

$$= 9.32 \qquad \textit{Ans. (b)}$$

Fin efficiency, $\eta_f = \dfrac{Q_f}{hA_s\theta_0}$, where $A_s = 2H(L + t) + Lt = 1.962 \times 10^{-3}$ m$^2$

$$= \frac{3 \cdot 915/8}{10 \times 1.962 \times 10^{-3} \times 25}$$

$$= 0.998 \qquad \textit{Ans. (c)}$$

For the heat flow to be within 95% of the infinite fin, $\dfrac{\sinh ml + \dfrac{h}{mk}\cosh ml}{\cosh ml + \dfrac{h}{mk}\sinh ml} \geq 0.95$

sin $h/mk << 1$, the above inequality simplifies to

$$\tanh ml \geq 0.95$$

$$\therefore \quad ml \geq 1.83$$

or, $\quad l \geq 295$ mm

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**Example 2.27** A steel rod, 15 cm dia, 90 cm long, $k$ = 40 W/mK, attached to its wall dissipates heat at the rate of 45 W with $h$ = 15 W/m$^2$K.
*To find:* The rate of heat loss from the rod including the tip.

**Solution** Heat loss from the rod is given by Eq. (2.92)

$$Q_0 = \sqrt{hPkA}\cdot\theta_0\cdot\frac{\tanh ml + \dfrac{h}{mk}}{1+\dfrac{h}{mk}\tanh ml}$$

where,
$$m = \sqrt{\frac{hP}{kA}} = \sqrt{\frac{15\times\pi\times5\times10^{-2}}{40\times\dfrac{\pi}{4}(5\times10^{-2})^2}}$$
$$= 5.48 \text{ m}^{-1}$$
$$ml = 5.48\times0.9 = 4.932$$
$$\tanh ml = 1.$$
$$\sqrt{hPkA} = \sqrt{15\times5\times10^{-2}\times\pi\times40\times\frac{\pi}{4}(5\times10^{-2})^2}$$
$$= 0.43$$

$\therefore$
$$Q_0 = 45 \text{ W} = 0.43\,\theta_0\,\frac{1+\dfrac{h}{mk}}{1+\dfrac{h}{mk}}$$

$\therefore$
$$\theta_0 = 104.65\text{°C} = T_0 - T_\infty$$

$\therefore$
$$T_0 = \text{The wall temperature} = 134.65\text{°C} \quad \textit{Ans.}$$

**Example 2.28** *Given:* Two long rods of the same dia, one of brass ($k$ = 85 W/mK) and the other of copper ($k$ = 375 W/mK) have one of their ends inserted in a furnace and the other ends exposed to the same atmospher. At a distance of 105 mm away from the furnace the temperature of the brass rod is 120°C.
*To find:* The distance from the furnace end the copper rod would have attained the same temperature.

**Solution** For a thin long rod the temperature distribution is given by Eq. (2.100),

$$\theta = \theta_0\, e^{-mx}$$

or,
$$\frac{T-T_\infty}{T_0-T_\infty} = e^{-mx}$$

For brass rod, at $x$ = 0.105 m, $T$ = 120°C.

$$\frac{120-T_\infty}{T_0-T_\infty} = e^{-m_1}\times0.105 \qquad (1)$$

For copper rod, at $x = L$, say, $T$ = 120°C

$\therefore$
$$\frac{120-T_\infty}{T_0-T_\infty} = e^{-m_2L} \qquad (2)$$

From Eqs (1) and (2) above,

$$m_2L = m_1\times0.105$$

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$$L = \frac{m_1}{m_2} \times 0.105 = \frac{\sqrt{hP/k_1 A}}{\sqrt{hP/k_2 A}} \times 0.105 = \sqrt{\frac{k_2}{k_1}} \times 0.105$$

$$= \sqrt{\frac{375}{85}} \times 0.105 = 0.2205 \text{ m} = 220.5 \text{ mm} \quad \textit{Ans.}$$

**Example 2.29** A solid sphere of radius 0.5 m has an internal heat generation rate of $2 \times 10^6$ W/m$^3$. If the thermal conductivity of the material is 40 W/mK and the convective heat transfer coefficient at the surface of the sphere is 10 W/m$^2$K, calculate the temperatures at the outer surface and at the centre. Take the ambient temperature as 30°C.

**Solution** Rate of heat generation

$$Q = q_G \times V = q_G \times \frac{4}{3}\pi R^3$$

$$= 2 \times 10^6 \times \frac{4}{3}\pi (0.5)^3 = 1.047 \times 10^6 \text{ W}$$

$$Q = hA(T_w - T_\infty)$$

$$\therefore \qquad T_w = \frac{1.047 \times 10^6}{10 \times 4\pi (0.5)^2} + 30 = 63.32\,°\text{C} \quad \textit{Ans.}$$

From Eq. (2.70), the temperature at the centre

$$T_{max} = T_w + \frac{q_G R^2}{6k} = 63.32 + 2 \times 10^6 \times (0.5)^2 (6 \times 40)$$

$$= 2146.65°\text{C} \quad \textit{Ans.}$$

**Example 2.30** A hollow sphere (inner radius 8 cm, outer radius 12 cm, K = 40 W/mK) has an internal heat generation rate of $2 \times 10^7$ W/m$^3$. The inside surface of the sphere is insulated and the temperature at the outer surface is 375°C. Calculate the maximum temperature in the solid and the convective heat transfer coefficient if the ambient temperature is 30°C.

**Solution** The temperature distribution in a hollow sphere is given by Eq. (2.71),

$$T = T_w + \frac{q_G}{6k}(r_2^2 - r^2) + \frac{q_G r_1^3}{3k}\left(\frac{1}{r_2} - \frac{1}{r}\right)$$

The maximum temperature will occur at the inner surface when $r = r_1$.

$$\therefore \qquad T_{max} = T_1 = 375 + \frac{2 \times 10^7}{6 \times 40}(0.12^2 - 0.08^2) + \frac{2 \times 10^7}{3 \times 40} \times (0.08)^3 \times \left(\frac{1}{0.12} - \frac{1}{0.08}\right)$$

$$= 375 + 666.67 - 355.56 = 686.11°\text{C} \quad \textit{Ans.}$$

Total heat generated, $Q = 2 \times 10^7 \times \frac{4}{3}\pi[(0.12)^3 - (0.08)^3]$

$$= 5.09 \times 10^{-3} \times 2 \times 10^7$$

$$= 101800 \text{ W}$$

$$Q = hA(T_w - T_\infty)$$

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$$101800 = h\,4\pi(0.12^2 - 0.08^2)\,(375 - 30)$$

$$\therefore \qquad h = 2935.12 \text{ W/m}^2 \text{ K} \quad \textit{Ans.}$$

**Example 2.31** Write down the Fourier equation for heat conduction in spherical coordinates. Hence, deduce an expression for steady state heat conduction in radial direction through a solid sphere of radius $R$ with a uniform volumetric heat generation of $q_G$ W/m$^3$ at the centre. Assume thermal conductivity of the spherical material to be uniform. (a) An approximately sphere shaped orange ($k$ = 0.23 W/mK), 90 mm in diameter undergoes ripening process and generates 5100 W/m$^3$ of energy (heat of respiration). If the external surface of the orange is at 8°C, determine (i) the temperature at the centre of the orange and (ii) the rate of heat flow from the outer surface of the orange.

**Solution** The general heat conduction equation in spherical coordinates is given as

$$\frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \phi^2} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left\{\sin\theta \frac{\partial T}{\partial \theta}\right\} + \frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{q_G}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau}$$

For steady state heat conduction in radial direction,

$$\frac{\partial T}{\partial \phi} = 0,\ \frac{\partial T}{\partial \theta} = 0 \text{ and } \frac{\partial T}{\partial \tau} = 0$$

Hence, the above equation reduces to

$$\frac{1}{r^2}\cdot\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}T}{\mathrm{d}r}\right) + \frac{q_G}{k} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}T}{\mathrm{d}r}\right) + \frac{q_G}{k} r^2 = 0$$

On integration,

$$r^2 \frac{\mathrm{d}T}{\mathrm{d}r} + \frac{q_G}{k}\frac{r^3}{3} = C_1 \qquad \text{(i)}$$

At the centre, $r = 0,\ \dfrac{\mathrm{d}T}{\mathrm{d}r} = 0$

$$\therefore \qquad C_1 = 0$$

Equation (i) reduces to

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{q_G}{k}\frac{r}{3} \qquad \text{(ii)}$$

Heat conduction in radial direction, at any radius $r$, is

$$Q = -kA_r \frac{\mathrm{d}T}{\mathrm{d}r}$$

where $A_r$ = surface area of a sphere = $4\pi r^2$

$$\therefore \qquad Q = -k(4\pi r^2)\cdot\left(-\frac{q_G}{k}\cdot\frac{r}{3}\right)$$

$$= \frac{4}{3}\pi r^3 \cdot q_G$$

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Hence, heat conducted at the surface

$$= \frac{4}{3}\pi R^3 \cdot q_G \quad Ans.$$

(a) Outside radius of the orange, $R$

$$= \frac{90}{2} = 45 \text{ mm} = 0.045 \text{ m}$$

Rate of heat generation, $q_G = 5100$ W/m$^3$
Temperature at the outer surface of the orange,

$$T_w = 8°\text{C}.$$

(i) Temperature at the centre of the orange, from Eq. (2.70),

$$T_{max} = T_w + \frac{q_G}{6k}R^2$$

$$= 8 + \frac{5100}{6 \times 0.23} \times (0.045)^2$$

$$= 15.48°\text{C} \quad Ans.$$

(ii) Heat flow from the outer surface of the orange, $Q$
Heat conducted = Heat generated

$$Q = q_G \times \frac{4}{3}\pi R^3$$

$$= 5100 \times \frac{4}{3}\pi \times (0.045)^3 = 1.946 \text{ W} \quad Ans.$$

**Example 2.32** A spherical thin-walled metallic container is used to store liquid nitrogen at 75 K. The container has a diameter of 0.5 m and is covered with an evacuated reflective insulation composed of silica powder ($k$ = 0.0017 W/mK). The insulation is 25 mm thick and its outer surface is exposed to ambient air at 30°C. The heat transfer coefficient on the outside is 20 W/m$^2$ K. The latent heat of vaporization of nitrogen is 2.05 × 10$^5$ J/kg and its density is 800 kg/m$^3$. (a) What is the rate of heat transfer to the liquid nitrogen? (b) What is the rate of liquid boil-off per day?

**Solution**

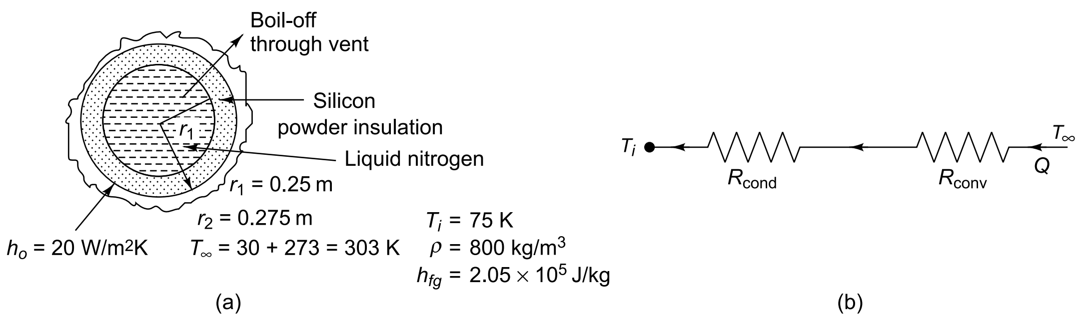


**Fig.** Ex. 2.32

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Rate of heat transfer [Fig. Ex. 2.32(b)],

$$Q = \frac{T_\infty - T_i}{R_{\text{conv}} + R_{\text{cond}}} = \frac{T_\infty - T_i}{\dfrac{1}{h\,4\pi r_2^2} + \dfrac{1}{4\pi k}\left(\dfrac{1}{r_1} - \dfrac{1}{r_2}\right)}$$

$$= \frac{303 - 75}{\dfrac{1}{20 \times 4\pi\,(0.275)^2} + \dfrac{1}{4\pi(0.0017)}\left(\dfrac{1}{0.25} - \dfrac{1}{0.275}\right)}$$

$$= \frac{228}{0.0526 + 17.022} = 13.353 \text{ W} \quad \textit{Ans.}$$

$$\text{Liquid boil-off} = \frac{Q}{h_{fg}}$$

$$= \frac{13.353 \text{ W}}{2.05 \times 10^5 \text{ J/kg}} = 6.514 \times 10^{-5} \text{ kg/s}$$

$$= 5.63 \text{ kg/day} \quad \textit{Ans.}$$

**Example 2.33** The body of an electric motor is 360 mm in diameter and 240 mm long. It dissipates 360 W of heat and its surface temperature should not exceed 55°C. Longitudinal fins of 15 mm thickness and 40 mm height are proposed. The heat transfer coefficient is 40 W/m$^2$ K when the ambient air is 30°C. Determine the number of fins required, if $k$ of the fin material is 40 W/mK.

**Solution**

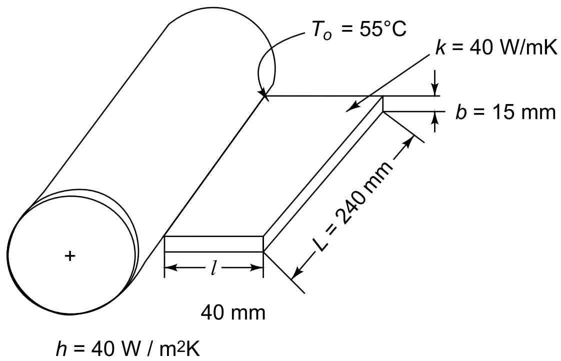


**Fig.** Ex. 2.33

The rate of heat transfer in one fin

$$Q_0 = \sqrt{hPkA}\,,\ \theta_0 \frac{\tanh ml + \dfrac{h}{mk}}{1 + \dfrac{h}{mk}\tanh ml}$$

where $$m = \sqrt{\frac{hP}{kA}} = \sqrt{\frac{h \cdot 2(L+b)}{k \cdot (L \times b)}} = \sqrt{\frac{40 \times 2(0.24 + 0.015)}{40 \times (0.24 \times 0.015)}} = 11.9 \text{ m}^{-1}$$

$\therefore$ $$ml = 11.5 \times 0.04 = 0.476$$

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$$Q_0 = \sqrt{40 \times (0.255 \times 2) \times 40 \times (0.24 \times 0.015)} \times (55 - 30) \times \frac{\tanh(0.476) + \dfrac{40}{11.9 \times 40}}{1 + \dfrac{40}{40 \times 11.9} \times \tanh(0.476)}$$

$$= 1.714 \times 25 \times \frac{0.443 + 0.084}{1 + 0.084 \times 0.443} = 21.77 \text{ W}$$

$$\text{Number of fins required} = \frac{360}{21.77} = 16.356 \text{ or } 17 \text{ fins} \quad Ans.$$

**Example 2.34** The engine cylinder of a motor cycle is made of aluminium alloy ($k$ = 186 W/mK) and is 0.15 m long and outside diameter 50 mm. The temperature of the outer surface is 500 K and the ambient air is 300 K. To increase the rate of heat transfer, five annual fins of the same material are provided. The length and thickness of the fins are 20 mm and 6 mm respectively. The fins are equally spaced. Estimate the percentage increase in the heat transfer rate due to fins if the convective heat transfer coefficient is 50 W/m$^2$K.

**Solution** Using Fig. 2.30 and Eq. (2.133), the data given are as follows:

$$k = 186 \text{ W/mK}, r_1 = 0.025 \text{ m}, r_2 = 0.045 \text{ m}, b = 0.006 \text{ m},$$
$$h = 50 \text{ W/m}^2\text{K},$$

$$m = \sqrt{\frac{2h}{kb}} = \sqrt{\frac{2 \times 50}{180 \times 0.006}} = 9.466 \text{ m}^{-1}$$

$$mr_1 = 9.466 \times 0.025 = 0.236$$
$$mr_2 = 9.466 \times 0.045 = 0.426$$

From Bessel Function Tables and by interpolation,

$$I_1(mr_2) = 0.2183, I_1(mr_1) = 0.1191$$
$$I_0(mr_1) = 1.0155, K_1(mr_2) = 2.07,$$
$$K_1(mr_1) = 4.31, K_0(mr_1) = 1.638$$

Using Eq. (2.133),

$$Q_0 = (2\pi \times 186 \times 9.466 \times 0.006 \times (500 - 300) \times 0.025) \times \frac{4.31 \times 0.2183 \times 0.1191 \times 2.07}{1.638 \times 0.2183 + 1.0155 \times 2.07}$$
$$= 93.7 \text{ W}$$

Total heat transfer from 5 fins

$$Q_f = 93.7 \times 5 = 468.5 \text{ W}$$

Heat transfer from unfinned surface

$$Q_{uf} = hA\,\theta_0 = 50 \times \pi \times 0.05 \times (0.15 - 0.03) \times (500 - 300)$$
$$= 185 \text{ W}$$

$$\text{Total heat transfer} = Q_f + Q_{uf}$$
$$= 468.5 + 185 = 653.5 \text{ W}$$

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Heat transfer without fins

$$= hA\,\theta_0 = 50\pi \times 0.05 \times 0.15 \times 200$$
$$= 235.65 \text{ W}$$

Increase in heat transfer due to incorporation of fins

$$= 653.5 - 235.65 = 417.85 \text{ W}$$

∴ % increase in heat transfer

$$= \frac{417.85}{235.65} \times 100 = 177.32\% \quad \textit{Ans.}$$

**Example 2.35** Annular steel ($k$ = 40 W/mK) fins, 15 mm long and 2 mm thick are provided on a tube of diameter 30 mm. The tube surface temperature is 210°C and the ambient air temperature is 25°C. Estimate the heat transfer from one fin if the convective heat transfer coefficient between the fin and the air is 30 W/m$^2$ K.

**Solution** Heat transfer from an annular fin is given by Eq. (2.133),

$$Q_0 = 2\pi k\, bm\, \theta_0\, r_1 \frac{K_1(mr_1)\, I_1(mr_2) - I_1(mr_1)\, K_1(mr_2)}{K_0(mr_1)\, I_1(mr_2) + I_0(mr_1)\, K_1(mr_2)}$$

where
$$m = \sqrt{\frac{2h}{kb}} = \sqrt{\frac{2 \times 30}{40 \times 2 \times 10^{-3}}} = 27.386 \text{ m}^{-1}$$

$$mr_1 = 27.386 \times 0.015 = 0.41,\ mr_2 = 27.386 \times 0.030 = 0.82$$

From Bessel Function Tables and by interpolation,

$$I_1(mr_2) = 0.477,\ I_1(mr_1) = 0.22,\ I_0(mr_1) = 1.043,$$
$$K_1(mr_2) = 0.836,\ K_1(mr_1) = 2.14,\ K_0(mr_1) = 1.1$$
$$Q_0 = 2\pi \times 40 \times 27.386 \times 2 \times 10^{-3}\,(210 - 25) \times 15 \times 10^{-3} \times \frac{2.14 \times 0.447 - 0.22 \times 0.836}{1.1 \times 0.447 + 1.043 \times 0.836}$$
$$= 21.65 \text{ W} \quad \textit{Ans.}$$

**Example 2.36** A steel tube carries steam at a temperature of 320°C. A thermometer pocket of iron ($k$ = 52.3 W/mK) of inside diameter 15 mm and 1 mm thick is used to measure the temperature. The error to be tolerated is 1.5% of maximum. Estimate the length of the pocket necessary to measure the temperature within this error. The diameter of the steel tube is 95 mm. Assume $h$ = 93 W/m$^2$K and the tube wall temperature is 120°C. Suggest a suitable method of locating the thermometer pocket.

**Solution**

$T_\infty$ = 320°C $d_i$ = 15 mm $h$ = 93 W/m$^2$K

$T_w$ = 120°C $\delta$ = 1 mm $k$ = 52.3 W/mK

$d_0 = d_i + 2\delta = 15 + 2 \times 1 = 17$ mm

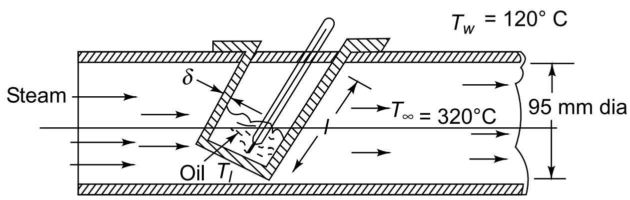


**Fig.** Ex. 2.36

Temperature recorded by the thermometer ($T_l$) is found by the relation

$$\frac{T_l - T_\infty}{T_w - T_\infty} = \frac{1}{\cosh ml} \tag{i}$$

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$$T_\infty - T_l = 0.015\, T_\infty, \quad \text{i.e., } 15\% \text{ of } T_\infty.$$

$\therefore$
$$T_l = 0.985\, T_\infty$$

Now,
$$m = \sqrt{\frac{hP}{kA}} = \sqrt{\frac{h}{k}\frac{\pi d_0}{\frac{\pi}{4}(d_0^2 - d_1^2)}}$$

$$= \sqrt{\frac{93}{52.3}\frac{4 \times 0.017}{0.017^2 - 0.015^2}} = 43.5 \text{ m}^{-1}$$

Substituting in Eq. (i),

$$\frac{0.985 \times 320 - 320}{120 - 320} = \frac{1}{\cosh(43.5\, l)} = \frac{-4.8}{-200}$$

$$\cosh 43.5\, l = 41.67$$

$\therefore$
$$43.5\, l = 4.423$$

$$l = 0.1016 \text{ m} = 101.6 \text{ mm} \quad \textit{Ans.}$$

As $l > D$(95 mm), the pocket should be fitted inclined as shown in Fig. Ex. 2.36.

**Example 2.37** Show that for small objects transferring heat to the surroundings, the minimum value of $hd/k' = 2$.

**Solution** Let us consider a small body as a sphere (Fig. Ex. 2.37).

Heat flow by conduction through a sphere is given by

$$Q = \frac{T_1 - T_2}{\dfrac{R_2 - R_1}{4\pi k (R_1 R_2)}} = \frac{T_1 - T_2}{R_{k,\text{th}}}$$

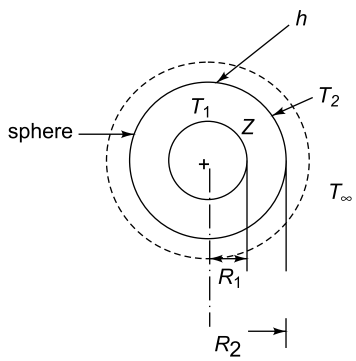

**Fig.** Ex. 2.37

Heat conducted at $r = R_2$ is further convected to the surrounding air and it is given by

$$Q = \frac{T_2 - T_\infty}{\dfrac{1}{4\pi R_2^2 h}} = \frac{T_2 - T_\infty}{R_{c,\text{th}}}$$

where $R_{k,\text{th}}$ and $R_{c,\text{th}}$ are conduction and convection resistance respectively

$$R_{k,\text{th}} = \frac{1}{4\pi k}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

Now the surrounding fluid will be considered a spherical shell of radius $r$ and infinite outside radius. The mechanism of heat transfer will be conduction through the small sphere and the thermal resistance is

$$R_{k,\text{th}} = \frac{1}{4\pi k}\left(\frac{1}{r} - \frac{1}{\infty}\right) = \frac{1}{4\pi k r}$$

$$Q = \frac{T_2 - T_\infty}{\dfrac{1}{4\pi k r}} \tag{i}$$

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Heat flow is also given by

$$Q = \frac{T_2 - T_\infty}{\dfrac{1}{4\pi r^2 h}} \quad \text{(ii)}$$

Equating (i) and (ii)

$$4\pi r^2 h = 4\pi k r$$

$$\frac{hr}{k} = 1$$

or, $$\frac{hd}{k} = 2 \quad \textbf{Proved.}$$

**Example 2.38** A square plate 1 kW electric heater (150 mm × 150 mm) is inserted between two slabs. Slab $A$ is 20 mm thick ($k$ = 50 W/mK) and slab $B$ is 10 mm thick ($k$ = 0.2 W/mK). The outside heat transfer coefficients on side $A$ and side $B$ are 200 W/m$^2$ K and 50 W/m$^2$ K respectively. The temperature of surrounding air is 25°C. Estimate (a) the maximum temperature in the system and (b) the outside surface temperatures of the two slabs. Draw the equivalent electrical circuit.

**Solution** In Fig. Ex. 2.38, $L_A$ = 0.02 m, $L_B$ = 0.01 m,
$k_A$ = 50 W/mK, $k_B$ = 0.2 W/mK, $h_1$ = 200 W/m$^2$K,
$h_2$ = 50 W/m$^2$K, $T_\infty$ = 25°C

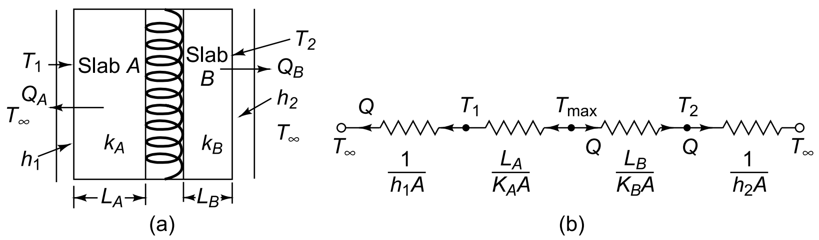


**Fig.** Ex. 2.38

Equivalent electrical circuit is shown in (b).

$$A = 0.15 \times 0.15 = 0.0225 \text{ m}^2$$

$$Q = 1 \text{ kW} = 1000 \text{ W}$$

$$Q = Q_A + Q_B = \frac{T_{max} - T_\infty}{\dfrac{L_A}{k_A A} + \dfrac{1}{h_1 A}} + \frac{T_{max} - T_\infty}{\dfrac{L_B}{k_B A} + \dfrac{1}{h_2 A}}$$

$$= A(T_{max} - T_\infty)\left[\frac{1}{\dfrac{L_A}{k_A} + \dfrac{1}{h_1}} + \frac{1}{\dfrac{L_B}{k_B} + \dfrac{1}{h_2}}\right]$$

$$\therefore \quad 1000 = 0.0225\,(T_{max} - 25)\left[\frac{1}{\dfrac{0.02}{50} + \dfrac{1}{200}} + \frac{1}{\dfrac{0.01}{0.2} + \dfrac{1}{50}}\right]$$

$$= 0.0225\,(T_{max} - 25) \times 199.47$$

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$$\therefore \quad T_{max} = 25 + \frac{1000}{0.0225 \times 199.47} = 247.81\,°C \quad Ans.$$

$$Q_A = \frac{k_A \cdot A \cdot (T_{max} - T_1)}{L_A} = h_1 A\,(T_1 - T_\infty)$$

$$\therefore \quad \frac{50(247.81 - T_1)}{0.02} = 200\,(T_1 - 25)$$

$$\therefore \quad T_1 = 231.3°C \quad Ans.$$

Similarly,

$$Q_B = \frac{0.2\,A(247.81 - T_2)}{0.01} = 50\,A(T_2 - 25)$$

$$20(247.81 - T_2) = 50(T_2 - 25)$$

or,

$$247.81 - T_2 = 2.5\,T_2 - 62.5$$

$$\therefore \quad T_2 = 88.66°C \quad Ans.$$

**Example 2.39** An electric hot plate is maintained at a temperature of 350°C and is used to keep a solution boiling at 95°C. The solution is contained in a cast iron ($k$ = 50 W/mK) vessel of wall thickness 25 mm which is enamelled to a thickness of 0.8 mm. The heat transfer coefficient for the boiling solution is 5.5 kW/m$^2$K and thermal conductivity of enamel is 1.05 W/mK. Calculate (a) the overall heat transfer coefficient and (b) the rate of heat transfer per unit area.

**Solution**

$$\frac{1}{U} = \frac{x_{C.I.}}{k_{C.I.}} + \frac{x_{enamel}}{k_{enamel}} + \frac{1}{h_{solution}}$$

$$= \frac{0.025}{50} + \frac{0.8 \times 10^{-3}}{1.05} + \frac{1}{5500}$$

$$= 1.444 \times 10^{-3}\ m^2\,K/W$$

$$\therefore \quad U = 692.5\ W/m^2K \quad Ans.\ (a)$$

$$q = U(T_0 - T_{sol.}) = 692.5\,(350 - 95) = 176587.5\ W/m^2$$

$$= 176.59\ kW/m^2 \quad Ans.\ (b)$$

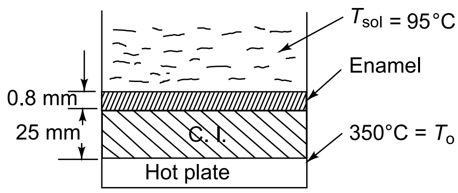


**Fig.** *Ex. 2.39*

**Example 2.40** An insulated steam pipe having outside diameter of 30 mm is to be covered with two layers of insulation, each having a thickness of 20 mm. The thermal conductivity of one material is 5 times that of the other.

Assuming that the inner and outer surface temperatures of composite insulation are fixed, how much heat transfer will be increased when the better insulation material is next to the pipe than it is to the outer layer?

**Solution**

***Case I*** When better insulation is inside

$$r_1 = 30/2 = 15\ mm,\ r_2 = 15 + 20 = 35\ mm,\ r_3 = 35 + 20 = 55\ mm$$

$$k_B = 5\,k_A$$

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$$Q_1 = \frac{2\pi L(T_1 - T_2)}{\dfrac{\ln \dfrac{r_2}{r_1}}{k_A} + \dfrac{\ln \dfrac{r_3}{r_2}}{k_B}} = \frac{2\pi L(T_1 - T_2)}{\dfrac{\ln \dfrac{35}{15}}{k_A} + \dfrac{\ln \dfrac{55}{35}}{5\,k_A}}$$

$$= \frac{2\pi\, k_A L\,(T_1 - T_2)}{0.8473 + 0.0904} = 1.066 \times 2\pi\, k_A\, L(T_1 - T_3) \qquad \text{(i)}$$

**Case 2** When the better insulation is outside

$$Q_2 = \frac{2\pi L(T_1 - T_2)}{\dfrac{\ln \dfrac{35}{15}}{5\,k_A} + \dfrac{\ln \dfrac{55}{35}}{k_A}} = \frac{2\pi\, k_A L\,(T_1 - T_2)}{0.1694 + 0.452}$$

$$= 1.609 \times 2\pi\, k_A \theta\, L(T_1 - T_2)$$

$$\frac{Q_2}{Q_1} = \frac{1.609 \times 2\pi\, k_A L(T_1 - T_2)}{1.066 \times 2\pi\, k_A L(T_1 - T_2)} = 1.509$$

As $Q_2 > Q_1$, therefore, putting the better insulation inside, next to the pipe, decreases the heat flow.

$\therefore$ % age decrease in heat transfer

$$= \frac{Q_2 - Q_1}{Q_1} = 1.509 - 1 = 0.509 \text{ or } 50.9\% \quad \textit{Ans.}$$

**Example 2.41** A gas-filled tube has 2 mm inside diameter and 250 mm length. The gas is heated by an electric wire of diameter 0.05 mm located along the axis of the tube. The current and voltage drop across the heating element are 0.5 amp and 4 volts respectively. If the measured wire and inside wall temperatures are 175°C and 150°C respectively, determine the thermal conductivity of the gas filling the tube.

**Solution**

$$r_1 = 1 \text{ mm}, L = 250 \text{ mm} = 0.25 \text{ m}, r_w = 0.025 \text{ mm},$$

$$T_1 = 150°\text{C}, T_2 = 175°\text{C}, I = 0.5 \text{ amp}, V = 4 \text{ V}$$

$$Q = \frac{2\pi\, kL\,(T_w - T_1)}{\ln \dfrac{r_1}{r_w}} = \frac{2\pi\, k \times 0.25\,(175 - 150)}{\ln \dfrac{1}{0.025}}$$

$$= (10.645\, k) \text{ W}$$

$$Q = V \times I = 4 \times 0.5 = 2 \text{ W}$$

$$\therefore \quad 10.645k = 2$$

$$k = 0.188 \text{ W/mK} \quad \textit{Ans.}$$

**Example 2.42** A wire of 6.5 mm dia. at 60°C is covered with insulation of $k$ = 0.174 W/mK.

*Given:* $r_i = \dfrac{6.5}{2} = 3.25$ mm, $k_i$ = 0.174 W/mK,

$h_i$ = 8.722 W/m$^2$K, $T_\infty$ = 30°C.

*To find:* 1. $(r_0)_{cr}$ and $(r_0)_{cr} - r_i$, 2. $Q_{max}$, 3. percentage increase in heat loss due to insulation.

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**Solution** From Eq. (2.73), for maximum heat loss, the critical radius of insulation.

$$(r_0)_{cr} = \frac{k_i}{h_0} = \frac{0.174}{0.722} = 0.01995 \text{ m}$$
$$= 19.95 \text{ mm}$$

$$\text{Insulation thickness} = (r_0)_{cr} - r_i$$
$$= 19.95 - 3.25 = 16.70 \text{ mm} \quad \textit{Ans.}$$

*Heat loss without insulation*

$$Q_1 = \frac{2\pi L(T_w - T_\infty)}{1/h_0 r_i} = \frac{2\pi \times 1(60-20)}{1/(8.722 \times 0.00325)}$$
$$= 7.124 \text{ W/m}$$

Heat loss (maximum) with insulation
From Eq. (2.72)

$$Q_2 = \frac{2\pi L(T_w - T_\infty)}{\frac{1}{k_i}\ln\frac{r_c}{r_i} + \frac{1}{h_0 r_0}} = \frac{2\pi \times 1(60-20)}{\frac{1}{0.174}\ln\frac{19.95}{0.00325} + \frac{1}{8.722 \times 0.00325}}$$
$$= 15.537 \text{ W/m}$$

Precentage increase in heat loss

$$= \frac{15.537 - 7.124}{7.124} \times 100 = 118.1\% \quad \textit{Ans.}$$

**Example 2.43** An oil film acts as a lubricant ($\mu$ = 10 N-s/m$^2$, $k$ = 230 W/mK, $\rho$ = 1220 kg/m$^3$) between two coaxial cylindrical surfaces (outer diameter = 100 mm and inner diameter = 99.5 mm). The outer cylinder rotates at 10,000 rpm. Calculate the maximum temperature in the oil film if both wall temperatures are maintained at 75°C.

The rate of heat generation per unit volume due to viscous dissipation may be assumed to be $\mu\left(\frac{du}{dy}\right)^2$.

**Solution** Since the clearance between two cylinders is very small, the velocity variation may be assumed to be linear.

$$\frac{d^2T}{dy^2} + \frac{q_G}{k} = 0$$

$$\frac{dT}{dy} = -\frac{q_G y}{k} + C_1$$

$$T = -\frac{q_G}{k}\frac{y^2}{2} + C_1 y + C_2$$

where $q_G = \mu(V/C)^2$

At $y = 0, T = 75°C, \quad \therefore C_2 = 75°C$

At $y = C, T = 75°C,$

$$75 = -\frac{\mu V^2}{c^2} \cdot \frac{c^2}{2k} + C_1 c + 75$$

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$$C_1 = \frac{\mu V^2}{2\,kc}, \quad V = \frac{2\pi N}{60} \times 0.05$$

For maximum temperature $\dfrac{\mathrm{d}T}{\mathrm{d}y} = 0.$

$$0 = \frac{q_G}{2\,k} \cdot y + C_1 = -\frac{q_G}{k} \cdot y + \frac{\mu V^2}{2k}$$

$$\therefore \qquad y = \frac{\mu V^2}{2k} \times \frac{k \times c^2}{\mu V^2} = \frac{c}{2}$$

$$\therefore \qquad T_{\max} = -\frac{q_G}{k} \cdot \frac{c^2}{4} \times \frac{1}{2} + \frac{\mu V^2}{2\,kc} \cdot \frac{c}{2} + 75$$

$$= -\frac{\mu V^2}{c^2} \times \frac{c^2}{8\,k} + \frac{\mu v^2}{4\,k} + 75$$

$$= \frac{\mu V^2}{8\,k} + 75 = 75 + 10 \times \left(\frac{2\pi \times 10^4 \times 0.05}{60}\right)^2 \frac{1}{8 \times 230}$$

$= 75 + 14.9 = 89.9$°C *Ans.*

## Summary

The general heat conduction equation for isotropic bodies is derived first in rectangular coordinates and then in cylindrical and spherical coordinates taking into consideration the internal heat generation. These equations are then reduced for steady state heat conduction in simple geometries with and without heat generation for plane, cylindrical and spherical walls. The critical radius of insulation for a cylinder and a sphere is separately derived and discussed with practical applications. Extended surfaces for enhancement of heat transfer are extensively reviewed, first for fins of uniform cross-sectional areas and then of variable cross-sectional areas. The fin performance parameters are analysed. Two- and three-dimensional steady state heat conduction in the absence of heat source is then discussed, first dealing with the analytical solution and then with graphical and numerical methods. The finite difference technique and the relaxation method for steady heat conduction are introduced.

## Important Formulae and Equations

| *Equation number* | *Equation* | *Remarks* |
|---|---|---|
| (2.6) | $\dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 T}{\partial y^2} + \dfrac{\partial^2 T}{\partial z^2} + \dfrac{q_G}{k} = \dfrac{1}{\alpha}\dfrac{\partial T}{\partial T}$ | Three-dimensional time dependent heat conduction equation with constant $k$ in rectangular co-ordinates (Fourier equation) |
| (2.7) | $\dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 T}{\partial y^2} + \dfrac{\partial^2 T}{\partial z^2} + \dfrac{q_G}{k} = 0$ | Poisson equation |

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| *Equation number* | *Equation* | *Remarks* |
|---|---|---|
| (2.8) | $\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=0$ | Laplace equation |
| (2.19) | $\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(y\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}+\frac{q_G}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$ | Three-dimensional time-dependent heat conduction equation with constant $k$ in cylindrical coordinates |
| (2.22) | $\frac{1}{r}\frac{d}{dr}\left(r\frac{\mathrm{d}T}{\mathrm{d}r}\right)+\frac{q_G}{k}=0$ | Steady state radial heat conduction in presence of heat source |
| (2.23) | $\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}T}{\mathrm{d}r}\right)=0$ | Steady state heat conduction in cylindrical coordinate radial direction |
| (2.30) | $\frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right)+$ $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{q_G}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$ | General heat conduction equation in spherical co-ordinates |
| (2.31) | $\frac{d^2 T}{dx^2}=0$ | Temperature distribution for one dimensional steady heat conduction through a wall |
| (2.33) | $T=-\frac{T_1-T_2}{b}x+T_1$ | Temperature at any distance $x$ in a wall of thickness $b$ |
| (2.36) | $T(x)=-\frac{q_G x^2}{2k}+\frac{T_2-T_1}{b}x+\frac{q_G b}{2k}+T_1$ | Temperature distribution in a wall with uniform heat generation |
| (2.42) | $T_{\max}=\frac{q_G}{8k}b^2+\frac{q_G b}{2h}+T_\infty$ | Maximum temperature in the mid-plane of a wall in presence of uniform heat generation |
| (2.45) | $Q_k=\frac{2\pi kL(T_i-T_0)}{\ln\frac{r_0}{r_i}}$ | Steady state heat flow through a cylindrical wall |
| (2.46) | $R_{\mathrm{th}}=\frac{\ln\frac{r_0}{r_i}}{2\pi kL}$ | Thermal resistance offered by a cylindrical wall |
| (2.49) | $\frac{1}{U_0 A_0}=\Sigma R_{\mathrm{th}}=\frac{1}{\frac{1}{h_i A_i}+\frac{\ln r_2/r_i}{2\pi k_1 L}+\frac{\ln r_3/r_2}{2\pi k_2 L}+\frac{1}{h_0 A_0}}$ | Overall heat transfer coefficient in a composite sylinder with convection at the interior and exterior surfaces |

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| Equation number | Equation | Remarks |
|---|---|---|
| (2.55) | $T(r) = \frac{q_G R^2}{4k}\left(1-\frac{r^2}{R^2}\right)+T_w$ | Temperature variation along the wall radius in presence of heat source |
| (2.63) | $\frac{T_{\max}}{T_\infty} = 1+\frac{q_G R}{4h_C T_\infty}\left(2+\frac{h_C R}{k}\right)$ | Maximum temperature in the cylindrical wall in the presence of heat source with external convection |
| (2.69) | $T = T_w + \frac{q_G (R^2-r^2)}{6k}$ | Temperature distribution in a solid sphere with heat generation |
| (2.71) | $T = T_w + \frac{q_G}{6k}(r_2^2 - r^2) + \frac{q_G r_1^3}{3k}\left(\frac{1}{r_2}-\frac{1}{r}\right)$ | Temperature distribution in a hollow sphere with heat generation |
| (2.73) | $(r_0)_{\text{cr}} = \frac{k_i}{h_a}$ | Critical radius of insulation for a tube or wire |
| (2.75) | $(r_0)_{\text{cr}} = \frac{2k_i}{h_o}$ | Critical radius of insulation for a spherical wall |
| (2.84) | $\theta = C_1 e^{mx} + C_2 . e^{-mx}$ | Temperature distribution in a straight fin |
| (2.90) | $\frac{\theta}{\theta_0} = \frac{\cosh m(l-x) + \frac{h}{mk}\sin h\, m(l-x)}{\cosh ml + \frac{h}{mk}\sin h\, ml}$ | Temperature distribution in a straight fin losing heat at the tip |
| (2.91) | $\frac{\theta_l}{\theta_0} = \frac{1}{\cosh ml + \frac{h}{mk}\sinh ml}$ | Tip temperature of a fin losing heat |
| (2.93) | $Q_0 = (hPkA)^{1/2}\theta \cdot \frac{\frac{h}{mk} + \tanh ml}{1+\frac{h}{mk}\tanh ml}$ | Heat transfer from the base of a fin losing heat at the tip |
| (2.95) | $\frac{\theta}{\theta_0} = \frac{\cosh m(l-x)}{\cosh ml}$ | Temperature distribution in a fin with insulated tip |
| (2.96) | $\frac{\theta_l}{\theta_0} = \frac{1}{\cosh ml}$ | Temperature of the tip with no heat loss from the tip |
| (2.97) | $Q_0 = mkA\theta_0 \tanh ml$ | Heat transfer from a straight fin with insulated tip |

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| *Equation number* | *Equation* | *Remarks* |
|---|---|---|
| (2.100) | $\frac{\theta}{\theta} = e^{-mx}$ | Temperature distribution in an infinitely long fin |
| (2.101) | $Q = mkA\theta_0$ | Heat flow from an infinitely long fin |
| (2.104) | $\varepsilon_f = \left(\frac{hP}{kA}\right)^{1/2} \tanh ml$ | Fin effectiveness with no heat loss from the tip |
| (2.110a) | $\eta_f = \frac{\tanh ml}{ml}$ | Fin efficiency of a fin with insulated tip |
| (2.114) | $Q_{\text{total}} = h\theta_0\left[A-(1-\eta_f)A_f\right]$ | Total heat transfer from finned and unfinned surfaces |
| (2.116b) | $\frac{l}{b/2} = 1.419\left(\frac{2k}{hb}\right)^{1/2}$ | Optimum ratio of fin height to half thickness for maximum heat flow for a given weight of a rectangular fin |
| (2.119) | $\frac{d^2\theta(x)}{dx^2} + \frac{dA/dx}{A}\frac{d\theta(x)}{dx} - \frac{hP}{kA}\theta(x) = 0$ | General form of steady energy equation for an extended surface |
| (2.145) | $T(x, y) = T_m \frac{\sinh(\pi y/L)}{\sinh(\pi b/L)} \sin\frac{\pi x}{L}$ | Temperature variation in a solid for two-dimensional steady state heat conduction |

## Objective Type Questions

2.1 The heat flow line at a point in a solid is along a path perpendicular to the
  (a) isotropic surface
  (b) isothermal surface
  (c) adiabatic surface
  (d) isobaric surface

2.2 The inside and outside heat transfer coefficients of a fluid across a brick wall of 15 cm thickness and thermal conductivity 0.10 W/mK are 30 W/m$^2$K. The overall heat transfer coefficient (W/m$^2$K) will be closer to
  (a) inverse of heat transfer coefficient
  (b) heat transfer coefficient
  (c) thermal conductivity of brick
  (d) none of the above

2.3 The relation $\nabla^2 T = 0$ is referred to as
  (a) Fourier heat conduction equation
  (b) Laplace equation
  (c) Poisson equation
  (d) Euler equation

2.4 If $k$ is the thermal conductivity, $\rho$ is the mass density and $c$ is the specific heat of a substance, then its thermal diffusivity is given by
  (a) $\frac{\rho c}{k}$ (b) $\frac{k}{\rho c}$
  (c) $\frac{kc}{\rho}$ (d) $\frac{k\rho}{c}$

2.5 Transient heat conduction means
  (a) heat conduction for a short time
  (b) conduction when the temperature at a point varies with time
  (c) very little heat transfer
  (d) heat conduction with a very small temperature difference

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2.6 A composite wall has two layers of different materials having thermal conductivities of $k_1$ and $k_2$. If each layer has the same thickness, the equivalent thermal conductivity of the wall is
(a) $k_1 + k_2$ (b) $(k_1 + k_2)/k_1k_2$
(c) $\frac{2k_1k_2}{k_1+k_2}$ (d) $k_1k_2$

2.7 In cylindrical coordinates the following relation
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}=0$$
is called
(a) Laplace equation (b) Poisson equation
(c) Fourier equation (d) none of the above

2.8 In rectangular coordinates the following equation
$$\nabla^2 T+\frac{q_G}{k}=0$$
is called
(a) Laplace equation
(b) Poisson equation
(c) Fourier heat conduction equation
(d) none of the above

2.9 The steady state heat conduction equation in radial direction in spherical coordinates in absence of any heat source is given by
(a) $\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}T}{\mathrm{d}r}\right)=0$
(b) $\frac{1}{r^2}\cdot\frac{\mathrm{d}T}{\mathrm{d}r}=0$
(c) $\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}T}{\mathrm{d}r}\right)=0$
(d) $\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}T}{\mathrm{d}r}\right)=0$

2.10 The temperature profile for heat conduction through a wall of constant thermal conductivity in the presence of a heat source is
(a) a straight line (b) parabolic
(c) logarithmic (d) hyperbolic

2.11 The temperature variation for heat conduction through a cylindrical wall having uniform $k$ is
(a) linear (b) parabolic
(c) logarithnic (d) hyperbolic

2.12 When the convective resistance is very low, i.e., $1/h_c \to 0$ or when the thermal resistance of a solid predominates, i.e., $k \to 0$, the Biot number approaches
(a) infinity (b) zero
(c) unity (d) none of the above

2.13 The maximum temperature at the centre of solid sphere of radius $R$ having heat generation of $q_G$ (W/m$^3$) is given by
(a) $T_w+\frac{q_G R^2}{2k}$ (b) $T_w+\frac{q_G R^2}{4k}$
(c) $T_w+\frac{q_G R^2}{6k}$ (d) $T_w+\frac{q_G R^2}{8k}$
where $T_w$ is the wall temperature.

2.14 Up to the critical radius of insulation,
(a) convective heat loss will be less than conduction heat loss
(b) heat flux will decrease
(c) added insulation will increase the heat loss
(d) addd insulation will decrease the heat loss

2.15 If the radius of a current carrying wire is less than the critical radius, then the addition of electrical insulation will enable the wire to carry a higher current because
(a) the thermal resistance of the insulation is reduced.
(b) the thermal resistance of the insulation is increased.
(c) the heat loss from the wire would decrease.
(d) the heat loss from the wire would increase.

2.16 For pipes, the radius of the pipe is taken higher than the critical radius, so that any insulation added will only
(a) decrease the heat loss from the pipe.
(b) increase the heat loss from the pipe.
(c) keep the heat loss unaltered.
(d) enable heat gain from the surroundings.

2.17 The critical radius of insulation for a spherical shell is equal to
(a) $h/k$ (b) $2h/k$
(c) $k/h$ (d) $2k/h$

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2.18 Fins are provided on heat transfer surface
(a) to enhance heat transfer by increasing the turbulence in flow.
(b) to increase surface area in promoting the rate of heat transfer.
(c) to increase the temperature gradient in augmenting heat transfer.
(d) to decrease the pressure drop of fluid.

2.19 When the convective heat transfer coefficient $h = mk$ where $m = \sqrt{\dfrac{hP}{kA}}$, the incorporation of an extended surface will
(a) increase the rate of heat transfer
(b) decrease the rate of heat transfer
(c) not alter the rate of heat transfer
(d) only increase the rate of heat transfer when the length of the fin is very large

2.20 The fin effectivencss is enhanced by
(a) the choice of a material of high thermal conductivity.
(b) increasing the ratio of the permeter to the cross-sectional area of the fin.
(c) the low value of heat transfer coefficient.
(d) all of the above.

2.21 The ratio of heat transfer rate of a fin to the heat transfer rate without fin is referred to as
(a) fin effectiveness (b) fin efficiency
(c) fin resistance (d) fin conductance

2.22 The ratio of actual heat transfer from a fin to the maximum possible heat transfer when the entire fin were at the base temperature is called
(a) fin effectiveness
(b) fin efficiency
(c) fin performance coefficient
(d) convective coefficient

2.23 The temperature distribution along a fin with insulated tip is equal to
(a) $\exp(-mx)$
(b) $\dfrac{\exp(mx)+\exp(-mx)}{2}$
(c) $\dfrac{\cosh m(l-x)}{\cosh ml}$
(d) $\cosh m(1-x) + \cosh ml$

2.24 For $h/mk > 1$, i.e., $h > mk$, adding an extended surface
(a) reduces the rate of heat transfer
(b) increases the rate of heat transfer
(c) does not alter the rate of heat transfer
(d) obeys none of the above

2.25 For a rectangular fin of thickness $b$, the fin efficiency is given by
(a) $\left(\dfrac{kh}{2b}\right)^{1/2} \tanh ml$ (b) $\left(\dfrac{2k}{hb}\right)^{1/2} \tanh ml$
(c) $\left(\dfrac{2h}{kb}\right)^{1/2} \tanh ml$ (d) $\left(\dfrac{kb}{2h}\right)^{1/2} \tanh ml$

2.26 It is desired to increase the heat dissipation rate from the surface of an electronic device of spherical shape of 5 mm radius exposed to convection $h$ = 10 W/m$^2$K by encasing it in a spherical sheath of conductivity $k$ = 0.04 W/m$^2$K. For maximum heat flow, the diameter of the sheath should be
(a) 18 mm (b) 16 mm
(c) 12 mm (d) 8 mm

2.27 Consider the following statements pertaining to heat transfer through fins:
1. Fins are equally effective irrespective of whether they are on the hot side or cold side of the fluid.
2. The temperature along the fin is variable and hence the rate of heat transfer varies along the fin.
3. The fins may be made of materials that have a higher thermal conductivity than the material of the wall.
4. Fins must be arranged at right angles to the direction of fluid flow

Of these statements,
(a) 1 and 2 are correct
(b) 2 and 4 are correct
(c) 1 and 3 are correct
(d) 2 and 3 are correct

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## Answers

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 2.1 | (b) | 2.2 | (c) | 2.3 | (b) | 2.4 | (b) | 2.5 | (b) |
| 2.6 | (c) | 2.7 | (a) | 2.8 | (b) | 2.9 | (c) | 2.10 | (b) |
| 2.11 | (c) | 2.12 | (a) | 2.13 | (c) | 2.14 | (c) | 2.15 | (d) |
| 2.16 | (a) | 2.17 | (d) | 2.18 | (b) | 2.19 | (c) | 2.20 | (d) |
| 2.21 | (a) | 2.22 | (b) | 2.23 | (c) | 2.24 | (a) | 2.25 | (d) |
| 2.26 | (b) | 2.27 | (b) | | | | | | |

## Open Book Problems

2.1 The variation of thermal conductivity of a wall material is given by

$$k = k_o \,.\, (1 + \alpha T + \beta T^2)$$

If the thickness of the wall is $L$ and its two surfaces are maintained at temperatures $T_1$ and $T_2$, find an expression for the steady state one-dimensional heat flow through the wall.

*Hints:* Use Fourier's equation $q = -k\dfrac{dT}{dx}$ and substitute $k = k_0\,(1 + \alpha T + \beta T^2)$. Then,

$$\int_0^L dx = \int_{T_1}^{T_2} k_0 (1 + \alpha T + \beta T^2) dT$$

On integrating, deduce the required expression.

2.2 An electric hot plate is maintained at a temperature of 350ºC and is used to keep a solution boiling at 95ºC. The solution is contained in a cast iron vesel of wall thickness 25 mm which is enamelled inside to a thickness of 0.8 mm. The heat transfer coefficient for the boiling solution is 5.5 kW/m$^2$K and the thermal conductivities of cast iron and enamel are 50 and 1.05 W/mK respectively. Calculate (a) the overall heat transfer coefficient, and (b) the rate of heat transfer per unit area.

*Hints:* (a) The overall heat transfer coefficient is found from

$$\frac{1}{U} = \frac{(\Delta x)_{CI}}{k_{CI}} + \frac{(\Delta x)_{\text{enamel}}}{k_{\text{emamel}}} + \frac{1}{h\ \text{solution}}$$

and then calculate $Q = UA\,(T_{\text{heater}} - T_{\text{solution}})$

2.3 A steam pipe, 10 cm I.D. and 11 cm O.D., is covered with an insulating material ($k$ = 1 W/mK). The steam temperature and the ambient temperature are 200ºC and 20ºC respectively. If the convective heat transfer coefficient is 8 W/m$^2$K, find the critical radius of insulation, and for this value of $r_0$, calculate the heat loss per metre of the pipe and the outer surface temperature. Neglect resistance of the pipe material.

*Hints:* Find $r_c = k/h$, and using $r_o = r_c$, find

$$Q/L = \frac{T_i - T_\infty}{\dfrac{\ln r_0 / r_i}{2\pi k} + \dfrac{1}{2\pi r_0 h}}$$

The outer surface temperature $T_0$ is found from the expression

$$T_o = (Q/L)R_0 + T_\infty$$

2.4 A thin hollow tube with 6 mm O.D. and 4 mm I.D. carries a current of 1000 amperes. Water at 30ºC is circulated inside the tube for cooling it. Taking the heat transfer coefficient of the water side as 35,000 W/m$^2$K and $k$ of the tube material as 18 W/mK, estimate the surface temperature of the tube if the outer surface is insulated. The electrical resistance of the material is 0.1 ohm –mm$^2$/m.

*Hints:* Use Eq. (2.51), $T = \dfrac{q_G r^2}{4k} + C_1 \ln r + C_2$

Now, $\left(\dfrac{dT}{dr}\right)_{r=r_o} = 0$ since outer surface is insulated.

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Heat generated is transferred to the fluid inside.

$$\therefore \quad Q = q_G(r_o^2 - r_i^2) = -k\left(\frac{dT}{dr}\right)_{r=r_i}$$

$$2\pi r_i = h2\pi r_i(T_i - T_f)$$

At $r - r_i$, $T = T_i$, find $C_2$ and substituting $C_1$ and $C_2$

$$T_o = T_f + \frac{q_G r_i^2}{4k}\left[\left(\frac{2k}{hr_i} - 1\right)\left\{\left(\frac{r_0}{r_i}\right)^2 - 1\right\} + 2\left(\frac{r_0}{r_i}\right)^2 \ln\frac{r_0}{r_i}\right]$$

Substituting $q_G = \dfrac{I^2R}{\pi(r_0^2 - r_i^2)}$, $T_f = 30$ºC,

$h = 35000$ W/m²K, $k = 18$ W/m²K, find $T_o$.

2.5 Two rods of diameter $D$ and length $L$ have one of the ends at 120ºC and are exposed to air at 30ºC. The conductivity of the material of one rod is 45 W/m²K and the temperature of the rod at the end is measured as 80ºC, while the end temperature of the other rod was 60ºC. Determine the conductivity of the other material.

*Hints:* Equation (2.96) $\dfrac{\theta_l}{\theta_o} = \dfrac{1}{\cosh ml}$ gives the temperature of the fin at the tip which is insulated.

Here, $\dfrac{T_l - T_\infty}{T_o - T_\infty} = \dfrac{1}{\cosh ml}$, $T_l = 80$ºC,

$T_\infty = 30$ºC, $T_o = 120$ºC for one rod. Find $m_1 l$ for this rod and $m_2 l$ for the other rod.

Therefore, $\dfrac{m_1}{m_2} = \dfrac{\sqrt{hP/k_1A}}{\sqrt{hP/k_2A}} = \sqrt{\dfrac{k_2}{k_1}}$.

If $k_1 = 45$ W/mK, find $k_2$.

2.6 The aluminium ($k$ = 200 W/m²K) square fins (0.5 mm × 0.5 mm) of 10 mm length are provided on a surface of a semiconductor electronic device to dissipate 1 W of energy generated. The temperature at the surface of the device should not exceed 80ºC when the surrounding temperature is 40ºC, and $h$ = 15 W/m²K. Neglecting heat loss from the tip, find the number of fins required to carry out the above duty.

*Hints:* First find $m = \sqrt{\dfrac{hP}{kA}}$ and then $ml$.

Equation (2.97) gives

$Q_{fin} = n[mkA\theta_0 \tanh ml] = 1$ watt.

where $n$ = number of fins required.

2.7 A longitudinal copper fin ($k$ = 380 W/mK) 600 mm long and 5 mm diameter is exposed to an air stream at 20ºC. The convective heat transfer coefficient is 20 W/m²K. If the fin base temprature is 150ºC, determine (a) the rate of heat transfer and (b) the efficiency of the fin.

*Hints:* Neglecting tip loss, $Q_0 = mkA\theta_0 \tanh ml$

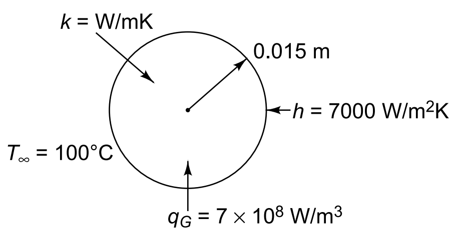
where $m = \sqrt{\dfrac{hP}{kA}} = \sqrt{\dfrac{h}{2} \times \dfrac{\pi d}{\frac{\pi}{4}d^2}} = \sqrt{\dfrac{4h}{kd}}$,

$T_o = 150$ºC, $T_\infty = 20$ºC,

$d = 0.005$ m, $L = 0.6$ m. Find $Q_o$ and then

$$\eta_{fin} = \frac{\tanh ml}{ml}$$

2.8 Nuclear fuel rods are to be clad with aluminium or stainless steel or zirconium with melting temperatures of 650ºC, 1400ºC and 1800ºC. The diameter is 30 mm and the heat generation rate is 7 × 10⁸ W/m³. The convection on the surface to a fluid at 100ºC has $h$ = 7000 W/m²K. The conductivity of the material is 52 W/mK. At shut-down times, coolant may not circulate and the surface temperature may reach the mean between the steady state surface and the centre temperature. Check for the material to be used for cladding.



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*Hints:* Energy balance gives

Heat generated = Heat convected away

$$q_G \times \pi r^2 l = h\, 2\pi r l\, (T_w - T_\infty)$$

Find $T_w$.

From Eq. (2.56)

$T_o - T_\infty = (q_G/4k)\, R^2$. Find $T_o = T_{max}$.

Check for the melting point and notice that zirconium is the most suitable material for cladding.

## Review Questions

2.1 What is an isotropic solid?

2.2 Why is there a negative sign in the Fourier's law of heat conduction?

2.3 Define thermal diffusivity. What is its dimension? How does it differ from thermal conductivity?

2.4 What is Laplacian? Express it in rectangular and cylindrical coordinates.

2.5 Define thermal conductivity and explain its significance in heat transfer. How do thermal conductivities of gases and liquids vary with temperature?

2.6 Show that the temperature profile for heat conduction through a wall of constant thermal conductivity is a straight line and in the presence of a heat source it becomes parabolic.

2.7 Show that the temperature variation for heat conduction through a cylindrical wall having uniform $k$ is logarithmic.

2.8 Show that the maximum temperature in a cylindrical rod with heat generation $q_G$ (kW/m$^3$) is given by

$$\frac{T_{max}}{T_\infty} = 1 + \frac{q_G R}{4 h_c T_\infty}\left(2 + \frac{h_c R}{k}\right)$$

where $h_c$ is the convection heat transfer coefficient and $T_\infty$ is the ambient temperature.

2.9 Show that the thermal resistance offered by a spherical wall of uniform $k$ is given by $(r_o - r_i)/(4\pi k r_o r_i)$.

2.10 What do you mean by critical radius of insulation? Show that it is given by $k_i/h_a$, where $k_i$ is the thermal conductivity of insulation and $h_a$ is the heat transfer coefficient.

2.11 A pipe is insulated to reduce the heat loss from it. However, measurements indicate that the rate of heat loss has increased instead of decreasing. Can the measurements be right?

2.12 Explain why an insulated small diameter wire has a higher current carrying capacity than an uninsulated one.

2.13 Explain the effect of extended surfaces on heat transfer.

2.14 If a fin is thin and long and tip loss is negligible, show that the heat transfer from the fin is given by

$$Q_o = mkA\theta_o \tanh ml$$

where $m = (hP/kA)^{1/2}$

2.15 Define fin effectiveness. When is the use of fins not justified?

2.16 What are the considerations in determining the proper length of the fins attached to a surface?

2.17 Discuss the criteria of selection of fins. What is the difference between the fin effectiveness and the fin efficiency?

2.18 How is the thermal performance of a fin measured? Define fin efficiency.

2.19 Show that the fin efficiency for a rectangular fin is given by

$$\eta_f = \frac{\tanh\,[(2h\,l_c^2)/kb]^{1/2}}{\left[(2h\,l_c^2)/kb\right]^{1/2}}$$

where $l_c$ = corrected length $= l + \dfrac{b}{2}$

or $$l_c = l + \frac{A}{P}$$

2.20 Show that the total heat transfer from a finned wall is given by

$$Q = h\,\theta\,[A - (1 - \eta_f)\, A_f]$$

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where $A$ = total area of fin and unfinned surfaces, $A_f$ = area of the finned surface, $\eta_f$ = fin efficiency and $\theta_0 = T_o - T_\infty$.

2.21 Show that for maximum heat flow from a rectangular fin of a given weight, the optimum ratio of fin height to half the fin thickness is

$$\frac{1}{b/2} = 1.419 \left(\frac{2k}{hb}\right)^{1/2}$$

and the excess temperature at the fin tip is

$$\theta_1 = 0.457\ \theta_0$$

where $\theta_0$ is the excess temperature at the fin root.

2.22 Determine the optimum shape of a fin having the minimum weight for a given heat flow. Explain how the triangular fin is of the best shape.

2.23 Use the separation-of-variables method to solve the Laplacian for a two-dimensional heat conduction problem. Why are the isotherms and adiabatics orthogonal?

2.24 Explain the graphical method of solving a two-dimensional heat conduction problem. Define the shape factor. How is it estimated?

2.25 Explain discretisation and stability with respect to the numerical method of solving a two-dimensional heat conduction problem.

2.26 Explain the matrix inversion method of determining the temperature distribution for steady heat conduction in a two-dimensional solid.

2.27 What is a conduction shape factor? How is it related to the thermal resistance?

2.28 Discuss Gauss–Siedel iteration technique to determine temperature at a nodal point in a two-dimensional solid.

2.29 What is the relaxation method of solving a set of algebraic equations? What do you mean by residuals?

2.30 How can a three-dimensional heat conduction problem be solved?

2.31 Explain the electrical analogy method of solving heat conduction problems.

## Problems for Practice

2.1 A plane wall of width $L$ has a constant thermal conductivity $k$. The surface temperatures are $T_1$ at $x = 0$ and $T_2$ at $x = L$. The heat generated per unit volume in the wall varies according to the expression $q_G = bx^2$. Determine (a) the steady temperature distribution, (b) the location of the plane of maximum temperature and (c) the heat flux leaving the wall at the surface $x = L$.

2.2 Determine the steady-state temperature distribution and the total radial heat flow in a hollow sphere in a region $a < r < b$ when the boundary surface at $r = a$ and $r = b$ are kept at uniform temperatures $T_o$ and $T_i$ respectively.

$$\left( Ans.\ T = \frac{1}{b-a}\left[aT_o\left(\frac{b}{r}-1\right)+bT_1\left(1-\frac{a}{r}\right)\right], \right.$$

$$\left. Q = \frac{(T_o - T_1)\,4\pi kab}{b-a} \right)$$

2.3 A plastic pipe ($k$ = 0.5 W/m K) carries a fluid such that the convective heat transfer coefficient is 300 W/m$^2$ K. The average fluid temperature is 100°C. The pipe has an inner diameter of 3 cm and outer diameter of 4 cm. If the heat transfer rate through the pipe per unit length is 500 W/m, calculate the external pipe temperature and the overall heat transfer coefficient based on outside area. (*Ans.* 36.53°C, 62.69 W/m$^2$ K)

2.4 A steel ($k$ = 15 W/m K) tube with 5 cm inner diameter and 7.6 cm outer diameter is covered with an insulation ($k$ = 0.2 W/m K) of thickness 0.2 cm. A hot gas at 330°C flows through the tube with $h_i$ = 400 W/m$^2$ K. The outer surface of the insulation is exposed to air at 30°C with $h_o$ = 60 W/m$^2$ K. Calculate (a) the heat loss from the tube which is 10 m long and (b) the temperature drops resulting from the thermal resistances of the hot gas

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flow, the steel tube, the insulation layer and the outside air.

(*Ans.* (a) 7251 W, (b) 12°C, 3.3°C, 253°C and 31.7°C)

2.5 Derive an expression for the one-dimensional steady-state temperature distribution in a slab of thickness $L$ where heat is generated at a constant rate of $q_G$ W/m$^3$. The boundary surface at $x = 0$ is insulated and that at $x = L$ is kept at zero temperature. Calculate the temperature of the insulated surface for $k = 40$ W/m K, $q_G = 10^6$ W/m$^3$ and $L = 0.1$ m.

(*Ans.* 125°C)

2.6 An electrical resistance wire of radius 1 mm with thermal conductivity 25 W/m K is heated by the flow of electric current which generates heat at the rate of $2 \times 10^9$ W/m$^3$. Determine the centreline temperature rise above the surface temperature of the wire if the surface is maintained at a constant temperature. (*Ans.* 20°C)

2.7 An industrial furnace is made of fireclay brick of thickness 25 cm and thermal conductivity $k_1 = 1$ W/m K. The outside surface is insulated with material ($k_2 = 0.05$ W/m K). Determine the thickness of the insulation layer in order to limit the heat loss from the furnace wall to $q = 1000$ W/m$^2$ when the inside surface of the wall is at 1030°C and the outside surface at 30°C.

(*Ans.* 3.75 cm)

2.8 A hollow steel sphere ($k = 10$ W/m K) has an inside radius of 10 cm and outside radius of 20 cm. The inside surface is maintained at a uniform temperature of 230°C and the outside surface dissipates heat by convection with $h = 20$ W/m$^2$ K into the ambient air at 30°C. Determine the thickness of asbestos insulation ($k = 0.5$ W/m K) required to reduce the heat loss by 50%. (*Ans.* 5.8 cm)

2.9 Two very long slender rods of the same diameter are given. One rod is of aluminium and has $k_1 = 200$ W/m K, but $k_2$ of the other rod is not known. To determine the $k_2$ of the other rod, both the rods are thermally attached to a metal surface maintained at a constant temperature $T_o$. Both rods lose heat by convection with a heat transfer coefficient $h$ into the ambient air at $T_\infty$. The surface temperature of each rod is measured at various distances from the hot base surface. The temperature of the aluminium rod at $x_1 = 40$ cm from the base is the same as that of the other rod at $x_2 = 20$ cm from the base. Determine the thermal conductivity $k_2$ of the second rod.

(*Ans.* 50 W/m K)

2.10 The inner and outer radii of a hollow cylinder are 5 cm and 10 cm respectively. The inside surface is maintained at 300°C, and the outside surface at 100°C. The thermal conductivity varies with temperature in the range $100 < T < 300$°C as $k = 0.5\,(1 + 10^{-3}\,T)$ where $T$ is in °C. Determine the heat flow rate per meter length of cylinder.

(*Ans.* 1.088 kW/m)

2.11 A copper rod of diameter 5 mm is heated by the flow of current. The surface of the rod is maintained at 175°C while it is dissipating heat by convection ($h = 150$ W/m$^2$ K) into the ambient air at 25°C. If the rod is covered with a 1 mm thick coating ($k = 0.6$ W/m K), will the heat loss from the rod increase or decrease?

(*Ans.* $r_c = 4$ mm, heat loss increases)

2.12 Determine the steady-state temperature distribution and the radial heat flow rate for a length $L$ in a hollow cylinder of inside radius $r_i$ and outside radius $r_o$, in which heat is generated at a constant rate of $q_G$ W/m$^3$, while the inside and outside surfaces are maintained constant at uniform temperatures $T_i$ and $T_o$ respectively.

$$\left( Ans. T = -\frac{q_G}{4k} r^2 + \frac{(T_o - T_i) + (q_G / 4k)\,(r_o^2 - r_i^2)}{\ln\,(r_o / r_i)} \right.$$

$$\ln r + \left( T_1 + \frac{q_G r_i^2}{4k} \right) - \left[ (T_o - T_i) + \frac{q_G (r_o^2 - r_i^2)}{4k} \right]$$

$$\left. \frac{\ln r_i}{\ln\,(r_o / r_i)} \text{ and } Q = \frac{T_i - T_o}{\dfrac{\ln\,(r_o / r_i)}{(2\pi\,k\,L)}} \right)$$

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2.13 Estimate the rate of evaporation of liquid oxygen from a spherical container with 1.8 m inner diameter covered with 30 cm of asbestos insulation. The temperatures of the inner and outer surfaces of the insulation are – 183°C and 0°C respectively. The boiling point of oxygen is – 183°C and the latent heat of vaporisation is 212.5 kJ/kg. The thermal conductivity of insulation is 0.157 and 0.125 W/m K at 0°C and – 185°C respectively. Assume that the thermal conductivity of the wall varies as

$$k = k_o + (k_1 - k_o)\left(\frac{T - T_o}{T_i - T_o}\right)$$

(*Ans.* 0.0055 kg/s)

2.14 A current of 200 amp is passed through a stainless steel wire 0.25 cm in diameter. The resistivity of steel may be taken as 70 Ω cm, and the length of wire is 1 m. If the outer surface temperature of the wire is maintained at 180°C, calculate the centre temperature. Assume $k$ for stainless steel as 30 W/m K.

(*Ans.* 195.2°C)

2.15 A turbine blade, 6.25 cm long, cross-sectional area 4.5 cm$^2$, perimeter 12 cm, is made of stainless steel ($k$ = 26.16 W/m K). The temperature of the root is 500°C. The blade is exposed to a hot gas at 800°C, and the average heat transfer coefficient is 0.465 kW/m$^2$ K. Determine the temperature and the rate of heat flow at the root of the blade. Assume that the tip is insulated.

(*Ans.* 243 W)

2.16 A 10 mm cable is to be laid in an atmosphere of 20°C ($h_a$ = 8.49 W/m$^2$ K). The surface temperature of the cable is likely to be 65°C due to heat generated within. Find the rate of heat loss (a) with critical radius of insulation, (b) without insulation.

(*Ans.* (a) 18.95 W/m, (b) 12.09 W/m)

2.17 A cylindrical transformer coil made of insulated copper wire has an inner diameter of 16 cm and an outer diameter of 24 cm. Sixty percent of the total construction of the coil is copper and the rest insulation. The density of the current in the conductor is 190 A/cm$^2$. The specific resistance of copper is 210 × 10$^{-6}$ W cm$^2$/m. The heat transfer coefficient on both surfaces of the coil, which are cooled by air at 20°C, is 23.26 W/m$^2$ K. The thermal conductivity of the coil is 0.35 W/m K. Find the maximum temperature in the coil. Assume the coil as a plane wall with the thickness $2l$ = 4 cm.

(*Ans.* 85.4°C)

2.18 An IC engine cylinder carries copper fins with an inner diameter of 10 cm and an outer diameter of 12.5 cm. The cylinder wall and the ambient temperatures are 180°C and 36°C respectively and the convective heat transfer coefficient may be taken as 69.77 W/m$^2$ K. The thermal conductivity and density of copper are 383.7 W/m K and 8800 kg/m$^3$ respectively. If the fins are designed to obtain maximum heat transfer for a given mass, find (a) the rate of heat transfer per fin, if the heat transfer is independent of fin material and (b) the saving in mass in kilogram per fin if aluminium were used in place of copper. Take the thermal conductivity and density of aluminium as 203.5 W/m K and 2670 kg/m$^3$ respectively.

(*Ans.* (a) 55.5 W/fin
(b) 42.7% saving in mass)

2.19 An electric motor is to be connected by a horizontal steel shaft of 25 mm diameter to the impeller of a pump circulating liquid metal at a temperature of 540°C. If the temperature of the electric motor is to be limited to a maximum value of 52°C with the ambient air at 27°C, what length of shaft should be specified between the motor and the pump? Take $k$ for steel = 42.56 W/m K and $h$ = 40.7 W/m$^2$ K.

(*Ans.* $l$ = 30 cm)

2.20 The two ends of a thin circular rod of diameter $D$ at $x$ = 0 and $x$ = $L$ are maintained at temperatures $T_o$ and $T_L$ respectively, while heat is generated at a uniform rate of $q_G$ W/m$^3$. Determine the steady-state

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temperature distribution in the rod for the cases when (a) the lateral surface of the rod is insulated and (b) the lateral surface dissipates heat by convection into a medium at temperature $T_L$ with a heat transfer coefficient $h$.

2.21 A 1 mm diameter electrical wire is covered with a 2-mm thick layer of plastic insulation ($k$ = 0.5 W/m K). The wire is surrounded by air with an ambient temperature of 25°C and $h$ = 10 W/m$^2$ K. The wire temperature is 100°C. Determine the rate of heat dissipated from the wire per unit length with and without insulation. Find the radius of insulation when the rate of heat dissipation is maximum. What is the maximum value of this heat dissipation?

(*Ans.* $Q_{\text{with insul}}$ = 10.90 W/m, $Q_{\text{without insul}}$ = 2.36 W/m, $(r_o)_{cr}$ = 50 mm, $Q_{\max}$ = 42.1 W/m

2.22 An electrically heated sphere of diameter 6 cm is exposed to the ambient air at 25°C with a convection heat transfer coefficient 20 W/m$^2$ K. The surface of the sphere is maintained at 125°C. Calculate the rate of heat loss (a) when the sphere is uninsulated and (b) when the sphere is insulated for maximum heat loss. (*Ans.* 22.6 W, 44.4 W)

2.23 Steam in a heating system flows through tubes whose outer diameter is 3 cm and whose walls are maintained at a temperature of 120°C. Circular aluminium fins ($k$ = 180 W/m K) of outer diameter 6 cm and constant thickness 2 mm are attached to the tube, as shown in Fig. P-2.23. The space between the fins is 3 mm, and thus there are 200 fins per meter length of the tube. Heat is transferred to the surrounding air at 25°C, with a combined heat transfer coefficient of 60 W/m$^2$ K. Determine the increase in heat transfer due to addition of fins.

(*Ans.* 4783 W/m)

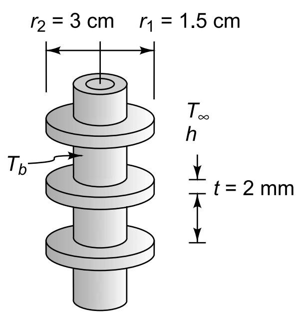


**Fig.** P-2.23

2.24 A stainless steel ($k$ = 20 W/m K) fin has a circular cross-sectional area with a diameter of 2 cm and a length of 10 cm. The fin is attached to a wall that has a temperature of 300°C. The fluid surrounding the fin has an ambient temperature of 50°C and the heat transfer coefficient is 10 W/m$^2$ K. The end of the fin is insulated. Determine (a) the rate of heat dissipation from the fin, (b) the temperature at the tip, (c) the rate of heat transfer from the wall area covered by the fin if the fin is not used and (d) the heat transfer rate from the same fin geometry if the stainless steel fin is replaced by a fictitious fin with infinite thermal conductivity.

(*Ans.* (a) 11.96 W, (b) 212°C, (c) 0.785 W, 15.2% less (d) 15.71 W, 24% more)

2.25 A long stainless steel tool bar of 20 mm × 20 mm cross-section is perfectly insulated on three sides and is maintained at a temperature of 400°C on the remaining side. Determine the maximum temperature in the bar when it is conducting a current of 1000 A. The thermal and electrical conductivity of stainless steel may be taken as 16 W/m K and 1.5 × 10$^4$ (Ω cm)$^{-1}$ and the heat flow at the ends may be neglected.

(*Ans.* $q_G$ = 4167 kW/m$^3$, $T_{\max}$ = 452.1°C)

2.26 Derive an equation giving the temperature at the centre of a circular rod conducting electric current in terms of the current density, the wall temperature and the material properties. What is the centre temperature of a stainless seel ($k$ = 16 W/m K, $\rho$ = 0.67 × 10$^{-4}$ Ω cm) rod of 20 mm diameter with an outer temperature of 400°C when conducting 1000 A?

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$$\left( Ans.\ T = \frac{q_G\, r^2}{4k} + T_w, T_o = 410.6°\text{C} \right)$$

2.27 A current of 200 amperes is passed through a stainless steel ($k$ = 19 W/m K) wire 3 mm in diameter. The resistivity of steel is 70 Ω cm and the length of the wire is 1 m. The wire is submerged in a liquid at 110°C and experiences a convection heat transfer coefficient of 4 kW/m$^2$ K. Calculate the centre-line temperature of the wire.

(*Ans.* 231.6°C)

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# Transient Heat Conduction

# 3

So far we have studied only steady-state heat conduction. After the heat transfer process is initiated some time must elapse before steady-state is reached. During this transient period the temperature changes and the analysis must take into account changes in the associated internal energy. Transient heat flow is of great practical importance in industrial heating and cooling.

In addition to unsteady heat flow when the system undergoes a transition from one steady state to another, there are also engineering problems involving periodic variations in heat flow and temperature e.g., periodic heat flow in a building between day and night and in regenerators, where the matrix is alternately heated and cooled. In periodic heat flow systems, the temperature at a point varies periodically or cyclically.

We shall first analyse the problems where the temperature is uniform throughout the system at any instant and it varies only with time. This type of analysis is called the *lumped-heat-capacity method*. Later, we shall deal with problems of unsteady heat flow where temperature not only depends on time but also varies with space coordinates.

## 3.1 LUMPED CAPACITANCE METHOD FOR BODIES OF INFINITE THERMAL CONDUCTIVITY

Even though no material in nature has an infinite thermal conductivity, many transient heat flow problems can be readily solved by assuming that the internal conductive resistance of the system is very small and the temperature within the system at any instant is uniform. This is justified when the external thermal resistance between the surface of the system and the surrounding fluid is so large compared to the internal thermal resistance that it controls the rate of heat transfer [Fig. 3.1(a)].

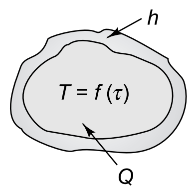

**Fig. 3.1(a)** *Cooling of a small billet*

A measure of the relative importance of the thermal resistance within a solid body is the Biot number Bi, which is the ratio of internal to external thermal resistance.

$$\text{Bi} = \frac{\text{Internal conductive resistance}}{\text{External convective resistance}} = \frac{L/k}{1/h} = \frac{hL}{k}$$

where $h$ is the average heat transfer coefficient, $L$ is a characteristic dimension obtained by dividing the volume of the body by its surface area and $k$ is the thermal conductivity of the solid body. For Bi < 0.1, i.e. when the internal resistance is less than 10% of the external resistance, the internal resistance can be ignored, and there is only one temperature at a certain instant that stands for the system as a whole, independent of space coordinates $x$, $y$ and $z$.

Let us consider the cooling of a small billet or metal casting in a quenching bath or air after it is removed from a hot furnace [Fig. 3.1(a)]. Neglecting any temperature gradient within the solid, an energy balance for the billet over a small time interval d$t$ gives.

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| | | |
|---|---|---|
| Change in internal energy of the billet during time d$t$ | = | Net heat flow from the billet to the bath or air during time d$t$ |

$$-\rho V c \, \mathrm{d}T = hA(T - T_\infty)\, \mathrm{d}t \tag{3.1}$$

where $\rho$ = density of billet, kg/m$^3$; $V$ = volume of billet, m$^3$; $c$ = specific heat of billet, J/kg $K$; $T$ = average temperature of billet, $K$; $T_\infty$ = surrounding fluid temperature, $K$; $h$ = average heat transfer coefficient, W/m$^2$ K; $A$ = surface area of billet, m$^2$ and d$T$ = temperature change, $K$, during time interval d$t$, $s$.

The minus sign indicates that the internal energy and hence temperature decrease with time. From Eq. (3.1)

$$\frac{\mathrm{d}T}{T - T_\infty} = \frac{\mathrm{d}\,(T - T_\infty)}{T - T_\infty} = -\frac{hA}{\rho c V}\mathrm{d}t \tag{3.2}$$

Let the initial temperature of the billet be $T_i$ when $t = 0$. After time $t$ has elapsed, the temperature of the billet falls to $T$. On integrating Eq. (3.2) between these limits

$$\ln \frac{T - T_\infty}{T_i - T_\infty} = -\frac{hA}{\rho c V} t$$

or

$$\frac{\mathbf{T} - \mathbf{T}_\infty}{\mathbf{T_i} - \mathbf{T}_\infty} = \mathbf{e}^{-\frac{\mathbf{hA}}{\rho \mathbf{cV}}\mathbf{t}} \tag{3.3}$$

Let Bi = $hL/k$ and Fo = Fourier number = $\dfrac{\alpha t}{L^2}$, where $\alpha$ = thermal diffusivity (m$^2$/s). The product

$$\text{Bi Fo} = \frac{hL}{k}\frac{\alpha t}{L^2} = \frac{hL}{k}\frac{k}{\rho c}\frac{t}{L^2} = \frac{ht}{\rho c L}$$

$$= \frac{hAt}{\rho c V} \text{ where } L = V/A$$

If excess temperature $\theta = T - T_\infty$, then Eq. (3.3) can be written as

$$\frac{\boldsymbol{\theta}}{\boldsymbol{\theta}_\mathbf{i}} = \mathbf{e}^{-\mathbf{Bi\ Fo}} \tag{3.4}$$

For a sphere, $\qquad L = \dfrac{(4/3)\pi r^3}{4\pi r^2} = \dfrac{r}{3}$

For a cylinder, $\qquad L = \dfrac{\pi r^2 l}{2\pi r l} = \dfrac{r}{2}$

For a cube of side $l$, $\qquad L = \dfrac{l^3}{6 l^2} = \dfrac{l}{6}$

The time-temperature history of the billet is shown in Fig. 3.1(b). The characteristic dimensionless numbers of transient heat conduction are the Fourier number, which may be called *dimensionless time*, and the Biot number. Rate of heat transfer at any time $t$ or instantaneous heat transfer rate can be obtained from Eq. (3.1).

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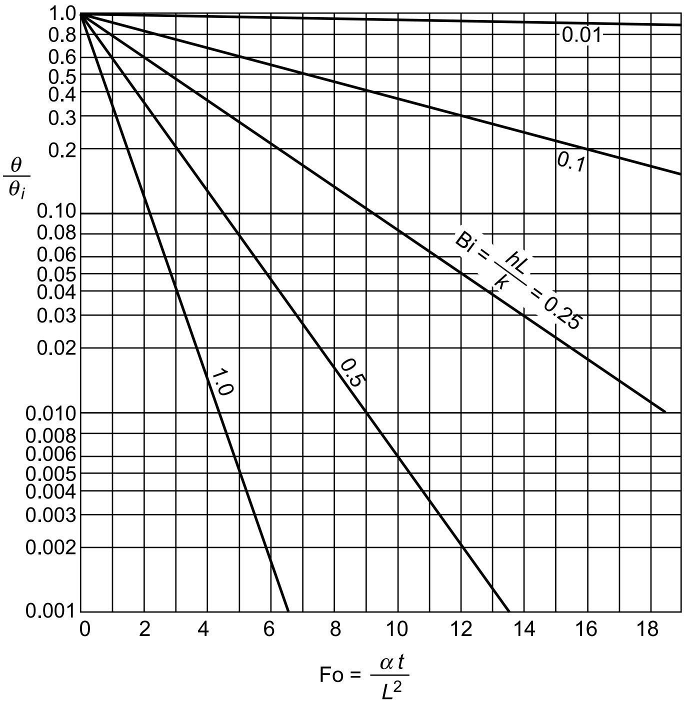


**Fig. 3.1 (b)** *Temperature-time history of the homogeneous billet*

$$Q = -\rho c V \frac{\mathrm{d}T}{\mathrm{d}t} = -\rho c V(T_i - T_\infty)\left(-\frac{hA}{\rho c V}\right)\exp\left[-\frac{hAt}{\rho c V}\right]$$

$$= \mathbf{hA\,\theta_i \exp\,(-Bi \cdot Fo)\,kW}$$

Total amount of heat transfer during a time interval $(O, t)$ is the change in internal energy of the body.

$$\Delta U = \int_0^t Q \cdot \mathrm{d}t = \int_0^t hA\,(T - T_\infty)\mathrm{d}t = \int_0^t hA(T_i - T_\infty)e^{-\mathrm{Bi}\cdot\mathrm{Fo}} \cdot \mathrm{d}t$$

$$= hA\,\theta_i \int_0^t e^{-\frac{hAt}{\rho c V}}\mathrm{d}t = hA\theta_i\,(1 - e^{-\mathrm{Bi}\cdot\mathrm{Fo}})\,\frac{t}{\mathrm{Bi}\cdot\mathrm{Fo}}$$

$$= hA\theta_i\,(1 - e^{-\mathrm{Bi}\cdot\mathrm{Fo}})\,\frac{t\,\rho c V}{hAt}$$

$$= \boldsymbol{\rho}\,\mathbf{cV\theta_i\,(1 - e^{-Bi \cdot Fo})\,kJ} \tag{3.5}$$

An electrical network analogous to the thermal network for a lumped-single-capacity system is shown in Fig. 3.2. The capacitor in this network is initially charged to the potential $T_i$ by closing the switch *S*. When the switch is opened, the energy stored in the capacitance is discharged through the resistance $1/hA$. The analogy between the thermal system and the electrical system is apparent. The thermal resistance is $R = 1/hA$, and the thermal capacitance is $C = \rho c V$, while $R_e$ and $C_e$ are the electrical resistance and capacitance respectively. To construct an electrical system that would behave exactly like the thermal system, the ratio $hA/\rho c V$ is just to be made equal to $1/R_eC_e$. In the thermal system internal energy is stored, whereas in the electrical system electric charge is stored. The flow of energy in the thermal system is heat,

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and the flow of charge is electric current. The quantity $\rho cV/hA$ is called the *time constant* of the system, since it has the dimension of time.

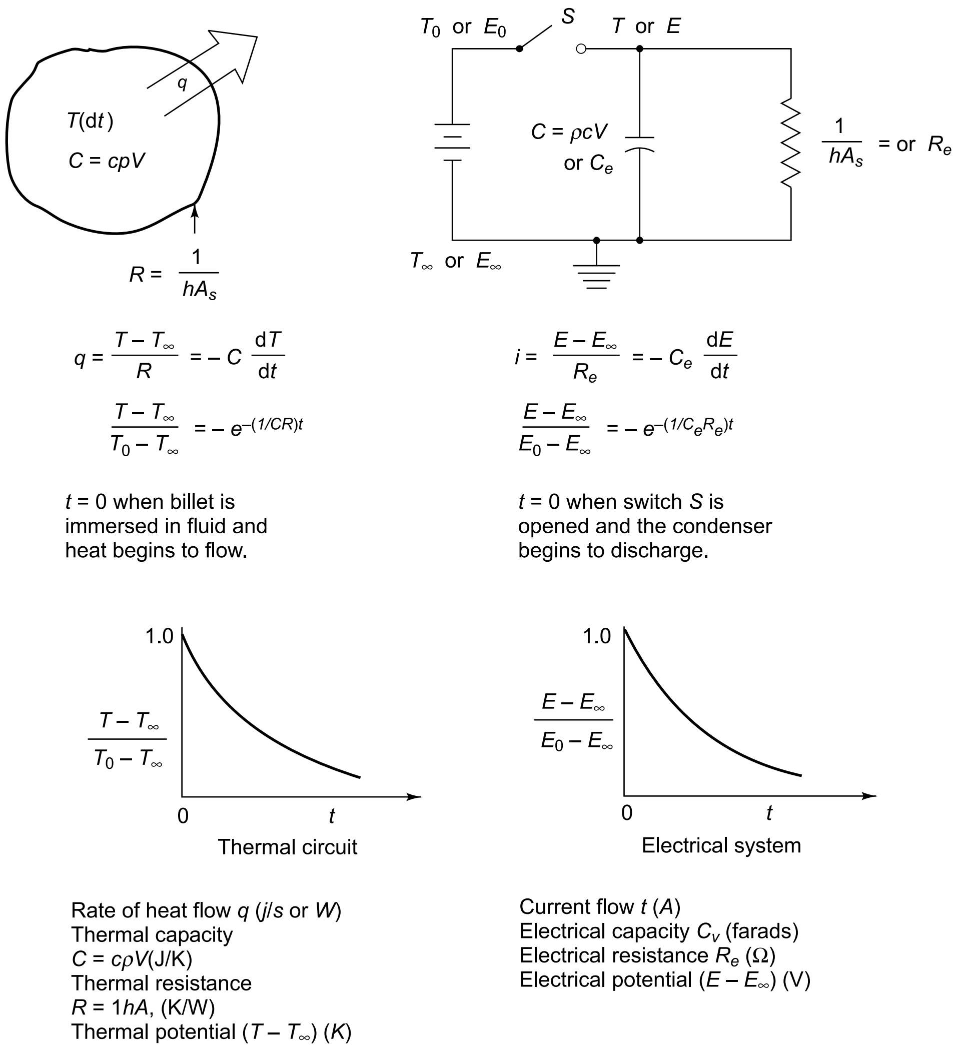


**Fig. 3.2** *Network and schematic of transient lumped-capacity system*

Let us consider a plane wall (Fig. 3.3), which is initially at a uniform temperature $T_i$ and experiences convection cooling when it is immersed in a fluid at $T_\infty < T_i$. We are interested in the temperature variation with position and time $T = T(x, t)$. This variation is a strong function of Biot number (Fig. 3.3). For Bi << 1, the temperature gradient in the solid is small and $T(x, t) \approx T(t)$, and the solid temperature remains nearly uniform. For Bi >> 1, the temperature difference across the solid is very large.

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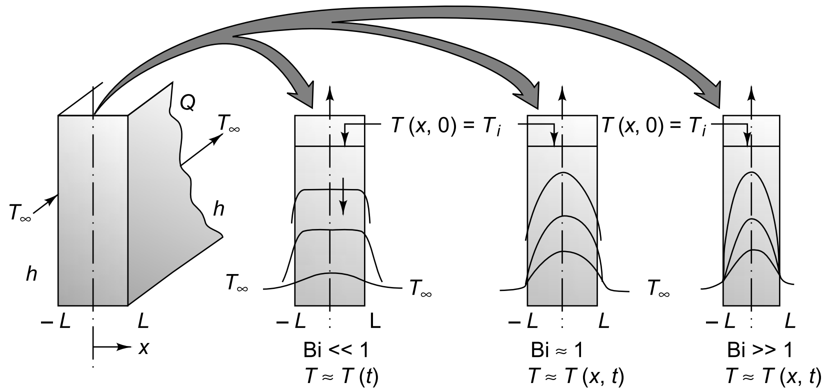


**Fig. 3.3** *Transient temperature distribution for different Biot numbers in a plane wall cooled by convection*

The inherent simplicity renders the lumped capacitance method the preferred method for solving transient conduction problems. The very first thing one should do is to calculate the Biot number. If Bi << 0.1, the error associated with the lumped capacitance method is small. This idealized assumption is possible if (a) the physical size of the body is very small, (b) $k$ of the material is very large, and (c) $h$ is very small.

### 3.1.1 Response Time of a Thermocouple

An important application of the lumped heat-capacity analysis is the measurement of temperature by a thermocouple or thermometer. A thermocouple should rapidly reach the temperature of the system which it is measuring i.e., it should come into thermal equilibrium with the system rapidly. The response time of a thermocouple is the time taken by it to reach thermal equilibrium. For a rapid response of the thermocouple, the term $hAt/\rho cV$ should be as large as possible, so that the exponential term reaches zero faster (when $T = T_\infty$). This can be achieved by decreasing the wire diameter (i.e. $V/A$), density and specific heat, or by increasing $h$. Hence, a thin wire should be used in a thermocouple for rapid response to reach thermal equilibrium quickly, particularly for measuring transient temperatures. The quantity $\rho cV/hA$, having the dimension of time, is often called the time constant of a thermocouple, $t^*$. When $t = t^*$, Eq. (3.3) becomes

$$\frac{T - T_\infty}{T_i - T_\infty} = e^{-t/t^*} = e^{-1} = \frac{1}{2 \cdot 718}$$
$$= \mathbf{0.368} \qquad (3.6)$$

Thus at the end of the time period equal to $t^*$, the temperature difference between the body and the ambient is 0.368 of the initial temperature difference. In other words, the temperature difference would be reduced by 63.2%. The time required by a thermocouple to reach 63.2% of the initial temperature difference is called the *sensitivity* of the thermocouple. The lower the value of the time constant, the better the response of the thermocouple. For all practical purposes a reading of the thermocouple should be taken after a period of $4t^*$. Thermocouple materials have nearly the same $\rho$ and $c_p$. Thus the response time essentially is a function of the wire diameter. The value of time constant varies between 0.04 s and 2.5 s for the thermocouples used in practice.

## 3.2 PLANE WALL WITH CONVECTION

Exact analytical solutions to transient conduction problems have been obtained for many simplified geometries and boundary conditions and are well documented in the literature (1–4).

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### 3.2.1 Infinite Plate with no External Thermal Resistance ($h = \infty$)

Let us consider a class of problems of heating and cooling of objects which have an appreciable internal thermal resistance. In these problems we shall simply assign a temperature for the surface of the object as $T_1$ for $t > 0$. This is to imply that in the practical situation either $T_1$ is known from actual measurement or the surface thermal resistance is negligible ($1/hA_1 = 0$ or $h = \infty$) so that $T_1$ is actually the ambient temperature $T_\infty$.

Let us consider the heating or cooling of a large plate (Fig. 3.4) of uniform thickness $L = 2\delta_1$, where $\delta_1$ = semi-thickness. Heat conduction equation in one dimension is

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{3.7}$$

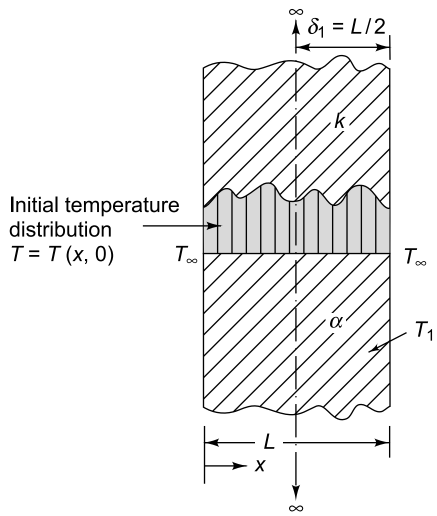


**Fig. 3.4** *Infinite solid plate with no external resistance*

The temperature distribution in the plate $T(x, t)$ is initially ($t = 0$) some arbitrary function of $x$, as $T(x, 0) = T_i(x)$. Then both surfaces $x = 0$ and $x = L$ are suddenly changed to, and maintained at, a uniform temperature $T_1$ for all $t > 0$. The problem now is to find out the temperature distribution after a certain time and the quantity of heat conducted during that time.

Let $\theta = T - T_\infty$, where $T_\infty$ is constant.

Then $\dfrac{\partial^2 \theta}{\partial x^2} = \dfrac{\partial^2 T}{\partial x^2}$ and $\dfrac{\partial T}{\partial t} = \dfrac{\partial \theta}{\partial t}$

Therefore, Eq. (3.7) becomes

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \theta}{\partial t} \tag{3.8}$$

When $t = 0$, $\theta(x) = T_1(x) - T_1$ for $0 < x < L$

At $x = 0$, $\theta = 0$ for $t > 0$

At $x = L$, $\theta = 0$ for $t > 0$

To solve the partial differential Eq. (3.8), we will use the method of separation of variables. Let

$$\theta = \theta(x, t)$$

which we can write as

$$\theta = X(x), Y(t)$$

where $X(x)$ is a function of $x$ only, and $Y$ is a function of $t$ only. Therefore, Eq. (3.8) can be written as

$$Y\frac{\partial^2 X}{\partial x^2} = X\frac{1}{\alpha}\frac{\partial Y}{\partial t}$$

or

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{\alpha Y}\frac{\partial Y}{\partial t} = -\lambda^2, \text{ the separation constant}$$

(since each side is a function of only a single variable). The negative sign of $\lambda^2$ is given to ensure a negative exponential solution in time.

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Let us take first the equation

$$\frac{1}{\alpha Y}\frac{dY}{dt} = -\lambda^2$$

or
$$\frac{dY}{Y} = -\lambda^2 \alpha \, dt$$

$$\ln Y = -\alpha\lambda^2 t + \ln A_1$$

$$Y(t) = A_1 e^{-\alpha\lambda^2 t}$$

Again
$$\frac{1}{X}\frac{d^2X}{dx^2} = -\lambda^2$$

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0$$

The characteristic equation is

$$m^2 + \lambda^2 = 0$$

$$m = \pm i\lambda$$

$$\begin{aligned} X(x) &= A_2 e^{i\lambda x} + A_3 e^{-i\lambda x} \\ &= A_2 (\cos \lambda x + i \sin \lambda x) + A_3 (\cos \lambda x - i \sin \lambda x) \\ &= (A_2 + A_3) \cos \lambda x + (iA_2 - iA_3) \sin \lambda x \\ &= B_1 \cos \lambda x + B_2 \sin \lambda x. \end{aligned}$$

$$\begin{aligned} \theta &= X(x)\, Y(t) \\ &= (B_1 \cos \lambda x + B_2 \sin \lambda x) A_1 e^{-\alpha\lambda^2 t} \end{aligned}$$

or
$$\theta = (C_1 \cos \lambda x + C_2 \sin \lambda x) e^{-\alpha\lambda^2 t} \tag{3.9}$$

At
$$x = 0, \text{ for } t > 0, \theta = 0$$

$$0 = C_1 e^{-\alpha\lambda^2 t}$$

Since $e^{-\alpha\lambda^2 t} \neq 0$,

$\therefore$
$$C_1 = 0$$

At $x = L$, for all $t > 0$, $\theta = 0$

$$0 = C_2 \sin \lambda L \, e^{-\alpha\lambda^2 t}$$

Since $C_2 \neq 0$, $\sin \lambda L = 0$

$\therefore$
$$\lambda L = n\pi$$

or
$$\lambda = \frac{n\pi}{L} \text{ where } n = 1, 2, 3, \ldots$$

($n \neq 0$, since if $n = 0$, $\lambda = 0$, $\theta = 0$, no solution)

The temperature distribution, Eq. (3.9), becomes

$$\theta = \sum_{n=1}^{\infty} e^{-(n\pi/L)^2 \alpha\tau} C_n \sin \frac{n\pi}{L} x \tag{3.10}$$

To satisfy the initial condition, at $t = 0$, $\theta = \theta_i$

$$\theta_i(x) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi}{L} x \tag{3.11}$$

This is a Fourier sine-series expansion of arbitrary function $\theta_i(x)$.

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$$C_n = \frac{2}{L}\int_0^L \theta_i(x)\sin\frac{n\pi}{L}x\,dx$$

The complete solution is

$$\theta = \sum_{n=1}^{\infty} C_n\, e^{-\left(\frac{n\pi}{L}\right)^2 \alpha t}\sin\frac{n\pi}{L}x$$

$$= \frac{2}{L}\sum_{n=1}^{\infty} e^{-(n\pi)^2 F_o}\sin\frac{n\pi}{L}x\int_0^L \theta_i(x)\sin\frac{n\pi}{L}x\,dx \tag{3.12}$$

This expression gives the temperature distribution in the slab as a function of time and depends on the specified initial temperature distribution.

Let us consider the special case in which a uniform initial temperature distribution, $T_i(x) = T_1$, exists throughout the thickness of the slab. The slab is initially heated to a uniform temperature $T_i$ and then dropped to a certain fluid medium with its temperature at $x = 0$ and $x = L$ kept fixed at $T_1$.

$$\theta_i(x) = T_i - T_1$$

$$\theta = \frac{2}{L}\theta_i\sum_{n=1}^{\infty} e^{-\left(\frac{n\pi}{L}\right)^2 \alpha t}\sin\frac{n\pi}{L}x\int_0^L \sin\frac{n\pi}{L}x\,dx$$

$$\frac{T - T_1}{T_i - T_1} = \frac{2}{L}\sum_{n=1}^{\infty} e^{-\left(\frac{n\pi}{L}\right)^2 \alpha t}\sin\frac{n\pi}{L}x\cdot\left(\frac{2L}{\pi n}\right)$$

$$= \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}e^{-\left(\frac{n\pi}{L}\right)^2 \alpha t}\sin\frac{n\pi}{L}x \tag{3.13}$$

The instantaneous rate at which heat is conducted

$$Q = -kA\frac{d\theta}{dx}$$

$$Q = -\frac{4kA}{L}(T_i - T_1)\sum_{n=1}^{\infty} e^{-\left(\frac{n\pi}{L}\right)^2 \alpha t}\cos\frac{n\pi}{L}x$$

$$\frac{Q}{A} = \frac{4}{\pi}\frac{kL}{\alpha}(T_1 - T_i)\sum_{n=1}^{\infty}\frac{1}{n^2}\left[1 - e^{-\left(\frac{n\pi}{L}\right)^2 \alpha t}\right]\cos\frac{n\pi}{L} \tag{3.14}$$

### 3.2.2 Infinite Solid Plate with Both Internal and External Resistances

Let us now consider the cases of heating and cooling of solids in which both the internal and surface resistances are present e.g., the quenching of a heated solid in a liquid bath. Here the temperature of the surrounding fluid is suddenly changed to, and maintained at, some temperature different from the initial solid temperature.

The convective heating or cooling of a large plate of uniform thickness $2l$ (Fig. 3.5) is considered. The plate is initially at a uniform temperature $T_i$ at $t = 0$. The plate is suddenly exposed to, or immersed in, a large mass of fluid at $T_\infty$, for all $t > 0$, with $T_\infty$ and $h$ remaining uniform throughout the heating or cooling period.

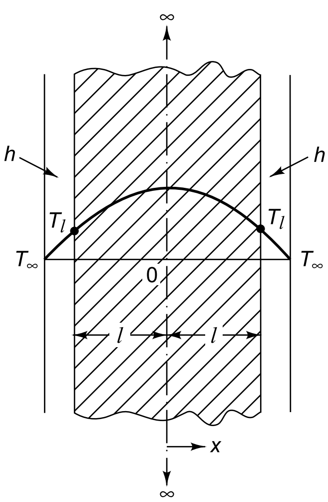


**Fig. 3.5** *Infinite solid plate: Temperature profile*

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In an effort to simplify the problem, we select the origin of the $x$-axis at the centre of the plate and thereby take advantage of the temperature symmetry about $x = 0$. The temperature-time history must then satisfy the equation $T = T(x, t)$ given by

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{3.15}$$

Let $\theta = T - T_\infty$, where $T_\infty$ is constant. Eq. (3.15) can be written as

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \theta}{\partial t} \tag{3.16}$$

To solve this equation, we will use the method of separation of variables. We can write

$$\theta = X(x)\, Y(t) \tag{3.17}$$

where $X(x)$ is a function of $x$ only, and $Y(t)$ is a function of $t$ only. Then

$$\frac{\partial \theta}{\partial x} = Y\frac{\partial X}{\partial x} \quad \text{and} \quad \frac{\partial \theta}{\partial t} = X\frac{\partial Y}{\partial t}$$

$$\frac{\partial^2 \theta}{\partial x^2} = Y\frac{\partial^2 X}{\partial x^2}$$

On substitution in Eq. (3.8),

$$Y\frac{\partial^2 X}{\partial x^2} = \frac{1}{\alpha}X\frac{\partial Y}{\partial t}$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{\alpha Y}\frac{\partial Y}{\partial t} = -\lambda^2 \text{ (say)} \tag{3.18}$$

Since each side of Eq. (3.18) is a function of only one variable, each side will be equal to a constant, called *separation constant*, $\lambda^2$. The negative sign has been given to get negative exponential solution in time. Taking each equation

$$\frac{1}{\alpha Y}\frac{dY}{dt} = -\lambda^2$$

$$\ln Y = -\alpha\lambda^2 t + \ln A_1$$

or

$$Y(t) = A_1\, e^{-\alpha\lambda^2 t} \tag{3.19}$$

where $A_1$ is a constant.

Again,

$$\frac{1}{X}\frac{d^2 X}{dx^2} = -\lambda^2$$

$$\frac{d^2 X}{dx^2} + \lambda^2 X = 0$$

The characteristic equation, $m^2 + \lambda^2 = 0$, or, $m = \pm i\lambda$.

$$\begin{aligned} X(x) &= A_2 e^{i\lambda x} + A_3\, e^{-i\lambda x} \\ &= A_2(\cos\lambda x + i\sin\lambda x) + A_3(\cos\lambda x - i\sin\lambda x) \\ &= \cos\lambda x\,(A_2 + A_3) + \sin\lambda x(iA_2 - iA_3) \\ &= B_1\cos\lambda x + B_2\sin\lambda x \end{aligned} \tag{3.20}$$

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Therefore,

$$\theta = X(x)\, Y(t)$$
$$= A_1\, e^{-\alpha\lambda^2 t}\, (B_1 \cos \lambda x + B_2 \sin \lambda x)$$
$$= e^{-\alpha\lambda^2 t}\, (C_1 \cos \lambda x + C_2 \sin \lambda x)$$
$$\boldsymbol{\theta = (C_1 \cos \lambda x + C_2 \sin \lambda x)\, e^{-\alpha\lambda^2 t}} \tag{3.21}$$

Here the constants $\lambda$, $C_1$ and $C_2$ are to be evaluated from the initial and boundary conditions.

(a) At $t = 0$, $\quad \theta = \theta_i = T_i - T_\infty$

(b) At $x = 0$, $\quad \dfrac{\partial \theta}{\partial x} = 0 \quad$ (no heat transfer across the midplane)

(c) At $x = l$, $\quad \bar{q} = -k \left(\dfrac{\partial \theta}{\partial x}\right)_{x=1} = h\,(T_1 - T_\infty) = h\theta_1$

$$\left(\frac{\partial \theta}{\partial x}\right)_{x=1} = -\frac{h}{k}\,\theta_1$$

Using the condition (b),

$$\left(\frac{\partial \theta}{\partial x}\right)_{x=0} = e^{-\alpha\lambda^2 t}\,(-\lambda C_1 \sin \lambda x + \lambda C_2 \cos \lambda x)_x = 0$$

$$\lambda C_2 \cos \lambda x = 0$$

or $$\lambda C_2 = 0, \quad \text{i.e. } C_2 = 0$$

The solution of Eq. (3.21) reduces to

$$\theta = C_1\, e^{-\alpha\lambda^2 t} \cos \lambda x \tag{3.22}$$

Using the condition (c),

$$\left(\frac{\partial \theta}{\partial x}\right)_{x=1} = -C_1\, e^{-\lambda^2 \alpha t}\, \lambda \sin \lambda l = -\frac{h}{k}\,\theta_1$$

$$-C_1\, e^{-\lambda^2 \alpha t}\, \lambda \sin \lambda l = \frac{h}{k}\,[-C_1\, e^{-\alpha\lambda^2 t} \cos \lambda l]$$

$$\lambda \sin \lambda l = \frac{h}{k} \cos \lambda l$$

$$\cot \lambda l = \frac{\lambda k}{h} \tag{3.23}$$

or $$\cot \lambda l = \lambda l\, \frac{k}{h\, l}$$

or $$\boldsymbol{\cot \lambda l = \frac{\lambda l}{\mathrm{Bi}}} \tag{3.24}$$

The points of intersection of the curves $Y = \cot \lambda l$ and $Y = \lambda l/\mathrm{Bi}$ will give the values of $\lambda l$ (Fig. 3.6). For a given $l$, we can determine $\lambda_1, \lambda_2, \lambda_3, \cdots$. The equation $\cot \lambda l = \lambda l/\mathrm{Bi}$ is satisfied for an infinite succession of values of the parameter $\lambda l$, so that for a given $\lambda$, the equation defines the values of $\lambda$. This succession of values of $\lambda$, called *eigen values*, will be denoted by $\lambda_n$ which depend on the Biot number.

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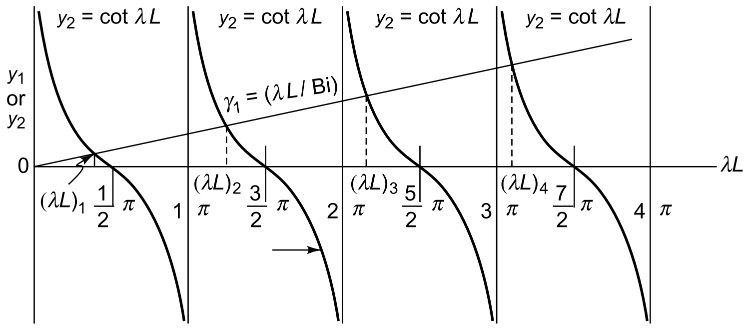


**Fig. 3.6** *Graphic solution of the transcendental equation* $\cot \lambda L = \dfrac{\lambda L}{\mathrm{Bi}}$

The temperature distribution, Eq. (3.22), thus becomes

$$\theta = \sum_{n=1}^{\infty} C_n\, e^{-\lambda_n^2 \alpha t} \cos \lambda_n x \tag{3.25}$$

where $\lambda_n$ is the $n$th root of the transcendental equation

$$\cot \lambda_n l = \frac{\lambda_n l}{\mathrm{Bi}}$$

or $$\boldsymbol{\lambda_n l \tan \lambda_n l - \mathrm{Bi} = 0} \tag{3.26}$$

The value of $C_n$ for each value of $\lambda_n$ is next to be determined. Using the condition (a), at $t = 0$, $\theta = \theta_i$

$$\theta_i = \sum_{n=1}^{\infty} C_n \cos \lambda_n x \tag{3.27}$$

If $\lambda_n$'s were simple integers 1, 2, 3, ..., then

$$\theta_i = \sum_{n=1}^{\infty} C_n \cos \lambda_n x$$

would have been a Fourier cosine series expansion.

$$\theta_i = C_1 \cos x + C_2 \cos 2x + C_3 \cos 3x + \ldots + C_n \cos nx + \ldots$$

and $C_n$ would have been

$$C_n = \frac{2}{1} \int_0^1 \theta_i \cos \lambda_n x \, \mathrm{d}x, \quad \lambda_n = 1, 2, 3, \ldots$$

But $\lambda_n$'s are not integers, and are the roots of the trigonometric equation

$$(\lambda_n l) \tan (\lambda_n l) - \mathrm{Bi} = 0$$

So some other method has to be found out to determine $C_n$'s

Let us multiply both sides of Eq. (3.27) by $\cos \lambda_m x \, \mathrm{d}x$, where $m \neq n$.

$$\theta_i \int_0^1 \cos \lambda_m x \, \mathrm{d}x = \sum_{n=1}^{\infty} C_n \int_0^1 \cos \lambda_m x \cos \lambda_n x \, \mathrm{d}x \tag{3.28}$$

$$\text{Integrand of L.H.S.} = \int_0^1 \cos \lambda_m x \, \mathrm{d}x = \left( \frac{\sin \lambda_m x}{\lambda_m} \right)_0^1 = \frac{\sin \lambda_m l}{\lambda_m}$$

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$$\text{Integrand of R.H.S.} = \frac{1}{2}\int_0^1 [\cos(\lambda_m x + \lambda_n x) + \cos(\lambda_m x - \lambda_n x)]\,dx$$

$$= \frac{1}{2}\left[\frac{\sin(\lambda_m + \lambda_n)x}{\lambda_m + \lambda_n} + \frac{\sin(\lambda_m - \lambda_n)x}{\lambda_m - \lambda_n}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{(\lambda_m - \lambda_n)[\sin\lambda_m x\cos\lambda_n x + \cos\lambda_m x\sin\lambda_n x] + (\lambda_m + \lambda_n)[\sin\lambda_m x\cos\lambda_n x - \cos\lambda_m x\sin\lambda_n x]}{\lambda_m^2 - \lambda_n^2}\right]_0^1$$

$$= \frac{1}{\lambda_m^2 - \lambda_n^2}(\lambda_m \sin\lambda_m l\cos\lambda_n l - \lambda_n\cos\lambda_m l\sin\lambda_n l] \tag{3.29}$$

Now, $(\lambda_n l)\tan\lambda_n l = \text{Bi} = (\lambda_m l)\tan\lambda_n l$

$$\lambda_n\frac{\sin\lambda_n l}{\cos\lambda_n l} = \lambda_m\frac{\sin\lambda_m l}{\cos\lambda_m l}$$

$\therefore$ $\lambda_m\sin\lambda_m l\cos\lambda_n l = \lambda_n\sin\lambda_n l\cos\lambda_m l$

Therefore, the integrand of R.H.S., Eq. (3.29), will be zero, unless $m = n$. Putting $\lambda_m = \lambda_n$, Eq. (3.28) becomes

$$\theta_i\frac{\sin\lambda_n l}{\lambda_n} = C_n\int_0^1\cos^2\lambda_n x\,dx = C_n\int_0^1\frac{1+\cos 2\lambda_n x}{2}\,dx \tag{3.30}$$

$$= C_n\left(\frac{1}{2} + \frac{\sin 2\lambda_n l}{4\lambda_n}\right)$$

$$= \frac{1}{2}C_n\left(l + \frac{2\sin\lambda_n l\cos\lambda_n l}{2\lambda_n}\right)$$

or

$$\theta_i\frac{\sin\lambda_n l}{\lambda_n} = C_n\frac{1}{2\lambda_n}(\lambda_n l + \sin\lambda_n l\cos\lambda_n l)$$

$$C_n = \frac{2\,\theta_i\sin\lambda_n l}{\lambda_n l + \sin\lambda_n l\cos\lambda_n l} \tag{3.31}$$

For convenience, let $\delta_n = \lambda_n l$, then

$$(\lambda_n l)\tan(\lambda_n l) = \delta_n\tan\delta_n = \text{Bi}$$

Therefore,

$$\theta = \sum_{n=1}^{\infty} C_n\, e^{-\lambda_n^2\alpha t}\cos\lambda_n x$$

or

$$\theta = \sum_{n=1}^{\infty} e^{-\delta_n^2\frac{\alpha t}{l^2}}\frac{2\theta_i\sin\delta_n}{\delta_n + \sin\delta_n\cos\delta_n}\cos\frac{\delta_n}{l}x$$

The temperature distribution is thus given by

$$\frac{\theta}{\theta_i} = \sum_{n=1}^{\infty} e^{-\delta_n^2\,\text{Fo}}\frac{2\sin\delta_n}{\delta_n + \sin\delta_n\cos\delta_n}\cos\frac{\delta_n}{l}x \tag{3.32}$$

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$$= \frac{T - T_\infty}{T_i - T_\infty}$$

Cumulative heat loss from the infinite slab is obtained from the Fourier heat conduction equation

$$dQ = -kA\left(\frac{\partial \theta}{\partial x}\right)_{x=l} dt \tag{3.33}$$

Differentiating Eq. (3.32) with respect to $x$ and putting the limit $x = l$,

$$\left(\frac{\partial \theta}{\partial x}\right)_{x=l} = 2\theta_i \sum_{n=1}^{\infty} e^{-\delta_n^2 \text{ Fo}} \frac{\sin \delta_n}{\delta_n + \sin \delta_n \cos \delta_n} \frac{\delta_n}{l}\left(-\sin \frac{\delta_n}{l} x\right)_{x=l}$$

$$= -\frac{2\theta_i}{l} \sum_{n=1}^{\infty} e^{-\delta^2{}_n \text{Fo}} \frac{\delta_n \sin^2 \delta_n}{\delta_n + \sin \delta_n \cos \delta_n}$$

Substituting in Eq. (3.33) and integrating

$$\frac{Q}{A} = \frac{2k\theta_i}{l} \int_0^t \sum_{n=1}^{\infty} e^{-\delta_n^2 \text{Fo}} \frac{\delta_n \sin^2 \delta_n}{\delta_n + \sin \delta_n \cos \delta_n} dt$$

$$\frac{Q}{A} = \frac{2k\theta_i}{l} \sum_{n=1}^{\infty} \frac{\delta_n \sin^2 \delta_n}{\delta_n + \sin \delta_n \cos \delta_n} - \frac{l^2}{\alpha \delta_n^2}\left(1 - e^{-\delta_n^2 \frac{\alpha t}{l^2}}\right)\int_0^t e^{-\delta_n^2 \frac{\alpha t}{l^2}} dt$$

or
$$\mathbf{\frac{Q}{A} = \frac{2kl}{\alpha}\theta_i \sum_{n=1}^{\infty} \frac{\sin^2 \delta_n (1 - e^{-\delta_n^2 Fo})}{\delta_n^2 + \delta_n \sin \delta_n \cos \delta_n}} \tag{3.34}$$

In order to make Eq. (3.34) dimensionless, we note that $c\rho l T_i$ represents initial internal energy per unit area of the slab. If we denote $c\rho l(T_i - T_\infty)$ by $Q_i/A$, we get

$$\frac{Q}{Q_i} = \frac{2kl\theta_i}{\alpha c \rho l \theta_i} \sum_{n=1}^{\infty} \frac{\sin^2 \delta_n}{\delta_n^2 + \delta_n \sin \delta_n \cos \delta_n}(1 - e^{-\delta_n^2 \text{ Fo}})$$

or
$$\frac{Q}{Q_i} = \sum_{n=1}^{\infty} \frac{2\sin^2 \delta_n}{\delta_n^2 + \delta_n \sin \delta_n \cos \delta_n}(1 - e^{-\delta_n^2 \text{ Fo}}) \tag{3.35}$$

The temperature distribution in the slab $T(x, t)$ is given by Eq. (3.32).

$$\frac{\theta}{\theta_i} = \sum_{n=1}^{\infty} e^{-\delta_n^2 \text{ Fo}} \frac{2\sin \delta_n}{\delta_n + \sin \delta_n \cos \delta_n} \cos \frac{\delta_n}{l} x$$

At $x = 0$, the centre-line temperature $T_c(t)$ varying with time is given by

$$\frac{\theta_c}{\theta_i} = \sum_{n=1}^{\infty} e^{-\delta_b^2 \text{ Fo}} \frac{2\sin \delta_n}{\delta_n + \sin \delta_n \cos \delta_n} \tag{3.36}$$

At $x = l$, the surface temperature $T_1(t)$ varying with time is given by

$$\mathbf{\frac{\theta_1}{\theta_i} = \sum_{n=1}^{\infty} e^{-\delta_n^2 Fo} \frac{2\sin \delta_n \cos \delta_n}{\delta_n + \sin \delta_n \cos \delta_n}} \tag{3.37}$$

The results using Eqs (3.36), (3.37) and (3.35) have been calculated for different cases and plotted in the form of charts for rapid use, by Gröber and Erk, Gurney–Lurie, Shack, Adams–Williamson, Heisler and others (1 – 4). Heisler's charts for $\theta_c/\theta_i$ are given in Fig. 3.7(a).

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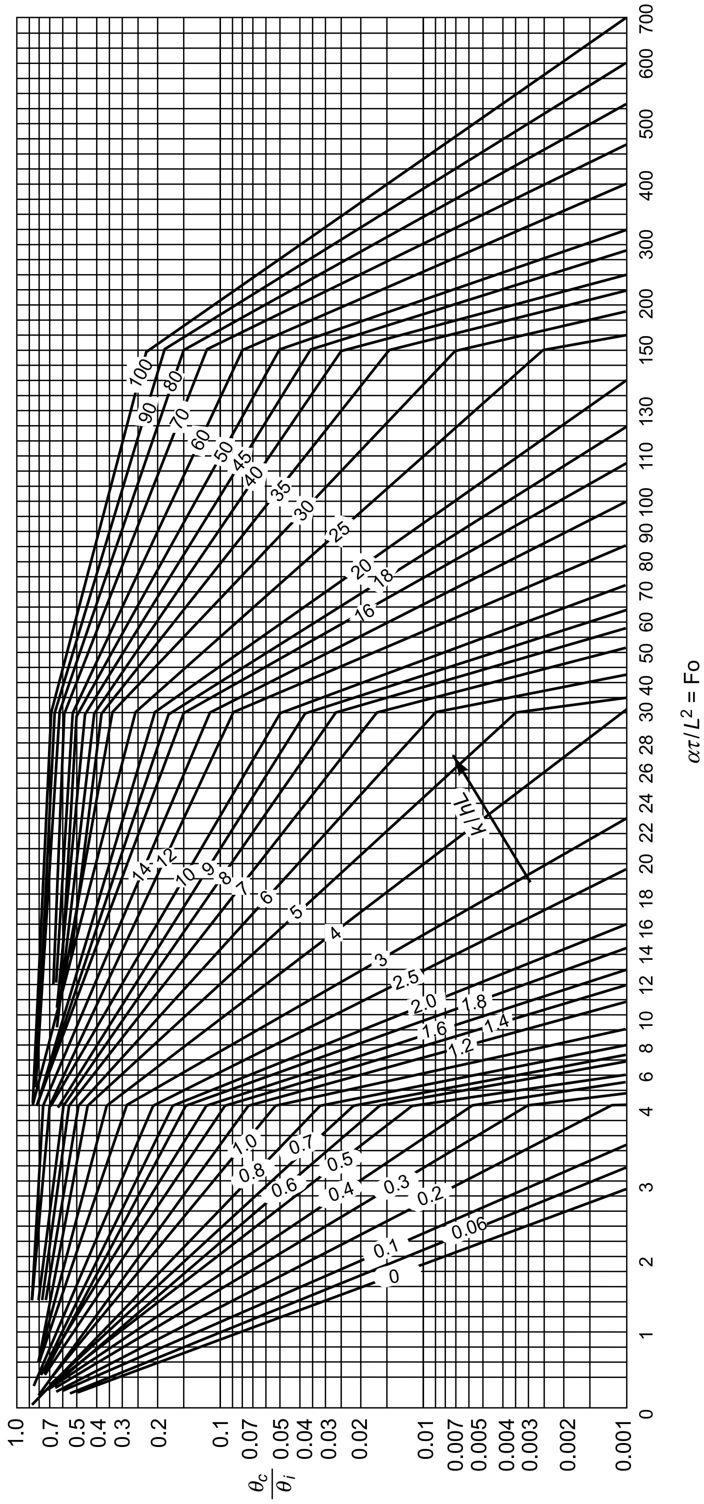


**Fig. 3.7(a)** *Centre-line temperature for an infinite plate of thickness 2L*

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The temperature at any distance $x$ from the mid-plane can be obtained from position-correction chart drawn with $\theta/\theta_c$ versus 1/Bi for various values of $x/l$ or $\xi$, so that

$$\frac{\theta}{\theta_c} \times \frac{\theta_c}{\theta_i} = \frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty}$$

from which temperature $T$ at any distance $x$ from the midplane can be estimated [Fig. 3.7(b)]. It is shown in an expanded scale in Fig. 3.7 (d).The cumulative heat losses $Q/Q_{\text{i}}$ for different values of Biot number are given in Fig. 3.7(c).

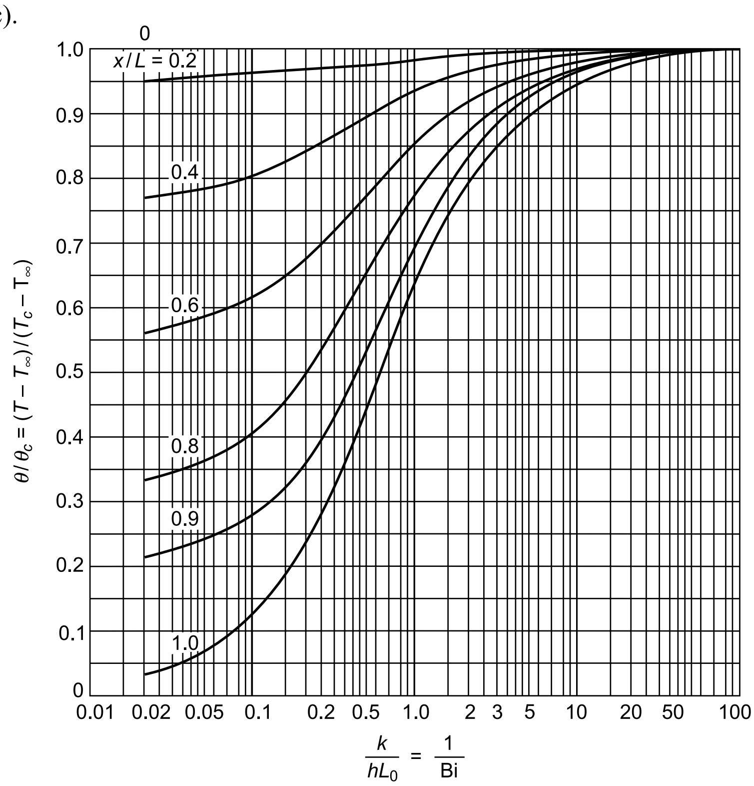


**Fig. 3.7(b)** *Temperature as a function of centre temperature in an infinite plate of thickness 2L*

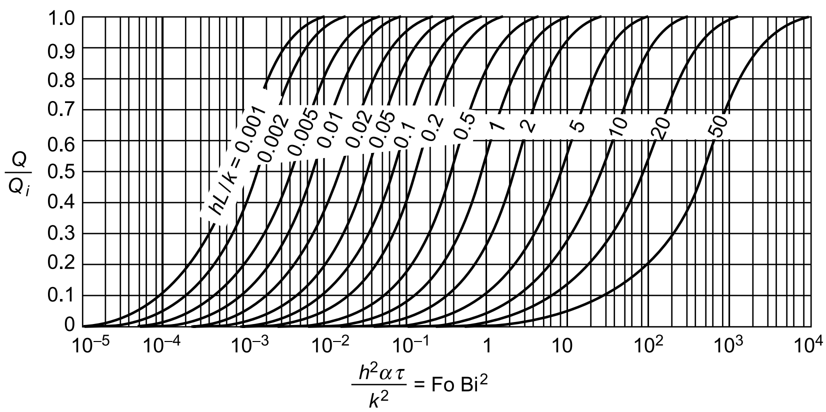


**Fig. 3.7(c)** *Heat loss of an infinite plate of thickness 2L with time*

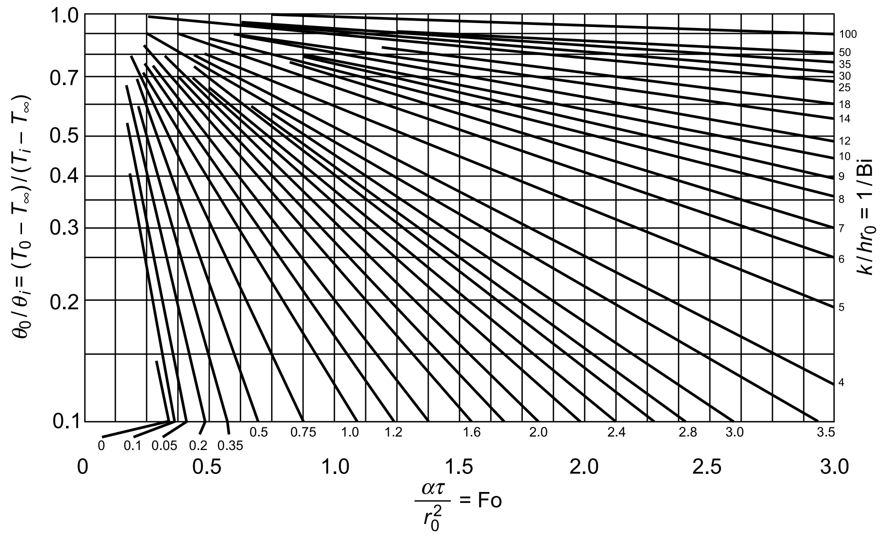


**Fig. 3.7(d)** *Continued: Expanded scale for 0 < Fo < 3.*

## 3.3 INFINITE CYLINDER AND SPHERE WITH CONVECTION

Similar to the solution of transient heat conduction in an infinite plate, the temperature distribution in an infinite cylinder at any instant with convective heating or cooling can be derived. Initially the temperature is uniform at $T_i$ throughout the cylinder (Fig. 3.8). The cylinder is suddenly immersed in a fluid at constant temperature $T_\infty$ with uniform heat transfer coefficient $h$. At any instant $t$, $\theta = T - T_\infty$ at a certain distance $r$ from the central plane. There is only radial conduction. In terms of $\theta$, the temperature distribution $T(r, t)$ in cylindrical coordinates is given by

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{1}{r}\frac{\partial \theta}{\partial r} = \frac{1}{\alpha}\frac{\partial \theta}{\partial t} \tag{3.38}$$

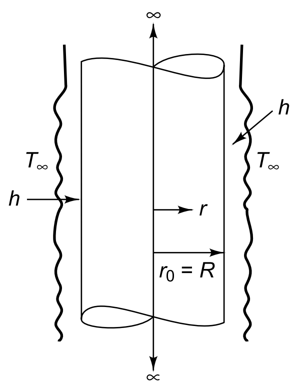


**Fig. 3.8** *Heating or cooling of an infinite cylinder by convection*

By separation of variables method, let

$$\theta = X(r)\, Y(t) \tag{3.39}$$

$$\frac{\partial^2 \theta}{\partial r^2} = Y\frac{\partial^2 X}{\partial r^2} \qquad \frac{\partial \theta}{\partial r} = Y\frac{\partial X}{\partial r} \qquad \frac{\partial \theta}{\partial t} = X\frac{\partial Y}{\partial t}$$

Substituting in Eq. (3.38),

$$Y\frac{\partial^2 X}{\partial r^2} + \frac{1}{r}Y\frac{\partial X}{\partial r} = \frac{1}{\alpha}X\frac{\partial Y}{\partial t}$$

$$\frac{1}{X}\left(\frac{\partial^2 X}{\partial r^2} + \frac{1}{r}\frac{\mathrm{d}X}{\mathrm{d}r}\right) = \frac{1}{\alpha Y}\frac{\mathrm{d}Y}{\mathrm{d}t} = -\lambda^2$$

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the separation constant

$$\frac{dY}{Y} = -\lambda^2 \alpha \, dt$$

$$Y(t) = A1\, e^{-\lambda^2 \alpha t} \tag{3.40}$$

Again
$$\frac{1}{X}\left(\frac{d^2 X}{dr^2} + \frac{1}{r}\frac{dX}{dr}\right) = -\lambda^2$$

$$\frac{d^2 X}{dr^2} + \frac{1}{r}\frac{dX}{dr} + \lambda^2 X = 0$$

Comparing this equation with

$$\mathbf{x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)\, y = 0} \tag{3.41}$$

When $n = 0$

$$\frac{d^2 y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0$$

which is Bessel function $J_0\,(x)$.

The solution of Eq. (3.41) is given by

$$X(x) = A_2 J_0\,(\lambda r) + A_3 Y_0(\lambda r) \tag{3.42}$$

Substituting Eqs (3.40) and (3.42) in Eq. (3.39)

$$\theta = [C_1 J_0\,(\lambda r) + C_2 Y_0\,(\lambda r)]\, e^{-\lambda^2 \alpha t} \tag{3.43}$$

At $t = 0,\ \theta = \theta_i$

At $t > 0$, at $r = R,\ \theta = \theta_R = T_R - T_\infty$,

$$\left(\frac{\partial \theta}{\partial r}\right)_{r=R} = -\frac{h}{k}\theta_R$$

At $r = 0,\ C_2 = 0,\ \theta = C_1 J_0\,(\lambda r)\, e^{-\lambda^2 \alpha t}$

The temperature distribution is given by [1]

$$\frac{\theta}{\theta_i} = 2\sum_{n=1}^{\infty} \frac{1}{\lambda_n R} e^{-\lambda_n^2 \alpha t} \frac{J_0\,(\lambda_n r)\, J_1\,(\lambda_n R)}{J_0^2\,(\lambda_n r) + J_1^2\,(\lambda_n R)}$$

At the centre line, $J_0(0) = 1$,

$$\frac{\theta_c}{\theta_i} = 2\sum_{n=1}^{\infty} \frac{1}{\lambda_n R} e^{-\lambda_n^2 \alpha t} \frac{J_1\,(\lambda_n R)}{J_0^2\,(\lambda_n R) + J_1^2\,(\lambda_n R)} \tag{3.44}$$

Heisler's charts for $\theta_c/\theta_i$ in infinite cylinders and the position-correction charts are given in Fig. 3.9(a) and (b) respectively. With the help of these the entire time–temperature history at any location in the cylinder can be found out. The heat transfer from or to the cylinder during time $t$ can be estimated with the help of Fig. 3.9(c).

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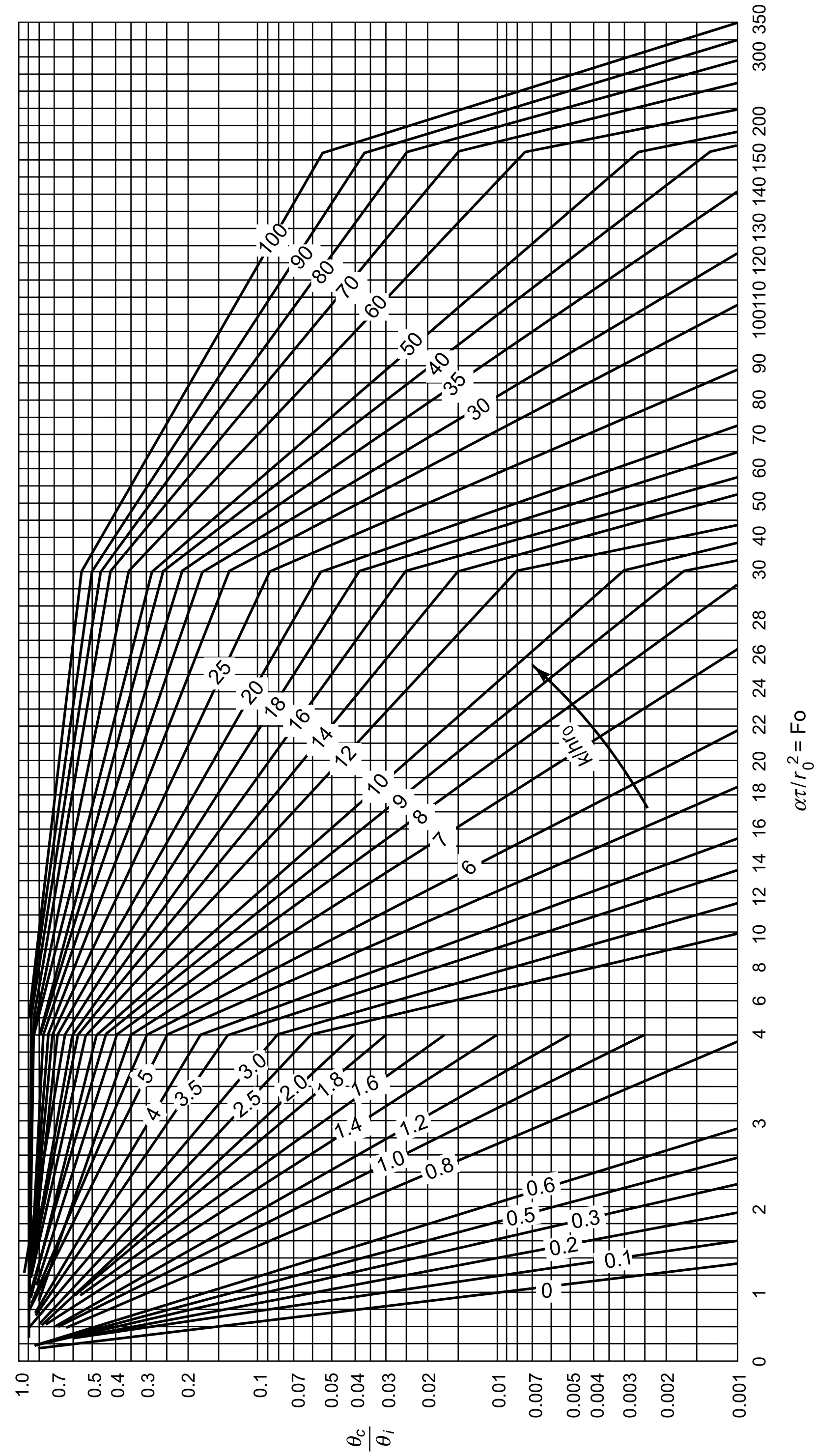


**Fig. 3.9(a)** *Axis temperature for an infinite cylinder of radius $r_0$*

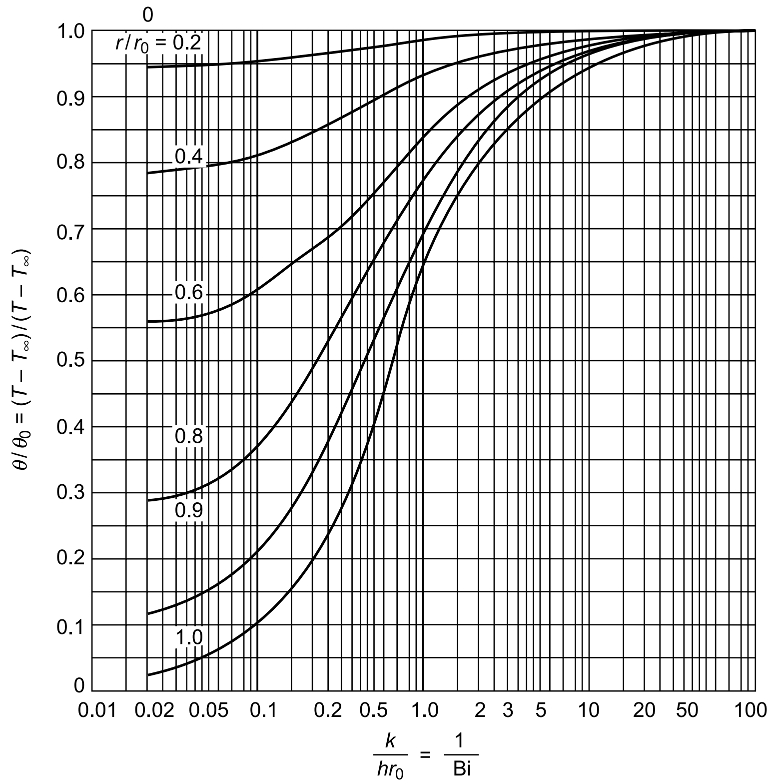


**Fig. 3.9(b)** *Temperature as a function of axis temperature in an infinite cylinder of radius $r_0$*

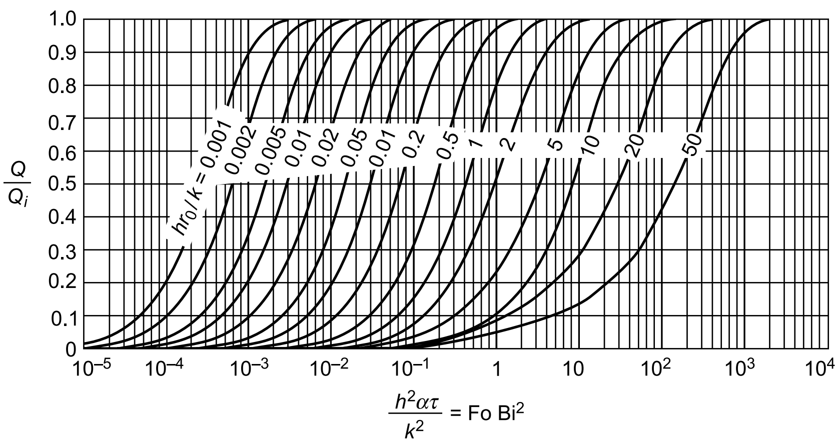


**Fig. 3.9(c)** *Heat loss $Q/Q_i$ of an infinite cylinder of radius $r_0$ with time*

Similar solutions can be obtained for a sphere of radius $R$, which have been derived by Schneider [1] by solving the temperature distribution in spherical coordinates, with Legendre's polynomials appearing in the final solution. Figure 3.10(a) and (b) show Heisler's charts for $\theta_c/\theta_i$ and position-correction charts to determine the temperature at any location in a sphere varying with time, while Fig. 3.10(c) gives the heat transfer during the time period $t$.

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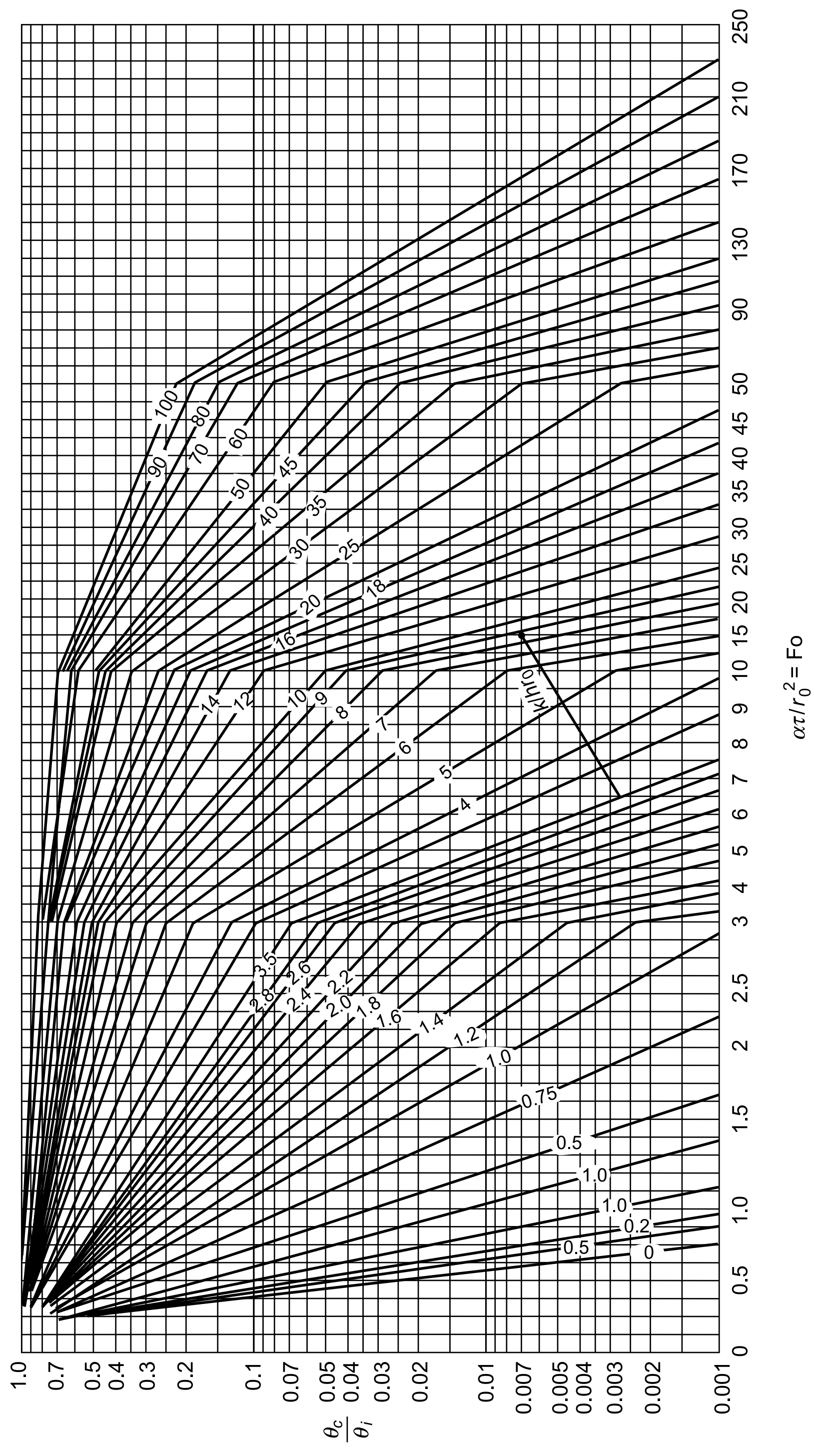


**Fig. 3.10(a)** *Centre temperature for a sphere of radius $r_0$*

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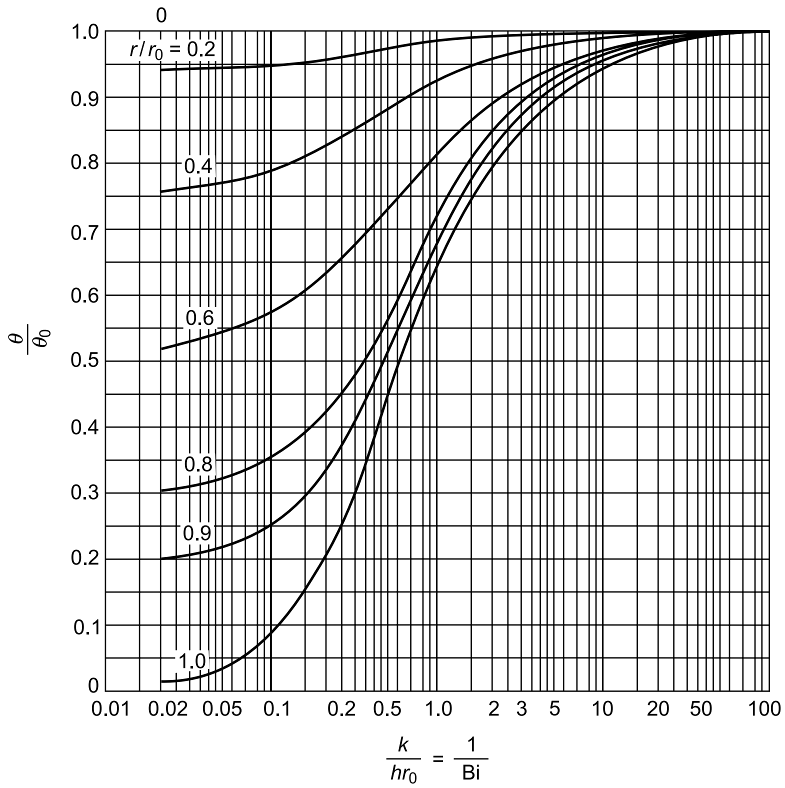


**Fig. 3.10(b)** *Temperature as a function of centre temperature for a sphere of radius $r_0$*

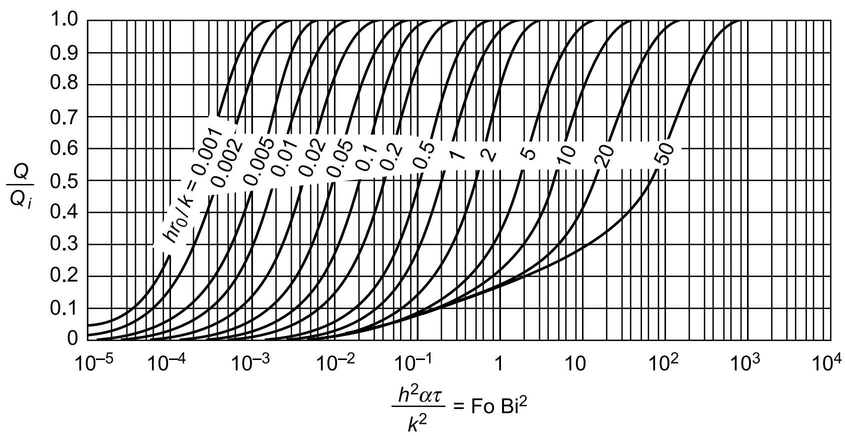


**Fig. 3.10(c)** *Heat loss $Q/Q_i$ of a sphere of radius $r_0$ with time*

### 3.3.1 One-term Approximation

The one-dimensional transient heat conduction problem can be solved exactly for any of the three geometries: plane wall, cylinder or sphere. But the solution involves infinite series, which are difficult to deal with. However, the terms in the solutions converge rapidly with increasing time, and for Fo > 0.2, keeping the first term and neglecting all the remaining terms in the series results in an error less than 2%. It is thus convenient to express the solution using this one-term approximation, which is given as follows:

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$$\text{Plane wall:}\quad \frac{\theta}{\theta_{\rm i}} = \frac{T - T_\infty}{T_{\rm i} - T_\infty} = A_1\, e^{-\delta_1^2\,\mathrm{Fo}} \cos \frac{\delta_1}{l}\, x\,,\ \mathrm{Fo} > 0.2$$

$$\text{Cylinder:}\quad \frac{\theta}{\theta_i} = A_1\, e^{-\delta_1^2\,\mathrm{Fo}}\, J_o\left(\frac{\lambda_1 r}{\lambda_0}\right),\ \mathrm{Fo} > 0.2$$

$$\text{Sphere:}\quad \frac{\theta}{\theta_i} = A_1\, e^{-\delta_1^2\,\mathrm{Fo}}\left(\frac{\sin(\lambda_1 r/r_0)}{\lambda_1 r/r_0}\right),\ \mathrm{Fo} > 0.2 \qquad (3.45)$$

where $A_1$ and $\lambda_1$ are functions of Biot number only, and their values are listed in Table 3.1 against Bi. The function $J_0$ is the zeroth-order Bessel function of the first kind, whose value can be determined from Table 3.2. Noting that $\cos 0 = J_0(0) = 1$ and the limit of $\sin x/x$ is also l, the above relations simplify to the following at the centre of the plane wall, cylinder or sphere.

**Table 3.1** *Coefficients used in the one-term approximate solution of transient one-dimensional heat conduction in plane walls, cylinders and spheres*

| *Plane slab* | | *Cylinder* | | *Sphere* | |
|---|---|---|---|---|---|
| *Bi* | $\lambda_1$ | $\lambda_1$ | $A_1$ | $A_i$ | $\lambda_i$ |
| 0.01 | 0.0998 | 1.0017 | 0.1412 | 1.0025 | 0.1730 |
| 0.02 | 0.1410 | 1.0033 | 0.1995 | 1.0050 | 0.2445 |
| 0.04 | 0.1987 | 1.0066 | 0.2814 | 1.0099 | 0.3450 |
| 0.06 | 0.2425 | 1.0098 | 0.3438 | 1.0148 | 0.4217 |
| 0.08 | 0.2791 | 1.0130 | 0.3960 | 1.0197 | 0.4860 |
| 0.1 | 0.3111 | 1.0161 | 0.4417 | 1.0246 | 0.5423 |
| 0.2 | 0.4328 | 1.0311 | 0.6170 | 1.0483 | 0.7593 |
| 0.3 | 0.5218 | 1.0450 | 0.7465 | 1.0712 | 0.9208 |
| 0.4 | 0.5932 | 1.0580 | 0.8516 | 1.0931 | 1.0528 |
| 0.5 | 0.6533 | 1.0701 | 0.9408 | 1.1143 | 1.1656 |
| 0.6 | 0.7051 | 1.0814 | 1.0184 | 1.1345 | 1.2044 |
| 0.7 | 0.7506 | 1.0918 | 1.0873 | 1.1539 | 1.3525 |
| 0.8 | 0.7910 | 1.1016 | 1.1490 | 1.1724 | 1.4320 |
| 0.9 | 0.8274 | 1.1107 | 1.2048 | 1.1902 | 1.5044 |
| 1.0 | 0.8603 | 1.1191 | 1.2558 | 1.2071 | 1.5708 |
| 2.0 | 1.0769 | 1.1785 | 1.5995 | 1.3384 | 2.0288 |
| 3.0 | 1.1985 | 1.2102 | 1.7887 | 1.4191 | 2.2889 |
| 4.0 | 1.2646 | 1.2287 | 1.9081 | 1.4698 | 2.4556 |
| 5.0 | 1.3138 | 1.2403 | 1.9898 | 1.5029 | 2.5704 |
| 6.0 | 1.3496 | 1.2479 | 2.0490 | 1.5253 | 2.6537 |
| 7.0 | 1.3766 | 1.2532 | 2.0937 | 1.5411 | 2.7165 |
| 8.0 | 1.3978 | 1.2570 | 2.1286 | 1.5526 | 2.7654 |
| 9.0 | 1.4149 | 1.2598 | 2.1566 | 1.5611 | 2.8044 |
| 10.0 | 1.4289 | 1.2620 | 2.1795 | 1.5677 | 2.8363 |
| 20.0 | 1.4961 | 1.2699 | 2.2880 | 1.5919 | 2.9857 |
| 30.0 | 1.5202 | 1.2717 | 2.3261 | 1.5973 | 3.0372 |
| 40.0 | 1.5325 | 1.2723 | 2.3455 | 1.5993 | 3.0632 |
| 50.0 | 1.5400 | 1.2727 | 2.3572 | 1.6002 | 3.0788 |
| 100.0 | 1.5552 | 1.2731 | 2.3809 | 1.6015 | 3.1102 |
| * | 1.5708 | 1.2732 | 2.4048 | 1.6021 | 3.1416 |

(Bi = $\dfrac{hL}{k}$ for a plane wall of thickness $2L$, and Bi = $\dfrac{h\,r_0}{k}$ for a cylinder or a sphere of radius $r_0$).

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**Table 3.2** *Zeroth = and first-order Bessel functions of the first kind*

| $\xi$ | $J_o(\xi)$ | $J_I(\xi)$ |
|---|---|---|
| 0.0 | 1.0000 | 0.0000 |
| 0.1 | 0.9975 | 0.0499 |
| 0.2 | 0.9900 | 0.0995 |
| 0.3 | 0.9776 | 0.1483 |
| 0.4 | 0.9604 | 0.1960 |
| 0.5 | 0.9385 | 0.2423 |
| 0.6 | 0.9120 | 0.2867 |
| 0.7 | 0.8812 | 0.3290 |
| 0.8 | 0.8463 | 0.3688 |
| 0.9 | 0.8075 | 0.4059 |
| 1.0 | 0.7075 | 0.4400 |
| 1.1 | 0.7130 | 0.4708 |
| 1.2 | 0.6711 | 0.4983 |
| 1.3 | 0.8201 | 0.5220 |
| 1.4 | 0.5669 | 0.5419 |
| 1.5 | 0.5118 | 0.5579 |
| 1.6 | 0.4554 | 0.5699 |
| 1.7 | 0.3980 | 0.5778 |
| 1.8 | 0.3400 | 0.5815 |
| 1.9 | 0.2818 | 0.5812 |
| 2.0 | 0.2239 | 0.5767 |
| 2.1 | 0.1000 | 0.5688 |
| 2.2 | 0.1104 | 0.5560 |
| 2.3 | 0.0555 | 0.5399 |
| 2.4 | 0.0025 | 0.5202 |

Plane wall: $\dfrac{\theta_c}{\theta_i} = A_1\, e^{-\delta_1^2\,\mathrm{Fo}}$

Cylinder: $\dfrac{\theta_c}{\theta_i} = A_1\, e^{-\delta_1^2\,\mathrm{Fo}}$

Sphere: $\dfrac{\theta_c}{\theta_i} = A_1\, e^{-\delta_1^2\,\mathrm{Fo}}$

Relations of Eq. (3.45) can be used to determine the temperature anywhere in the medium.

The boundary conditions and the initial conditions for all three geometries are similar. One boundary condition requires that the temperature gradient at the midplane of the plate, the axis of the cylinder and the centre of the sphere be equal to zero. Physically, this corresponds to no heat-flow at these locations.

The other boundary condition requires that the heat conducted to or from the surface be transferred by convection to or from a fluid at temperature $T_\infty$ through a uniform and constant heat transfer coefficient $\bar{h}$, or

$$h(T_w - T_\infty) = -k\left(\frac{\partial T}{\partial n}\right)_w$$

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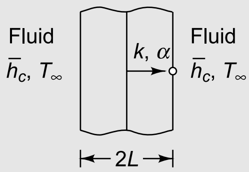
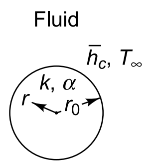
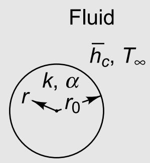
where the subscript $w$ refers to conditions at the wall or surface and $n$ to the coordinate direction normal to the surface. It should be noted that the limiting case of Bi $\to \infty$ corresponds to a negligible thermal resistance at the surface ($h \to \infty$) so that the surface temperature $T_w$ is equal to $T_\infty$ for $t > 0$.

The initial conditions for all three chart solutions require that the solid initially be at a uniform temperature $T_i$ and that when the transient begins at time zero ($t = 0$), the entire surface is contacted by fluid at $T_\infty$. The solutions of all three cases are plotted in terms of dimensionless parameters, summarized in Table 3.3. To find the centreline or midplane temperature, Figs 3.7(a), 3.9(a) and 3.10(a) have to be used. To find the local temperatures as a function of time, Figs 3.7(b), 3.9(b) and 3.10(b) are to be used.

**Table 3.3** *Summary of dimensionless parameters for use with transient heat conduction charts*

| | *Infinite plate, width 2L* | *Infinite long cylinder radius $r_0$* | *Sphere, radius $r_0$* |
|---|---|---|---|
| Geometry |  |  |  |
| Dimensionless position | $\dfrac{x}{L}$ | $\dfrac{r}{r_0}$ | $\dfrac{r}{r_0}$ |
| Biot number | $\dfrac{\bar{h}_c L}{k}$ | $\dfrac{\bar{h}_c r_0}{k}$ | $\dfrac{\bar{h}_c r_0}{k}$ |
| Fourier number | $\dfrac{\alpha t}{L^2}$ | $\dfrac{\alpha t}{r_0^2}$ | $\dfrac{\alpha t}{r_0^2}$ |
| Dimensionless centreline temperature $\dfrac{\theta(0, t)}{\theta_i}$ | Fig. 3.7(a) | Fig. 3.9(a) | Fig. 3.10(a) |
| Dimensionless local temperature $\dfrac{\theta(x, t)}{\theta(0, t)}$ or $\dfrac{\theta(r, t)}{\theta(0, t)}$ | Fig. 3.7(b) | Fig. 3.9(b) | Fig. 3.10(b) |
| Dimensionless heat transfer $\dfrac{Q''(t)}{Q_i''}, \dfrac{Q'(t)}{Q_i'}, \dfrac{Q(t)}{Q_i}$ | Fig. 3.7(c)<br>$Q_i'' = \rho c L(T_i - T_2)$ | Fig. 3.9(c)<br>$Q_i' = \rho c \pi r_0^2 (T_i - T_2)$ | Fig. 3.10(c)<br>$Q_i = \rho c \dfrac{4}{3} \pi r_0^3 (T_i - T_2)$ |

The instantaneous rate of heat transfer to or from the surface of the solid can be evaluated from Fourier's law once the temperature distribution is known. Each heat transfer value $Q(t)$ is the total amount of heat that is transferred from the surface to the fluid during the time from $t = 0$ to $t = t$. The normalising factor $Q_i$ is the initial amount of energy in the solid at $t = 0$.

Two general classes of transient problems can be solved by using the charts.

1. The time is known while the local temperature at that time is unknown.
2. The local temperature is known and the time required to reach that temperature is unknown.

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## 3.4 TWO- AND THREE-DIMENSIONAL SOLUTIONS OF TRANSIENT HEAT CONDUCTION

For regular shaped bodies, analytical methods in the form of Heisler's charts can be used to determine the time-temperature history.

Let us consider a rectangular parallelepiped with dimensions $2A$, $2B$ and $2C$, which is initially at a uniform temperature $T_i$ and then subjected to a constant surrounding temperature $T_\infty$ (Fig. 3.11).

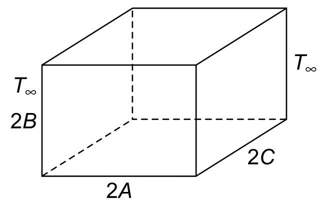


**Fig. 3.11** *A parallelepiped being heated or cooled*

Fourier's heat conduction equation in absence of any heat source is given by

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{3.46}$$

The solution may be broken up into factors so that

$$T = f_1(x, t)\, f_2(y, t)\, f_3(z, t) \tag{3.47}$$

Substituting in Eq. (3.46),

$$f_2 f_3 \frac{\partial^2 f_1}{\partial x^2} + f_3 f_1 \frac{\partial^2 f_2}{\partial y^2} + f_1 f_2 \frac{\partial^2 f_3}{\partial z^2}$$

$$= \frac{1}{\alpha}\left( f_2 f_3 \frac{\partial f_1}{\partial t} + f_3 f_1 \frac{\partial f_2}{\partial t} + f_1 f_2 \frac{\partial f_3}{\partial t} \right)$$

By re-arranging,

$$f_2 f_3 \left( \frac{\partial^2 f_1}{\partial x^2} - \frac{1}{\alpha}\frac{\partial f_1}{\partial t} \right) + f_3 f_1 \left( \frac{\partial^2 f_2}{\partial y^2} - \frac{1}{\alpha}\frac{\partial f_2}{\partial t} \right) + f_1 f_2 \left( \frac{\partial^2 f_3}{\partial z^2} - \frac{1}{\alpha}\frac{\partial f_3}{\partial t} \right) = 0 \tag{3.48}$$

Equation (3.48) will be satisfied only if

$$\frac{\partial^2 f_1}{\partial x^2} - \frac{1}{\alpha}\frac{\partial f_1}{\partial t} = 0$$

$$\frac{\partial^2 f_2}{\partial y^2} - \frac{1}{\alpha}\frac{\partial f_2}{\partial t} = 0$$

$$\frac{\partial^2 f_3}{\partial z^2} - \frac{1}{\alpha}\frac{\partial f_3}{\partial t} = 0$$

But these are simply three one-dimensional forms of Fourier's equation, so that the solution to the three-dimensional problem may be expressed as the product of three solutions of one-dimensional problems.

From Heisler's charts for infinite plates, $(\theta_c/\theta_i)_A$, $(\theta_c/\theta_i)_B$ and $(\theta_c/\theta_i)_C$ are found out. The product of these three will give the centre temperature of the parallelepiped (Fig. 3.12).

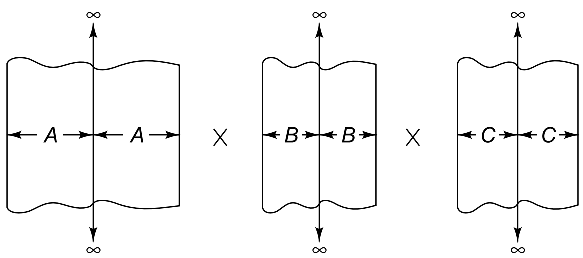


**Fig. 3.12** *Product solution of the parallelepiped in terms of three infinite plates*

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$$\frac{\theta_c}{\theta_i} = \left(\frac{\theta_c}{\theta_i}\right)_A \left(\frac{\theta_c}{\theta_i}\right)_B \left(\frac{\theta_c}{\theta_i}\right)_C = \frac{T_c - T_\infty}{T_i - T_\infty}$$

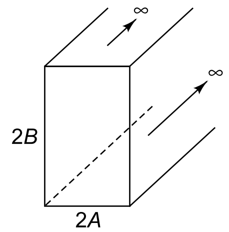


**Fig. 3.13** *Long rectangular bar*

from which $T_c$ at any time $t$ can be estimated.

Similarly, for any other point inside the parallelepiped the time–temperature history can be found out by using the position-correction charts.

In a similar manner, the temperature of a long rectangular bar can be obtained as a product of two one-dimensional results (Fig. 3.13).

$$\frac{\theta_c}{\theta_i} = \left(\frac{\theta_c}{\theta_i}\right)_A \left(\frac{\theta_c}{\theta_i}\right)_B$$

or, at any other point.

For a square bar, $A = B$ (Fig. 3.14)

$$\frac{\theta_c}{\theta_i} = \left(\frac{\theta_c}{\theta_i}\right)_A^2$$

For a cube, $A = B = C$,

$$\frac{\theta_c}{\theta_i} = \left(\frac{\theta_c}{\theta_i}\right)_A^3$$

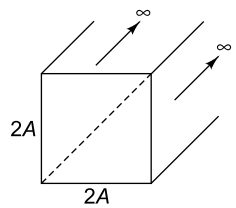


**Fig. 3.14** *Infinite square bar*

It can similarly be shown that for a finite circular cylinder of radius $R$ and length $2L$, the temperature anywhere can be obtained as the product of two one-dimensional results, one for an infinitely long cylinder of radius $R$ and the other for an infinite plate of thickness $2L$ (Fig. 3.15).

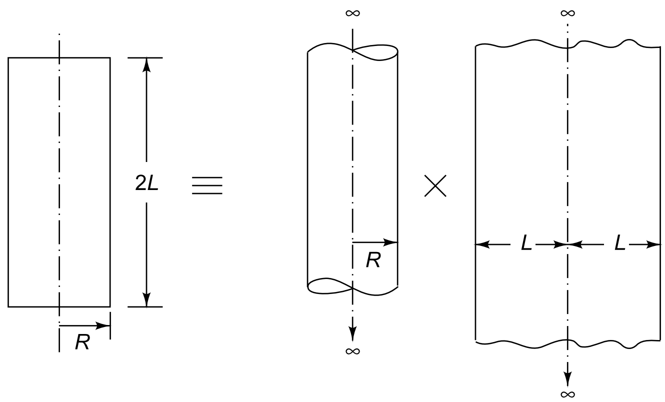


**Fig. 3.15** *Time–temperature history of a finite cylinder of radius R and length 2L is equivalent to the product solution for an infinite cylinder of radius R and an infinite plate of thickness 2L*

Similarly, at any other point in the cylinder the temperature at any instant can be found out with the help of position-correction charts.

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## 3.5 SEMI-INFINITE SOLID

Another simple geometry for which analytical solutions may be obtained is the *semi-infinite solid*. In principle, such a solid extends to infinity in all but one direction and it is characterised by a single identifiable surface (Fig. 3.16). If a sudden change of conditions is imposed at this surface, transient one-dimensional conduction will occur within the solid. The semi-infinite solid provides a useful idealization for many practical problems. It may be used to determine transient heat transfer near the surface of the earth or approximate the transient response of a finite solid such as a thick slab. The transient conduction in a semi-infinite solid is given by

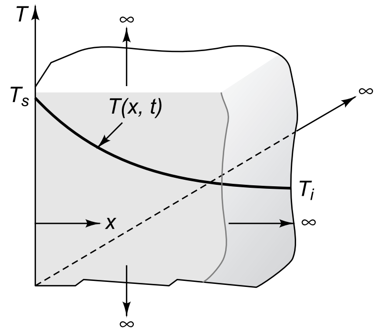


**Fig. 3.16** *Transient conduction in a semi-infinite solid*

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{3.49}$$

The initial condition is: $T(x, 0) = T_i$ and the interior boundary condition is

$$T(x \to \infty, t) = T_i$$

Closed form solutions have been obtained by Schneider [1] for three important surface conditions, instantaneously applied at $t = 0$. These conditions are shown in Fig. 3.17(a). They include application of a constant surface temperature $T_s \neq T_i$, application of a constant heat flux $q''_0$, and exposure of the surface to a fluid characterized by $T_\infty \neq T_i$ and the convection coefficient $h$.

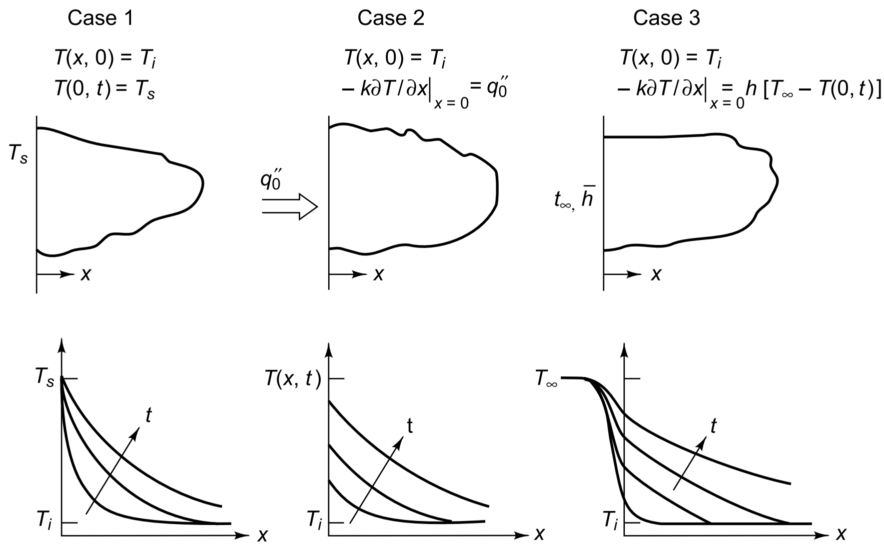


**Fig. 3.17(a)** *Transient temperature distributions in a semi-infinite solid for three surface conditions: (1) Constant surface temperature (2) constant surface heat flux and (3) surface convection*

The solution for case 1 may be obtained by recognizing the existence of a *similarity variable* $\eta$, through which the heat equation may be transformed from a partial differential equation with two independent variables ($x$ and $t$) to an ordinary differential equation in terms of a single similarity variable. To confirm that it is satisfied by $\eta = \dfrac{x}{\sqrt{4\alpha t}} = \dfrac{x}{2\sqrt{\alpha t}}$, we first transform the pertinent differential operators, such that

$$\frac{\partial T}{\partial x} = \frac{\mathrm{d}T}{\mathrm{d}\eta} \cdot \frac{\partial \eta}{\partial x} = \frac{1}{(4\,\alpha t)^{1/2}} \cdot \frac{\mathrm{d}T}{\mathrm{d}\eta}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{\mathrm{d}}{\mathrm{d}\eta}\left[\frac{\mathrm{d}T}{\mathrm{d}x}\right]\frac{\partial \eta}{\partial x} = \frac{1}{4\,\alpha t}\,\frac{\mathrm{d}^2 T}{\mathrm{d}\eta^2}$$

$$\frac{\partial T}{\partial t} = \frac{\mathrm{d}T}{\mathrm{d}\eta}\,\frac{\partial \eta}{\partial t} = -\frac{x}{2t(4\,\alpha t)^{1/2}} \cdot \frac{\mathrm{d}T}{\mathrm{d}\eta}$$

Substituting into Eq. (3.49), the heat equation becomes

$$\frac{\mathbf{d^2T}}{\mathbf{d\eta^2}} + \mathbf{2\eta}\frac{\mathbf{dT}}{\mathbf{d\eta}} = \mathbf{0} \tag{3.49a}$$

With $x = 0$ corresponding to $\eta = 0$, the surface condition becomes

$$T(\eta \simeq 0) = T_s$$

and with $x \to \infty$ and $t = 0$ corresponding to $\eta \to \propto$, both the initial and boundary conditions combine to a single condition

$$T(\eta \to \propto) = T_i$$

Irrespective of the values of $x$ and $t$, the temperature may be represented as a unique function of $\eta$, so that by separating the variables in Eq. (3.49a),

$$\frac{\mathrm{d}(\mathrm{d}T/\mathrm{d}\eta)}{(\mathrm{d}T/\mathrm{d}\eta)} = -2\eta\,\mathrm{d}\eta$$

On integration,

$$\ln(\mathrm{d}T/\mathrm{d}\eta) = -\eta^2 + C_1'$$

or,

$$\frac{\mathrm{d}T}{\mathrm{d}\eta} = C_1 \exp(-\eta^2)$$

Integrating a second time,

$$T = C_1 \int_0^{\eta} \exp(-u^2)\,\mathrm{d}u + C_2$$

where, $u$ is a dummy variable. Applying the condition at $\eta = 0$, it follows that $C_2 = T_s$.

$\therefore$

$$T = C_1 \int_0^{\eta} \exp(-u^2)\,\mathrm{d}u + T_s$$

and the definite integral gives

$$C_1 = \frac{2(T_i - T_s)}{\pi^{1/2}}$$

Hence, the temperature distribution may now be expressed as

$$\frac{T - T_s}{T_i - T_s} = \frac{2}{\sqrt{\pi}} \int_0^{\eta} \exp(-u^2)\,\mathrm{d}u = \mathrm{erf}(\eta) \tag{3.49b}$$

where the *Gaussian error function*, erf ($\eta$), is a mathematical function, which is tabulated in Appendix B-2. The surface heat flux may be obtained by applying Fourier's law at $x = 0$, in which case

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$$q_o'' = k\left.\frac{\partial T}{\partial x}\right|_{x=0} = -k\,(T_i - T_s)\frac{\mathrm{d}\,\mathrm{erf}\,(\eta)}{\mathrm{d}\eta}\left.\frac{\partial T}{\partial x}\right|_{\eta=0}$$

$$q_o'' = k(T_2 - T_i)\left.\left(\frac{2}{\sqrt{\pi}}\exp\,(-\eta^2)\;(4\,\alpha\,t\right)^{1/2}\right|_{\eta=0}$$

$$q_o'' = \frac{h(T_s - T_i)}{(\pi\,\alpha\,t)^{1/2}}$$

Analytical solutions may also be obtained for the case 2 and case 3 surface conditions, and results of all these cases are summarized as follows.

**Case (a)**

Change in surface temperature

$$T(0,\,t) = T_s$$

$$\frac{T(x,\,t) - T_s}{T_i - T_s} = \mathrm{erf}\,\frac{x}{2\,(\alpha t)^{1/2}} = \theta(x, t) \tag{3.50}$$

The dimensionless parameter $\xi = x/2\sqrt{\alpha t}$ is plotted in Fig. 3.17 against the dimensionless temperature $\theta(x, t)$.

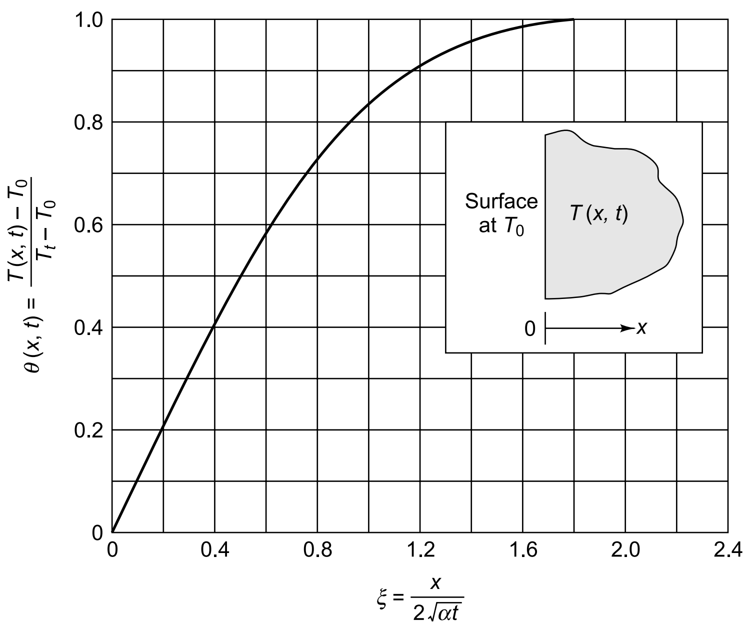


**Fig. 3.17** *Temperature distribution T(x, t) in a semi-infinite solid which is initialty at $T_o$ for t > 0 the surface at x = 0 is maintained at $T_o$*

For a given value of $x$, the graph represents the variation of temperature with time at that particular location.

$$q_s''(t) = -k\left.\frac{\partial T}{\partial x}\right|_{x=0} = \frac{k(T_s - T_i)}{(\pi\alpha t)^{1/2}}$$

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**Case (b)**

Constant surface flux

$$q''_s = q''_0$$

$$T(x, t) - T_i = \frac{2q''_0(\alpha t/\pi)^{1/2}}{k_s}\exp\left(-\frac{x^2}{4\alpha t}\right) - \frac{q''_0 x}{k_s}\,\mathrm{erf}\,c\left[\frac{x}{2(\alpha t)^{1/2}}\right] \tag{3.51}$$

**Case (c)**

Surface convection

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=0} = \bar{h}[T_\infty - T(0, t)]$$

$$\frac{T(x,t) - T_i}{T_\infty - T_i} = \mathrm{erf}\,c\left[\frac{x}{2(\alpha t)^{1/2}}\right] - \exp\left(\frac{\bar{h}x}{k} + \frac{\bar{h}^2\alpha t}{k^2}\right)$$

$$\mathrm{erf}\,c\left[\frac{x}{2(\alpha t)^{1/2}} + \frac{\bar{h}(\alpha t)^{1/2}}{k}\right] \tag{3.52}$$

It may be noted that $(h^2\alpha t)/k^2$ is equal to the product of $\mathrm{Bi}^2\mathrm{Fo}$, where $\mathrm{Bi} = \bar{h}x/k$ and $\mathrm{Fo} = \alpha t/x^2$. The function erfc is the *Gaussian error function* defined as

$$\mathrm{erf}\left[\frac{x}{2(\alpha t)^{1/2}}\right] = \frac{2}{\pi}\int_0^{\pi/2(\alpha t)^{1/2}} e^{-\eta^2}\,d\eta \tag{3.53}$$

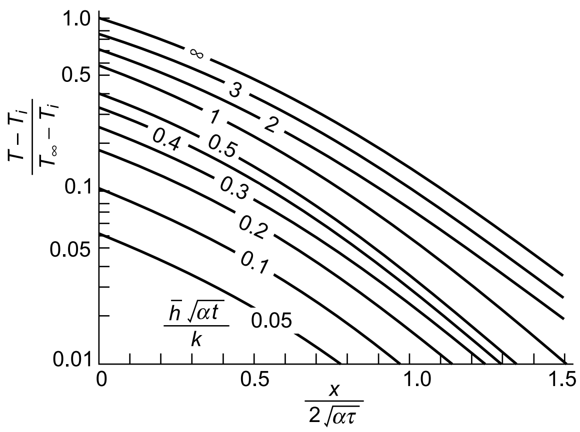


**Fig. 3.18** *Transient temperatures for a semi-infinite solid with surface convection*

Values of this error function are tabulated in the appendix. The complementary error function, erf $c(w)$ is defined as

$$\mathrm{erf}\,c(w) = 1 - \mathrm{erf}\,(w)$$

Temperature histories for the three cases are illustrated qualitatively in Fig. 3.17. For case (c), the specific temperature histories computed from Eq. (3.52) are plotted in Fig. 3.18. The curve corresponding to $\bar{h} = \infty$ is equivalent to the result that would be obtained for a sudden change in the surface temperature

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to $T_s = T(x, 0)$, because when $\bar{h} = \infty$, the second term on the r.h.s. of Eq. (3.52) is zero, and the result is equivalent to Eq. (3.50) for case (a).

## 3.6 NUMERICAL AND GRAPHICAL METHODS

Analytical treatment of transient heat conduction problems is possible only for regular shaped bodies, and for such bodies only, the various charts computed from analytical solutions are very useful in determining the temperature history and heat transfer rates. But for bodies of irregular geometric shapes and in cases where the boundry conditions are varying with time (e.g., the ambient temperature $T_\infty$, the heat transfer coefficient $h$ or the physical properties of the solid may vary with time), and in cases where the initial temperature distribution is not uniform, or there is an internal heat source and so on, analytical treatment is difficult to use. Such complex cases can be readily handled by employing numerical methods. The results yielded are approximate, but the accuracy obtained is sufficient in most practical cases.

There are two basic graphical methods in frequent use, viz., (a) the mapping method and (b) the finite difference equation method. The mapping method was first described by Lehman for electrical fields, and later adapted by others for heat conduction problems. The two-dimensional space is divided into meshes of approximately square form, representing isothermal and adiabatic lines, which are orthogonal. The mapping method was explained in the last chapter, and it is, however, not suitable for transient problems.

The method based on finite differences was indicated first by Binder, then developed anew and in more detail by Schmidt, and further improved by Nessi and Nissolle, and Dusinberre [6].

The relaxation method, introduced by Southwell, which is also a finite difference method of solving the Laplace equation, cannot be used in most cases of unsteady problems, because the relaxation procedure requires that the boundary conditions are completely specified, and so it is used only for steady-state problems, as explained in the last chapter. Iteration methods are adopted to solve unsteady state problems.

We will first consider the case in which there is no convective resistance, and later extend the method to cases where there is the external resistance.

### 3.6.1 Schmidt's Method with No Convective Resistance

Let us take a large slab of thickness $x$ and uniform cross-sectional area and divide it into a number of equal finite slices of thickness $\Delta x$ by temperature reference planes and the time into intervals $\Delta t$ (Fig. 3.19). The differential equation (in one dimension).

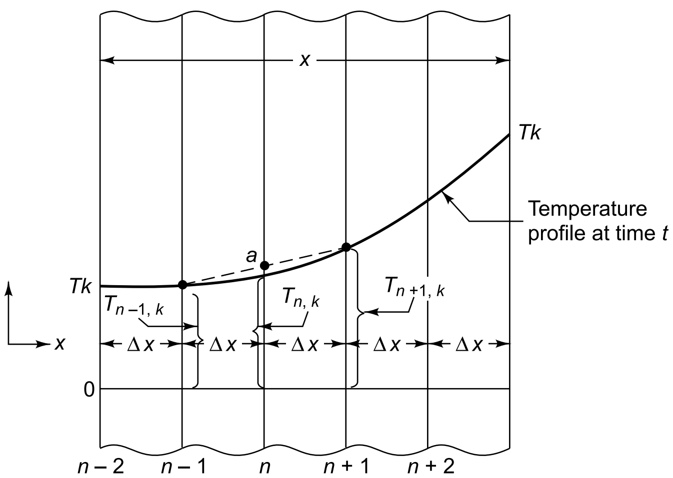


**Fig. 3.19** *Unsteady heat flow in a plane wall by the Schmidt's method*

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$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

can be written as a difference equation

$$\frac{\Delta_x^2 T}{(\Delta x)^2} = \frac{1}{\alpha}\frac{\Delta_t T}{\Delta t} \tag{3.54}$$

$$\Delta_x T_+ = T_{n+1,\,k} - T_{n,\,k} \quad \Delta_x T_- = T_{n,\,k} - T_{n-1,\,k}$$

$$\Delta_t T = T_{n,\,k+1} - T_{n,\,k}$$

$\Delta_x^2 T$ is the difference of two successive differences.

$$\begin{aligned}\Delta_x^2 T &= \Delta_x T_+ - \Delta_x T_- \\ &= (T_{n+1,\,k} - T_{n,\,k}) - (T_{n,\,k} - T_{n-1,\,k}) \\ &= T_{n+1,\,k} - 2T_{n,\,k} + T_{n-1,\,k}\end{aligned}$$

Substituting in Eq. (3.54),

$$\frac{T_{n+1,\,k} - 2T_{n,\,k} + T_{n-1,\,k}}{(\Delta x)^2} = \frac{1}{\alpha}\frac{T_{n,\,k+1} - T_{n,\,k}}{\Delta t}$$

$$T_{n,\,k+1} = \frac{\alpha\,\Delta t}{(\Delta x)^2}\,(T_{n+1,\,k} - 2T_{n,\,k} + T_{n-1,\,k}) + T_{n,\,k} \tag{3.55}$$

If we know the temperature at a point $n$ at a given instant $k$, we can determine the temperature at that point at the time $k + 1$ i.e., after a lapse of time $\Delta t$. By continual application of Eq. (3.55), the development of the temperature field with time can be determined from a known initial temperature distribution. The term $(\alpha\Delta t)/(\Delta x)^2$ in Eq. (3.55) is the Fourier number. Schmidt took Fo = 1/2, so that

$$T_{n,\,k+1} = \frac{1}{2}\,(T_{n+1,\,k} + T_{n-1,\,k}) - T_{n,\,k} + T_{n,\,k}$$

or

$$T_{n,\,k+1} = \frac{T_{n+1,\,k} + T_{n-1,\,k}}{2} \tag{3.56}$$

The temperature at a point after time $\Delta t$ is the arithmetic mean of the two temperatures at time $t$ ($k$th instant) $2\Delta x$ apart, $\Delta x$ distance on either side of the point. The point $a$ represents the temperature at the instant $k + 1$ in which the time increment later than $k$ is

$$\Delta t = \frac{(\Delta x)^2}{2\alpha}$$

In the same manner by drawing additional straight lines, other points of the temperature field for the time trace $(k + 1)$ can be obtained, and thus the entire temperature field can be determined (Fig. 3.20). In order to use the graphical solution, the temperature field must be known at some specific time.

### 3.6.2 Dusinberre's Method

Dusinberre [6] derived the same difference equation by a direct energy balance of a finite element. Let us consider the slab *abcd* divided equally into a number of finite slices of thickness $\Delta x$ (Fig. 3.21). Let us take a finite element *abcd* symmetrical about plane 1. Here $Q_1$ = heat flowing into *abcd*, $Q_2$ = heat flowing out from *abcd* and entering *bcde*.

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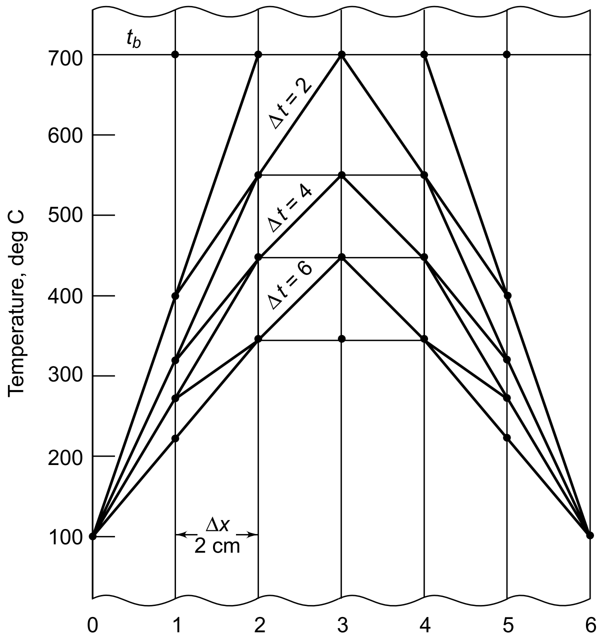


**Fig. 3.20** *Running picture of temperature distribution in a slab varying with time*

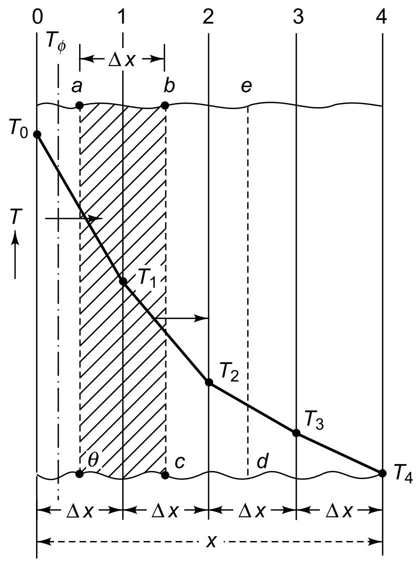


**Fig. 3.21** *Dusinberre's method of obtaining 1-D transients in a slab*

$$Q_1 = -kA\,\frac{T_1 - T_0}{\Delta x}$$

$$Q_2 = -kA\,\frac{T_2 - T_1}{\Delta x}$$

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The net heat transfer $(Q_1 - Q_2)$ will increase the internal energy of the element and the temperature increases from $T_1$ to $T_1'$ in time $\Delta t$. Therefore,

$$\Delta E = \frac{\rho\,(\Delta x A)\, c_P\,(T_1' - T_1)}{\Delta t}$$

By energy balance,

$$\frac{kA\,(T_0 - T_1)}{\Delta x} - \frac{kA\,(T_1 - T_2)}{\Delta x} = \frac{\rho \Delta x A\, c_P\,(T_1' - T_1)}{\Delta t}$$

$$T_1' - T_1 = \frac{\alpha \Delta t}{(\Delta x)^2}(T_0 - 2T_1 + T_2) \tag{3.57}$$

Here, $T_1'$ is the temperature after the lapse of time $\Delta t$. This equation is the same as that of Schmidt, Eq. (3.55).

Let $\dfrac{(\Delta x)^2}{\alpha \Delta T} = \dfrac{1}{\text{Fo}} = M$, a dimensionless modulus

Equation (3.57) becomes

$$T_1' = \frac{1}{M}(T_0 - 2T_1 + T_2) + T_1$$

$$= \frac{T_0 + (M-2)\,T_1 + T_2}{M} \tag{3.58}$$

There will always be some errors in numerically solving the partial differential equation. As $\Delta x$ or $\Delta t$ are chosen smaller and smaller, closer will be the approximation to real values and less will be the errors. It can be shown from the convergence and stability of the equation that the solution behaves satisfactorily if $M$ is taken large enough, $M \geq 2$ i.e., Fo $\leq 1/2$. Too small a value of $M$ will be indicated by a pronounced oscillation or divergence of solution. And too large a value of $M$ will consume too much of time in computation.

When $\quad M = 2,\ T_1' = \dfrac{T_0 + T_2}{2}$

$$M = 3,\ T_1' = \frac{T_0 + T_1 + T_2}{3}$$

$$M = 4,\ T_1' = \frac{T_0 + 2\,T_1 + T_2}{4}$$

Schmidt took $M = 2$, while $M > 2$ was adopted by Nessi and Nissolle, and Dusinberre [7].

### 3.6.3 Schmidt's Method with Convective Resistance

To account for the convective resistance in Schmidt's method, a fictitious half-slice having no heat capacity is added outside the surface and the reference planes for temperatures are located at distances of 0.5 $\Delta x$, 1.5 $\Delta x$, 2.5 $\Delta x$ and so on. The outer fictitious boundary is at a distance of – 0.5 $\Delta x$ from the surface (Fig. 3.22).

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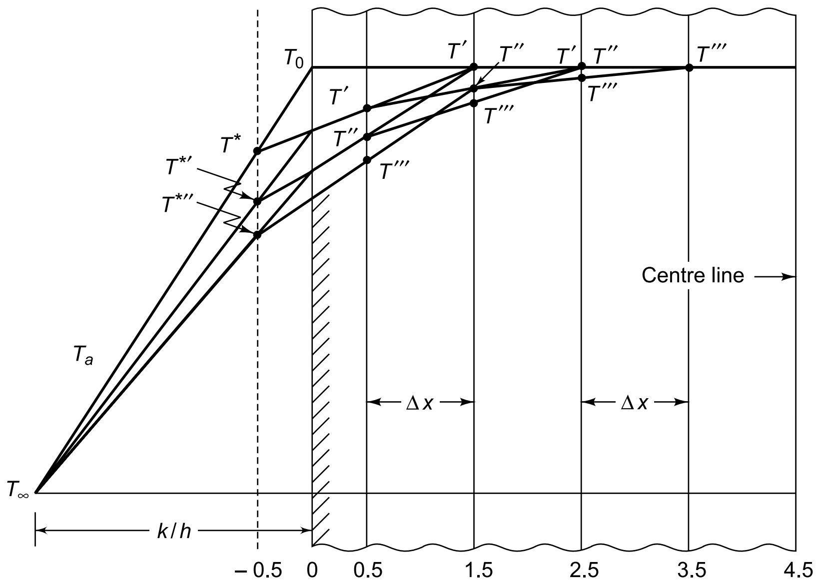


**Fig. 3.22** *Schmidt's method for a one-dimensional transient in a slab with heat transfer at one surface*

A heat balance at the surface gives

$$\bar{q} = -k\left(\frac{dT}{dx}\right)_{x=0} = h\,(T_\infty - T_0)$$

$$\therefore \qquad \left(\frac{dT}{dx}\right)_{x=0} = -\frac{T_\infty - T_0}{k/h} \tag{3.59}$$

This is the slope of the temperature profile at the surface (Fig. 3.22). The ambient temperature $T_\infty$ is plotted as ordinate at a distance $k/h$ from the surface. The term $(T_\infty/T_0)/(k/h)$ is the ratio of two distances. A straight line from $T_\infty$ to $T_0$ gives the slope $-(dT/dx)_{x=0}$ at surface and the ordinate at the intersection with the – 0.5 plane is designated as $T^*$. As an approximation $(dT/dx)_{x=0}$ is replaced by the chord slope $(T^* - T_{0.5})/\Delta x$, and the heat balance on the slice from plane 0 to plane 1 gives the Schmidt rule

$$T'_{0.5} = \frac{T^* + T_{1.5}}{2}$$

Heat balances on the interior give

$$T'_{1.5} = \frac{T'_{0.5} + T_{2.5}}{2} \qquad T'_{2.5} = \frac{T_{1.5} + T_{3.5}}{2}$$

and so on.

Figure 3.22 shows the graphical construction for a finite slab originally at uniform temperature, cooled at one face ($x$ = 0) and adiabatic on the other face ($x = 4.5\,\Delta x$). Since there is no heat transfer at the adiabatic wall, the temperature gradient is always zero at this point.

If during the course of time, $T_\infty$ or $h$ changes, these changes can be accommodated by displacing appropriately the guide point $r$. It can also be applied for heat transfer at both surfaces.

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When $k/h \leq \Delta x/2$, we can omit the fictitious half-slice altogether, as a measure of approximation (Fig. 3.23).

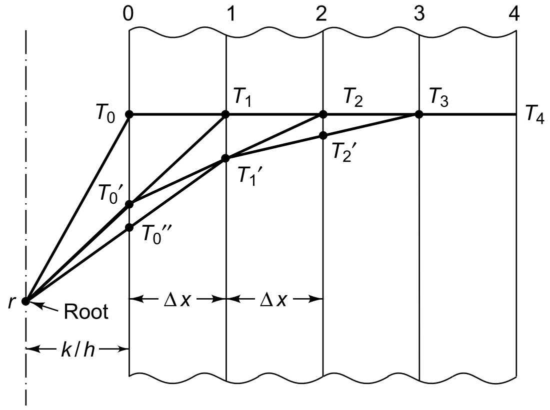


**Fig. 3.23** *No fictitious half slice is required for $k/h \leq \Delta x/2$*

Graphical construction provides a continuous picture of temperature changes taking place inside the material with time.

### 3.6.4 Dusinberre's Method with Convective Resistance

Dusinberre developed the difference equation by making energy balance for half slice (Fig. 3.24).

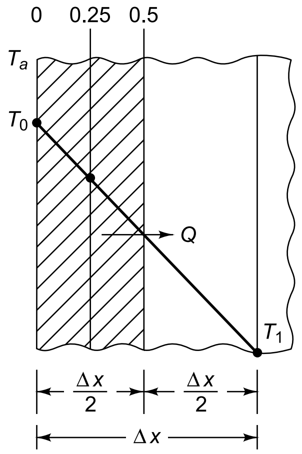


**Fig. 3.24** *Dusinberre's method with convective resistance*

$$Q_1 = hA\,(T_\infty - T_0)$$

$$Q_2 = -kA\,\frac{(T_1 - T_0)/2}{\Delta x/2} = -kA\,\frac{T_1 - T_0}{\Delta x}$$

$$\therefore \qquad \Delta E = \frac{\rho A \dfrac{\Delta x}{2} c_p\,(T'_{0.25} - T_{0.25})}{\Delta t}$$

Since $\qquad Q_1 - Q_2 = \Delta E,$

$$hA(T_\infty - T_0) - \frac{kA\,(T_0 - T_1)}{\Delta x} = \frac{\rho A \dfrac{\Delta x}{2} c_p\,(T'_{0.25} - T_{0.25})}{\Delta t}$$

By approximation,

$$T'_{0.25} - T_{0.25} = T'_0 - T_0$$

$$T'_0 - T_0 = \frac{2\,\Delta t}{\rho \Delta x\, c_p}\left[h(T_\infty - T_0) - \frac{k}{\Delta x}(T_0 - T_1)\right]$$

$$= \frac{2\,(\Delta t)\,\Delta x k}{k\,\rho c_p\,(\Delta x)^2}\left[h(T_\infty - T_0) - \frac{k}{\Delta x}(T_0 - T_1)\right]$$

$$= \frac{2\,\alpha \Delta t}{k\,(\Delta x)^2}\,\Delta x\left[h(T_\infty - T_0) - \frac{k}{\Delta x}(T_0 - T_1)\right]$$

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$$= \frac{1}{M}\left[\frac{2\,h\,\Delta x}{k}(T_\infty - T_0) - 2(T_0 - T_1)\right]$$

$$= \frac{1}{M}\,[2N\,(T_\infty - T_0) - 2(T_0 - T_1)]$$

where $\quad \dfrac{h\,\Delta x}{k} = N$, the Biot number.

$$T_0' = \frac{1}{M}\,[2NT_\infty - 2NT_0 - 2T_0 + 2T_1 + MT_0]$$

$$= \frac{2NT_\infty + [M - (2N+2)]\,T_0 + 2T_1}{M} \tag{3.60}$$

To prevent oscillatory results, $M \geq 2N + 2$

$$T_1' = \frac{T_0 + (M-2)\,T_1 + T_2}{M}$$

$T_2'$, $T_3'$ etc. can similarly be found out.

## 3.7 PERIODIC FLOW OF HEAT IN ONE DIMENSION

Here the transient boundary condition is a regular harmonic which leads to a periodic type of heat flow. The temperature varies with time in a periodic manner. Examples of this phenomenon are evident in the cylinders of internal combustion engines, in cyclic regenerators and in the earth as the result of daily and annual temperature changes that repeat themselves. Water pipes in cold places are buried in soil at a sufficient depth from the surface to prevent freezing in winter. Since there is an increase in volume as water freezes into ice, the water pipe wall may crack while freezing. It is difficult to treat such transient systems by Schmidt's graphical method or numerical methods. Solution can, however, be secured by analytical methods.

Let us consider a semi-infinite solid extending from the surface $x = 0$ to $x = \infty$, where the surface temperature of the exposed surface at $x = 0$ is varying periodically with time, which may vary either

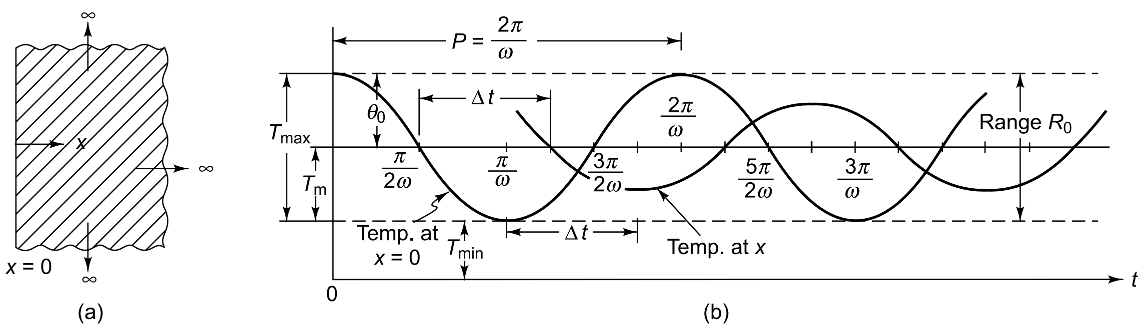


**Fig. 3.25** *Temperature oscillating about a mean temperature $T_m$*

sinusoidally or consinusoidally [Fig. 3.25 (a) and (b)]. Let the temperature oscillate about a mean temperature $T_m$, which is equal to $(T_{max} + T_{min})/2$, as shown in Fig. 3.25 (b). Here,

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The amplitude at $x = 0,\ \theta_0 = \dfrac{T_{max} - T_{min}}{2}$

Time period of oscillation $= P$

Frequency $f = \dfrac{1}{P} = \dfrac{\omega}{2\pi}$

where, $\omega = 2\pi f = \dfrac{2\pi}{P}$

At any time $t$, $\theta = T - T_m$

Let us assume that at the surface $x = 0$

$$\theta = \theta_0 \cos \omega t$$

where $\omega$ is the angular velocity in simple harmonic motion.

At $t = 0,\ \theta = \theta_0$.

One-dimensional heat conduction equation

$$\frac{\partial^2 \mathbf{T}}{\partial \mathbf{x}^2} = \frac{\mathbf{1}}{\boldsymbol{\alpha}} \frac{\partial \mathbf{T}}{\partial \mathbf{t}}$$

In terms of variable $\theta$,

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha} \frac{\partial \theta}{\partial t}$$

Let us again assume

$$\theta = X(x)Y(t)$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial^2 X}{\partial x^2} \quad \text{and} \quad \frac{\partial \theta}{\partial t} = X \frac{\partial Y}{\partial t}$$

Therefore,

$$Y \frac{\partial^2 X}{\partial x^2} = \frac{1}{\alpha} X \frac{\partial Y}{\partial t}$$

$$\frac{1}{\alpha Y} \frac{\partial Y}{\partial t} = \frac{1}{X} \frac{\partial^2 X}{\partial x^2} = \pm i\lambda^2, \text{ the separation constant}$$

where $\lambda$ is real,

$$\frac{1}{\alpha Y} \frac{\partial Y}{\partial t} = \pm i\lambda^2$$

and

$$\frac{1}{X} \frac{\partial^2 X}{\partial x^2} (\pm i\lambda^2) = 0$$

We will first take the positive sign.

$$\frac{\partial Y}{Y} = i\lambda^2 \alpha\, \delta t$$

$$\ln Y = i\, \lambda^2\, \alpha t + \ln C$$

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$$Y(t) = B_1\, e^{i\lambda^2 \alpha t}$$

and $$\frac{\partial^2 X}{\partial x^2} - i\lambda^2 X = 0$$

Characteristic equation is

$$m^2 - i\lambda^2 = 0 \quad m = \pm\, (i)^{1/2}\, \lambda$$

$$X(x) = B_2\, e^{\sqrt{i}\lambda x} + B_3\, e^{-\lambda x}$$

Now, $$(1 + i)^2 = 1 + 2i + i^2 = 2i$$

$$\sqrt{i} = \frac{1+i}{\sqrt{2}}$$

$$\theta_+ = e^{i\lambda^2 \alpha t}\left[C_1 e^{(1+i)\frac{\lambda x}{\sqrt{2}}} + C_2 e^{-\frac{\lambda x}{\sqrt{2}}(1+i)}\right]$$

$$\theta_+ = e^{-\frac{\lambda x}{\sqrt{2}}}\left[C_1 e^{\sqrt{2}\,\lambda x} + i\left(\lambda^2 \alpha t + \frac{\lambda x}{2}\right) + C_2 e^{+i\left(\lambda^2\, \alpha t - \frac{\lambda x}{\sqrt{2}}\right)}\right] \tag{3.61}$$

Similarly, for negative sign

$$\theta_- = e^{-\frac{\lambda x}{\sqrt{2}}}\left[C_3\, e^{\sqrt{2}\lambda x}\, e^{i\left(\lambda^2 \alpha t + \frac{\lambda x}{\sqrt{2}}\right)} + C_4\, e^{-i\left(\lambda^2 \alpha t - \frac{\lambda x}{\sqrt{2}}\right)}\right]$$

$$\theta = \theta_+ + \theta_-$$

$$= e^{-\frac{\lambda x}{2}}\left[C_1\, e^{\sqrt{2}\,\lambda x} e^{i\left(\lambda^2 \alpha t + \frac{\lambda x}{\sqrt{2}}\right)} + C_2\, e^{i\left(\lambda^2 \alpha t - \frac{\lambda x}{\sqrt{2}}\right)} + C_3\, e^{\sqrt{2}\,\lambda x}\, e^{i\left(\lambda^2\, \alpha + \frac{\lambda x}{\sqrt{2}}\right)} + C_4 e^{-i\left(\lambda^2\, \alpha t - \frac{\lambda x}{\sqrt{2}}\right)}\right]$$

Now we have, as $x \to \infty$, $\theta = 0$, $T = T_m$

But if we apply here, $x \to \infty$, $\theta \to \infty$

$$C_1 = 0 \quad C_3 = 0$$

Then

$$\theta = e^{-\frac{\lambda x}{\sqrt{2}}}\left[C_2 e^{i\left(\lambda^2\, \alpha t - \frac{\lambda x}{\sqrt{2}}\right)} + C_4 e^{i\left(\lambda^2\, \alpha t - \frac{\lambda x}{\sqrt{2}}\right)}\right]$$

$$= e^{-\frac{\lambda x}{\sqrt{2}}}\,[C_2\, e^{iu} + C_4\, e^{-iu}] \text{ where } i = \lambda^2 \alpha t - \frac{\lambda x}{\sqrt{2}}$$

$$= e^{-\frac{\lambda x}{\sqrt{2}}}\,[C_2\,(\cos u + i \sin u) + C_4\,(\cos u - i \sin u)]$$

$$= e^{-\frac{\lambda x}{\sqrt{2}}}\,(A_1 \cos u + A_2 \sin u)$$

$$\theta = e^{-\frac{\lambda x}{\sqrt{2}}}\left[A_1 \cos\left(\lambda^2 \alpha t - \frac{\lambda x}{\sqrt{2}}\right) + A_2 \sin\left(\lambda^2 \alpha t - \frac{\lambda x}{\sqrt{2}}\right)\right] \tag{3.62}$$

At $x = 0$,

$$\theta_{x=0} = A_1 \cos \lambda^2 \alpha t + A_2 \sin \lambda^2 \alpha t$$

But $\theta_{x=0} = \theta_0 \cos \omega t$ (3.63)

By comparing these two equations, we find

$$A_1 = \theta_0 \quad A_2 = 0$$

$$\omega t = \lambda^2 \alpha t$$

$$\lambda^2 = \frac{\omega}{\alpha}$$

or

$$\lambda = \left(\frac{\omega}{\alpha}\right)^{1/2} = \left(\frac{2\pi}{P\alpha}\right)^{1/2}$$

Therefore, the temperature variation is given by

$$\theta = e^{-\frac{\lambda x}{\sqrt{2}}}\left[A_1 \cos\left(\lambda^2 \alpha t - \frac{\lambda x}{\sqrt{2}}\right)\right]$$

$$= e^{-\left(\frac{\omega}{2\alpha}\right)^{1/2} x} \theta_0 \cos\left[\frac{\omega}{\alpha} \alpha t - \left(\frac{\omega}{2\alpha}\right)^{1/2} x\right]$$

At a depth $x$,

$$\boldsymbol{\theta = \theta_0 e^{-\left(\frac{\omega}{2\alpha}\right)^{1/2} x} \cos\left[\omega t - \left(\frac{\omega}{2\alpha}\right)^{1/2} x\right]} \tag{3.64}$$

where $\theta_0$ is the amplitude at $x = 0$.

This equation expresses the temperature at any time $t$ and distance $x$ from the surface.

Range, $R_{x=0} = 2\theta_0$

At any depth $x$, $R_x = 2\theta_x = 2\theta_0\, e^{-\left(\frac{\omega}{2\alpha}\right)^{1/2} x}$

or

$$\frac{R_{x_1}}{R_{x_2}} = \frac{e^{-\left(\frac{\omega}{2\alpha}\right)^{1/2} x_1}}{e^{-\left(\frac{\omega}{2\alpha}\right)^{1/2} x_2}}$$

Amplitude at any distance $x$ is

$$\theta_x = \theta_0\, e^{-\left(\frac{\omega}{2\alpha}\right)^{1/2} x}$$

The amplitude gets diminished by the term $e^{-\left(\frac{\omega}{2\alpha}\right)^{1/2} x}$.

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At the surface, from Eq. (3.64),

$$\theta_{x=0} = \theta_0 \cos \omega t$$

$$\theta_{x=0} = T_m - T_m = 0$$

$$\cos \omega t_1 = \cos \frac{\pi}{2}$$

$$t_1 = \frac{\pi}{2\omega}$$

At depth $x$, $\quad T = T_m, \theta = 0$

when $\omega t_2 - \left(\dfrac{\omega}{2\alpha}\right)^{1/2} x = \dfrac{\pi}{2}$

$$t_2 = \frac{\pi}{2\omega} + \frac{1}{(2\alpha\omega)^{1/2}} x$$

Time lag $\quad \Delta t = t_2 - t_1 = \dfrac{\pi}{2\omega} + \dfrac{1}{(2\alpha\omega)^{1/2}} x - \dfrac{\pi}{2\omega}$

$$= \frac{1}{(2\alpha\omega)^{1/2}} x = \frac{1}{\left(2\alpha \dfrac{2\pi}{P}\right)^{1/2}} x$$

$$= \frac{1}{2}\left(\frac{P}{\pi\alpha}\right)^{1/2} x$$

Velocity of propagation of the thermal wave into the solid

$$v = \frac{x}{\Delta t} = \frac{x}{x/(2\alpha\omega)^{1/2}} = (2\,\alpha\omega)^{1/2}$$

Wave length $\quad \lambda = vP = (2\alpha\omega)^{1/2} \dfrac{2\pi}{\omega} = 2\pi\left(\dfrac{2\alpha P}{2\pi}\right)^{1/2}$

$$= 2(\pi\alpha P)^{1/2}$$

= Distance travelled in one complete oscillation

The temperature distribution as given by Eq. (3.64) is plotted in Figs 3.26 and 3.27. Figure 3.26 shows temperature variation with time at any depth $x$, whereas the temperature distribution as a function of depth at $\omega t = 0$ and at $\omega t = \pi$ is depicted in Fig. 3.27. The surface amplitude is diminished at depth $x$ by the factor exp $[-(\omega/2\alpha)^{1/2} x]$. The depth of penetration is shown in Fig. 3.28.

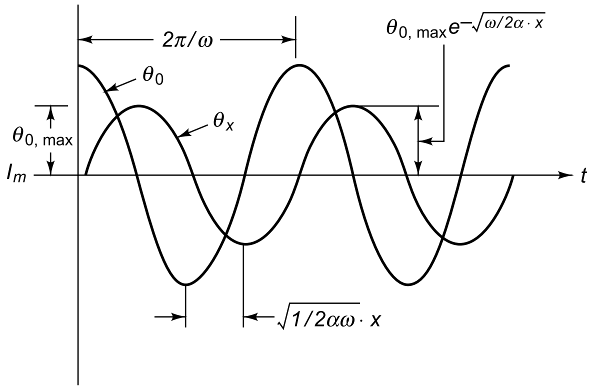


**Fig. 3.26** *Periodic temperature variation with time*

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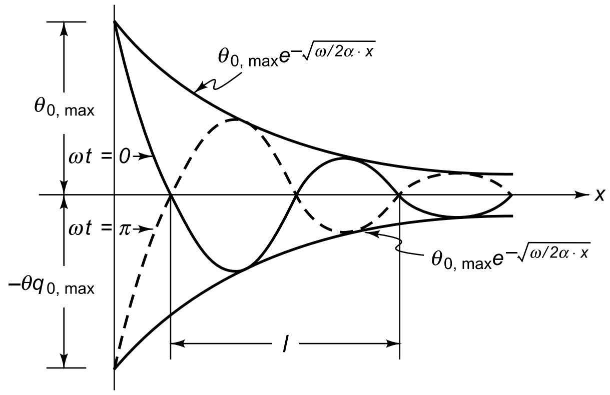


**Fig. 3.27** *Periodic variation of temperature with depth*

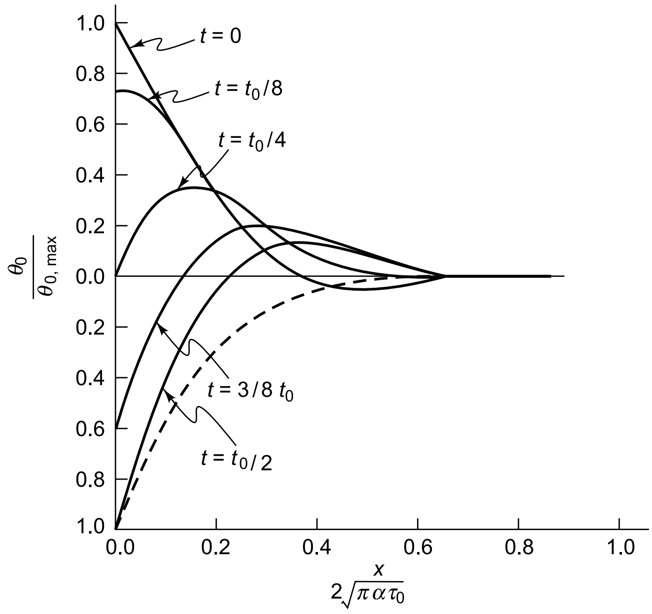


**Fig. 3.28** *Penetration of a temperature oscillation*

Several other important factors can be obtained from Eq. (3.63):

### (i) Amplitude at any Depth

The maximum excess temperature or amplitude at any depth would be obtained by putting

$$\cos\left[\omega t - \left(\frac{\omega}{2\alpha}\right)^{1/2} x\right] = 1$$

$$(\theta_x)_{max} = \theta_0\, e^{-(\omega/2\alpha)1/2x} \tag{3.65}$$

which shows that the advancing wave is decreasing in amplitude with increasing depth exponentially. It is also seen from Eq. (3.65) that the higher the frequency (the larger $\omega$), the less the penetration of

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thermal wave. Thus, high frequency thermal oscillations are rapidly damped out in comparison with lower harmonic oscillations. The penetration depth also depends upon thermal diffusivity; the lower the value of $\alpha$, the smaller will be the depth at which the amplitude $(\theta_x)_{max}$ is negligible. It has been observed that the temperature oscillations into an infinitely thick wall die off when $x/2\,(\pi \alpha t_o)^{1/2} = 0.8$.

### (ii) Time Lag or Phase Difference at any Depth

Maximum fluctuation in surface temperature, as from Eq. (3.63), occurs when

$$\cos \omega t = 1$$

or
$$t = 0, \frac{2\pi}{\omega}, \frac{4\pi}{\omega}, \cdots \quad \text{or} \quad t = \frac{2n\pi}{\omega} \text{ for } n = 0, 1, 2, \ldots \tag{3.66}$$

whereas the maximum fluctuation at any depth $x$, according to Eq. (3.64)

$$\cos\left[\omega t - \left(\frac{\omega}{2\alpha}\right)^{1/2} x\right] = 1$$

or
$$\omega t - x\left(\frac{\omega}{2\alpha}\right)^{1/2} = 2n\pi$$

$\therefore$
$$t = x\left(\frac{1}{2\alpha\omega}\right)^{1/2} + \frac{2n\pi}{\omega} \tag{3.67}$$

For
$$n = 0,\ t = \left(\frac{1}{2\alpha\omega}\right)^{1/2} x$$

Comparing Eqs (3.66) and (3.67) it is seen that while the temperature excess at the surface is maximum when $t = (2n\pi)/\omega$, it is maximum at depth $x$ at a later time

$$t = \frac{2n\pi}{\omega} + x\left(\frac{1}{2\alpha\omega}\right)^{1/2}$$

The difference or time lag $\Delta t$ is given by

$$\Delta t = \left(\frac{1}{2\alpha\omega}\right)^{1/2} x \tag{3.68}$$

The time lag decreases with increasing $\alpha$ and $\omega$, and increases with increasing depth $x$. It is because of this time lag that the inner surfaces of the walls of a building, whose outside is exposed to the sun, attain maximum temperature much after the noon.

### (iii) Frequency and Time Period of Wave at any Depth

By comparing the values of time when the temperature maxima occur at the surface ($x = 0$) and at a depth $x$, it is seen that the oscillations have the same period ($2\pi/\omega$) at each depth. Thus, the periodic time and frequency of the wave do not change with depth.

### (iv) Wave Length and Wave Velocity

The wavelength is the distance between two adjacent crests. At $x = 0$, the maximum occurs at $t = 0$; the next maximum occurs at $t_1 = 2\pi/\omega$. But at $t_1 = 2\pi/\omega$, a maximum at depth $x$ occurs when

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$$\cos\left[\omega t_1 - \left(\frac{\omega}{2\alpha}\right)^{1/2} x\right] = 1$$

or
$$\omega t_1 - \left(\frac{\omega}{2\alpha}\right)^{1/2} x = 0$$

or
$$x = \omega t_1 \left(\frac{2\alpha}{\omega}\right)^{1/2} x$$

Wavelength $\lambda$ of temperature wave is

$$\lambda = x = 2\pi\left(\frac{\omega}{2\alpha}\right)^{1/2}$$

Velocity of the temperature wave is

$$v = f\lambda = \frac{\omega}{2\pi} 2\pi \left(\frac{2\alpha}{\omega}\right)^{1/2} = (2\alpha\omega)^{1/2} \tag{3.69}$$

### *(v) Heat Flow Rate and Energy Storage*

The heat in a periodic system flows into or out of the body periodically, since the temperature gradient at the surface is sometimes positive and sometimes negative. The heat absorbed or rejected is given by

$$Q = -kA\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)_{x=0}$$

From Eq. (3.64)

$$\theta = \theta_0\, e^{-\mu x} \cos(\omega t - \mu x)$$

where
$$\mu = \left(\frac{\omega}{2\alpha}\right)^{1/2}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \theta_0\, e^{-\mu x}\,[-\sin(\omega t - \mu x)]\,(-\mu) + \theta_0 \cos(\omega t - \mu x)\,(-\mu)\, e^{-\mu x}$$
$$= \theta_0\, e^{-\mu x} \,.\, \mu[\sin(\omega t - \omega x) - \cos(\omega t - \omega x)]$$

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)_{x=0} = \theta_0\,\mu\,(\sin\omega t - \cos\omega t)$$

Now,
$$\cos\left(\omega t + \frac{\pi}{4}\right) = \cos\omega t \cos\frac{\pi}{4} - \sin\omega t \sin\frac{\pi}{4}$$
$$= \frac{\cos\omega t - \sin\omega t}{\sqrt{2}}$$

$$\frac{Q}{A} = -k\theta_0\,\mu\left[-\cos\left(\omega t + \frac{\pi}{4}\right)\sqrt{2}\right]$$

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$$= k\theta_0 \left(\frac{\omega}{2\alpha}\right)^{1/2} \sqrt{2}\cos\left(\omega t + \frac{\pi}{4}\right)$$

$$\frac{Q}{A} = k\left(\frac{\omega}{\alpha}\right)^{1/2} \theta_0 \cos\left(\omega t + \frac{\pi}{4}\right) \tag{3.70}$$

It is seen that $Q$ is positive within limits

$$\omega t + \frac{\pi}{4} = -\frac{\pi}{2} \text{ to } -\frac{\pi}{2}$$

and negative in the limits

$$\omega t + \frac{\pi}{4} = +\frac{\pi}{2} \text{ to } +\frac{\pi}{2}$$

$Q$ is positive for

$$t = -\frac{3\pi}{4\omega} \text{ to } +\frac{\pi}{4\omega}$$

and negative for

$$t = +\frac{\pi}{4\omega} \text{ to } \frac{3\pi}{4\omega}$$

Integrating $Q$ between these limits in half cycle is

$$Q = kA\left(\frac{\omega}{\alpha}\right)^{1/2} \theta_0 \int_{-3\pi/4\omega}^{\pi/4\omega} \cos\left(\omega t + \frac{\pi}{4}\right) dt$$

$$= kA\left(\frac{\omega}{\alpha}\right)^{1/2} \theta_0 \left[\frac{1}{\omega}\sin\left(\omega t + \frac{\pi}{4}\right)\right]_{-3\pi/4}^{\pi/4\omega}$$

$$= \frac{2}{(\omega\alpha)^{1/2}} kA\,\theta_0 \tag{3.71}$$

The above analysis has been carried out by taking a *cosine variation* of surface temperature with time. Any other trigonometric variation could be taken and analysed by following the same procedure.

## Solved Problems

**Example 3.1** A load of peas at a temperature of 25°C is to be cooled down in a room at a constant air temperature of 1°C. (a) How long the peas will rquire to cool down to 2°C when the surface heat transfer coefficient of the peas is 5.81 W/m$^2$ K? (b) What is the temperature of the peas after a lapse of 10 min from the start of cooling? (c) What air temperature must be used if the peas were to be cooled down to 5°C in 30 min? The peas are supposed to have an average diameter of 8 mm. Their density is 750 kg/m$^3$ and specific heat 3.35 kJ/kg K.

**Solution** We have for a lumped heat-capacity system

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = e^{-\frac{hAt}{\rho c V}} = e^{-\mathrm{Bi}\cdot\mathrm{Fo}}$$

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Since the diameter of the peas is only 8 mm, we can neglect any temperature variation within the pea.

$$\frac{\rho V}{A} = \frac{\rho \times 4\pi (d/2)^3}{3 \times 4\pi (d/2)^2} = \frac{\rho d}{3 \times 2} = \frac{750 \times 0.008}{3 \times 2} = 1$$

(a) $\ln \frac{2-1}{25-1} = -\frac{ht}{c} = -\frac{5.81\,t}{3.35 \times 10^3}$

$\ln 24 = 3.178 = \frac{5.81\,t}{3.35 \times 10^3}$

$t = 1832.4$ s = 30.54 min *Ans.* (a)

(b) $\frac{T-1}{25-1} = e^{-\frac{5.81 \times 600}{3350}} = 0.353$

$T = 9.48°C$ *Ans.* (b)

(c) $\frac{5-T_\infty}{25-T_\infty} = e^{-\frac{5.81 \times 30 \times 60}{3350}} = 0.044$

$1.1 - 0.044\,T_\infty = 5 - T_\infty$

$0.956\,T_\infty = 3.9$

$T_\infty = 4.08°C$ *Ans.* (c)

**Example 3.2** A steel tube of length 20 cm with internal and external diameters of 10 and 12 cm is quenched from 500°C to 30°C in a large reservoir of water at 10°C. Below 100°C the heat transfer coefficient is 1.5 kW/m$^2$ K. Above 100°C it is less owing to a film of vapour being produced at the surface, and an effective mean value between 500°C and 100°C is 0.5 kW/m$^2$ K. The density of steel is 7800 kg/m$^3$ and the specific heat is 0.47 kJ/kg K. Neglecting internal thermal resistance of the steel tube, determine the quenching time.

**Solution** $d_i = 10$ cm, $d_0 = 12$ cm, $l = 20$ cm, $T_\infty = 10°C$

$$A = \pi(d_0 + d_i)l = \pi(10 + 12) \times 20 = 1382 \text{ cm}^2$$

$$V = \frac{\pi}{4}(d_0^2 - d_i^2)\,l = \frac{\pi}{4}(144 - 100) \times 20 = 691 \text{ cm}^3$$

Cooling from 500°C to 100°C

$$-\ln \frac{T - T_\infty}{T_i - T_\infty} = +\frac{hAt}{\rho c V}$$

$$-\ln \frac{100-10}{500-10} = \ln \frac{490}{90} = \frac{500 \times 1382 \times 10^{-4} \times t}{7800 \times 470 \times 691 \times 10^{-6}}$$

$\therefore$ $t = 62.12$ s

Cooling from 100°C to 30°C

$$-\ln \frac{30-10}{100-10} = \frac{1500 \times 1382 \times 10^{-4} \times t}{7800 \times 470 \times 691 \times 10^{-6}}$$

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$\therefore \qquad t = 18.38\text{s}$

Total time for quenching = 62.12 + 18.38 = 80.5 s *Ans.*

**Example 3.3** Steel ball bearings ($k$ = 50 W/m K, $\alpha$ = 1.3 × $10^{-5}$ $m^2$/s) having a diameter of 40 mm are heated to a temperature of 650°C and then quenched in a tank of oil at 55°C. If the heat transfer coefficient between the ball bearings and oil is 300 W/$m^2$ K, determine (a) the duration of time the bearings must remain in oil to reach a temperature of 200°C, (b) the total amount of heat removed from each bearing during this time and (c) the instantaneous heat transfer rate from the bearings when they are first immersed in oil and when they reach 200°C.

**Solution** To determine whether the bearings have negligible resistance, we first check the magnitude of the Biot number.

$$\text{Bi} = \frac{hL}{k} = \frac{h\frac{r}{3}}{k} = \frac{300 \times 0.02}{3 \times 50} = 0.04$$

Since Bi < 0.1, internal resistance may be neglected.

$$\text{Fo} = \frac{\alpha t}{L^2} = \frac{\alpha t}{(r/3)^2} = \frac{1.3 \times 10^{-5} t}{(0.02/3)^2} = 0.2925\, t$$

(a) The time required for the ball bearings to reach 200°C is

$$\frac{\theta}{\theta_i} = e^{-\text{Bi}\cdot\text{Fo}}$$

$$\frac{200 - 55}{650 - 55} = e^{-0.04 \times 0.2925\, t} = \frac{145}{595} = e^{-0.0117\, t}$$

$\therefore$ $t$ = 120.67 s, which corresponds to a Fourier number of 35.3 *Ans.*

(b) Total amount of heat removed from each bearing during 120.67 s

$$Q = hA(T - T_\infty) = \int_0^t hA(T_i - T_\infty)\, e^{-\text{Bi}\times\text{Fo}}\, dt$$

$$= hA\theta_i \int_0^t e^{-\frac{hAt}{\rho c V}}\, dt = hA\theta_i\,(1 - e^{-\text{Bi}\,.\,\text{Fo}})\, \frac{t}{\text{Bi}\cdot\text{Fo}}$$

$$= 300 \times 4\pi\,(0.02)^2\,(650 - 55)\,(1 - e^{-0.04 \times 35.3}) \times \frac{120.5}{0.04 \times 35.31}$$

$$= 5.79 \times 10^4 \text{ W s or } J = 57.9 \text{ kJ} \quad \textit{Ans.}$$

(c) Instantaneous heat transfer rate at $t$ = 0 (or Fo = 0) is

$$Q = hA(T_i - T_\infty) = 300 \times 4\pi \times (0.02)^2 \times (650 - 55) = 897 \text{ W}$$

and at $\qquad t$ = 120 s (Fo = 35.3),

$$Q = hA(T_i - T_\infty)\, e^{-\text{Bi Fo}}$$

$$= 300 \times 4\pi \times (0.02)^2\,(650 - 50)\, e^{-(0.04)\,(35.3)}$$

$$= 218 \text{ W} \quad \textit{Ans.}$$

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**Example 3.4** The heat transfer coefficients for the flow of air at 28°C over a 12.5 mm diameter sphere are measured by observing the temperature–time history of a copper ball of the same dimension. The temperature of the copper ball ($c$ = 0.4 kJ/kg K, $\rho$ = 8850 kg/m$^3$) was measured by two thermocouples, one located in the centre, the other near the surface. Both of the thermocouples registered within the accuracy of the recording instruments the same temperature at a given instant. In one test the initial temperature of the ball was 65°C and in 1.15 min the temperature decreased by 11°C. Calculate the heat transfer coefficient for this case.

**Solution** Since there was no temperature difference recorded by the two thermocouples inserted at the centre and the surface of the copper ball, its thermal conductivity may be considered infinitely large offering no internal thermal resistance, for which case, the lumped system analysis may be used.

*Given:* $T_i$ = 65°C, $T_\infty$ = 28°C, $T_f$ = 65 – 11 = 54°C.

Using Eq. (3.4),

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = e^{-\frac{hAt}{\rho cV}} = \frac{54-28}{65-28} = \frac{26}{37}$$

$$\ln\frac{37}{26} = \frac{hAt}{\rho cV} = 0.353$$

$$\therefore \qquad h = \frac{0.353 \times 8850 \times 0.4 \times \frac{4}{3}\pi \times \left(\frac{0.0125}{2}\right)^3}{4\pi \times (0.0125)^2 \times 1.15 \times 60}$$

$$= 0.0377 \text{ kW/m}^2\text{K} = 37.7 \text{ W/m}^2\text{K} \quad \textit{Ans.}$$

**Example 3.5** A thermocouple, the junction of which can be approximated as a 1 mm diameter sphere is used to measure the temperature of a gas stream. The properties of the junction are $\rho$ = 8500 kg/m$^3$, $c$ = 320 J/kg K and $k$ = 35 W/m K. The heat transfer coefficient between the junction and the gas is 210 W/m$^2$ K. Determine how long it will take for the thermocouple to read 99% of the initial temperature difference.

**Solution** The schematic of the thermocouple is shown in Fig. Ex. 3.5

$$L = \frac{V}{A} = \frac{\frac{1}{6}\pi D^3}{D^2} = \frac{1}{6}D = \frac{1}{6} \times 0.001 = 1.67 \times 10^{-4} \text{ m}$$

$$\text{Bi} = \frac{hL}{k} = \frac{210 \times 1.67 \times 10^{-4}}{35} = 0.001$$

Since Bi < 0.1, lumped system analysis can be used.

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = 0.01 = e^{-\frac{hAt}{\rho cV}}$$

$$\frac{hAt}{\rho cV} = \ln 100 = 4.605$$

$$t = \frac{4.605 \times 8500 \times 320 \times 1.67 \times 10^{-4}}{210} = 9.96 \text{ s}$$

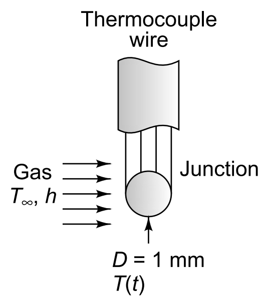


**Fig.** *Ex. 3.5*

Therefore, we must wait at least 10 s for the temperature of the thermocouple junction to approach within 1% of the initial junction – gas temperature difference.

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**Example 3.6** An average human body modeled as a 30 cm diameter, 170 cm long cylinder has 72% water by mass, so that its properties may be taken as those of water at room temperature: $\rho$ = 1000 kg/m$^3$, $c$ = 4180 J/kg K and $k$ = 0.608 W/m K. A person is found dead at 5 am in a room the temperature of which is 20°C. The temperature of the body is measured to be 25°C when found, and the heat transfer coefficient is estimated to be 8 W/m$^2$ K. Assuming the body temperature of a living man is 37°C, estimate the time of death of the above person.

**Solution** Characteristic length

$$L = \frac{\pi r_0^2 L}{2\pi\, r_0 L + 2\pi r_0^2}$$

$$= \frac{\pi \times (0.15)^2 \times 1.7}{2\pi (0.15)(1.7) + 2\pi\,(0.15)^2}$$

$$= 0.0689 \text{ m}$$

$$\text{Bi} = \frac{hL}{k} = \frac{8 \times 0.0689}{0.608} = 0.92$$

Since Bi > 0.1, the lumped system analysis is not applicable. However, we can still use it to obtain a rough estimate of the time of death.

$$\frac{\theta}{\theta_i} = e^{-\frac{hAt}{\rho c V}} = \frac{25 - 20}{37 - 20} = \frac{5}{17}$$

$$\frac{hAt}{\rho c V} = \ln \frac{17}{5} = 1.2238$$

$$t = 44057 \text{ s} = 12.24\ h$$

The person died about 12.5 h before the body was found, and thus the time of death is 4.30 pm.

**Example 3.7** A rocket engine nozzle is constructed from a piece of high temperature steel 0.64 cm thick, having a uniform thermal conductivity of 29 W/m K and a uniform thermal diffusivity of 6.39 × 10$^{-6}$ m$^2$/s. The flame side surface film coefficient is 8.37 kW/m$^2$ K. The uniform flame temperature is 2200°C. If the nozzle temperature is initially 25°C throughout and the maximum allowable operating temperature of this steel is 1100°C, then what is the permissible combustion duration?

**Solution** Since the thickness of the wall is very small compared to diameter, we can regard the wall as an infinite plane wall of thickness 0.64 cm (Fig. Ex. 3.7). We will assume the outer surface of the nozzle to be insulated against heat loss so that the axis $x = 0$ may be chosen as the outer surface.

The temperature distribution is given by Eq. (3.32)

$$\frac{\theta}{\theta_i} = \sum_{n=1}^{\infty} e^{-\delta_n^2 \text{ Fo}} \frac{2 \sin \delta_n}{\delta_n + \sin\, \delta_n \cos \delta_n} \cos \frac{\delta_n}{l} x$$

$$\frac{\theta_1}{\theta_i} = \frac{T_1 - T_\infty}{T_i - T_\infty} = \sum_{n=1}^{\infty} e^{-\delta_n^2 \text{ Fo}} \frac{2 \sin \delta_n \cos \delta_n}{\delta_n + \sin \delta_n \cos \delta_n}$$

$$\frac{\theta_1}{\theta_i} = \frac{1100 - 25}{2200 - 25} = \frac{1075}{2175} = 0.494$$

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$$\frac{1}{\text{Bi}} = \frac{k}{hl} = \frac{29}{8370 \times 0.64 \times 10^{-2}} = 0.541$$

$$\delta_n \tan \delta_n = \text{Bi}$$

When $\delta = \lambda l = \dfrac{\pi}{4} = 45°$

$$\tan \delta = \frac{\text{Bi}}{\delta} = \frac{4}{0.541 \times \pi} = 2.353$$

$\delta = 66.98°$, which is different from the assumed value of 45°.

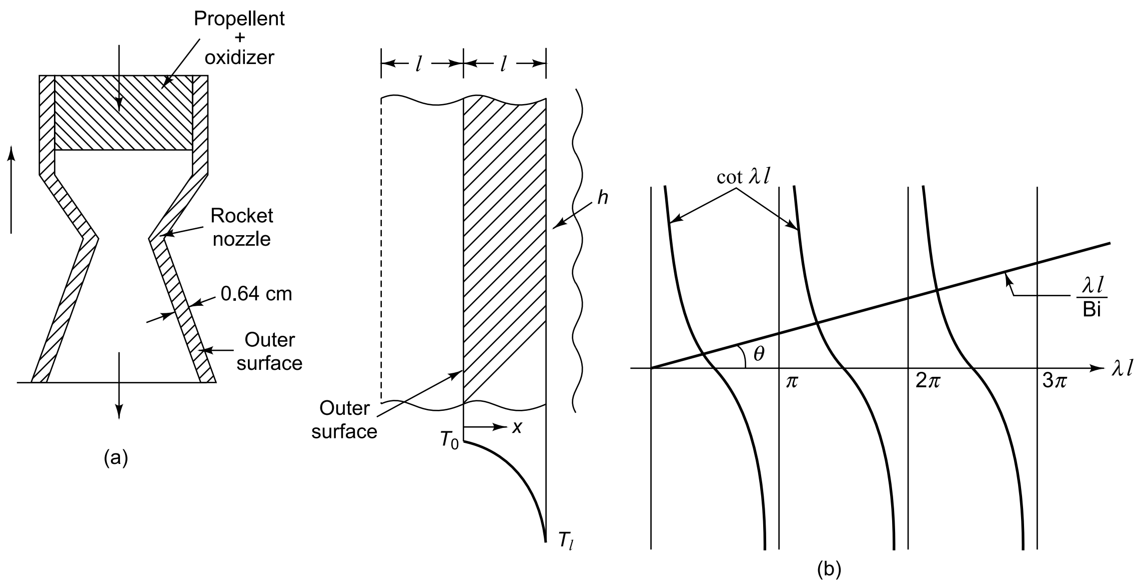


**Fig.** *Ex. 3.7*

When $\delta = \dfrac{\pi}{3} = 60°$

$$\tan \delta = \frac{3}{0.541 \times \pi} = 1.765$$

$\therefore\ \delta = 60° = \dfrac{\pi}{3}$, which tallies with the assumption.

Taking $n = 1$,

$$\frac{\theta_1}{\theta_i} = e^{-\delta_n^2 \text{ Fo}} \frac{\sin 2\delta_1}{\delta_1 + \sin \delta_1 \cos \delta_1}$$

$$0.494 = e^{-\left(\frac{\pi}{3}\right)^2 \text{Fo}} \frac{\sin 120°}{\pi/3 + \sin\ 60° \cos 60°}$$

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$$= e^{-1.0966\,\text{Fo}} \frac{0.866}{1.0472 + 0.866 \times 0.5}$$

$$= e^{-1.0966\,\text{Fo}} \times 0.585$$

$$e^{1.0966\,\text{Fo}} = \frac{0.585}{0.494} = 1.184$$

$$\text{Fo} = 0.154 = \frac{\alpha t}{l^2}$$

$$t = \frac{0.154 \times (0.64)^2 \times 10^{-4}}{6.39 \times 10^{-6}}$$

$$= 0.9888 = 0.99 \text{ s}$$

Combustion must be completed within about 1 s. *Ans.*

**Example 3.8** An egg is being cooked in boiling water. Its initial temperature is 5°C. It is dropped into boiling water at 95°C.

*Assumptions:* 1. The egg is spherical in shape with a radius of $r_0$ = 0.025 m. 2. Heat conduction in the egg is one-dimensional (radial only) because of thermal symmetry about the centre. 3. The thermal properties of the egg and the heat transfer coefficient are constant. 4. The Fourier number $F_0$ > 0.2 so that the one term approximate solutions are applicable. 5. The thermal conductivity and diffusivity of eggs are approximated as those of water. $k$ = 0.6 W/mK, $\alpha$ = 0.14 × $10^{-6}$ m$^2$/s. The heat transfer coefficient is 1200 W/m$^2$K.

*To find:* The time for the centre of the egg to reach 70°C.

**Solution** The temperature within the egg varies with radial distance as well as time, and the temperature at a specified location at a given time can be determined from the Heisler charts or the one-term solutions. Here, we will use the latter to demonstrate their use. The Biot number is equal to

$$\text{Bi} = \frac{hr_0}{k} = \frac{1200 \text{ W/m}^2\text{K} \times 0.025 \text{ m}}{0.6 \text{ W/mK}} = 50$$

which is much greater than 0.1. Actually, for a sphere Bi = $\frac{h.r_o/3}{k}$ is somewhat less. Thus the lumped system analysis is not applicable. From Table 3.1, the coefficients $A_1$ and $\lambda_1$ for a sphere corresponding to Bi = 50 are $A_1$ = 1.6002 and $\lambda_1$ = 3.0788. Substituting in Eq. (3.45),

$$\frac{T_o - T_\infty}{T_i - T_\infty} = A_1 . e^{-\lambda_1^2 F_o}$$

$$\frac{70 - 95}{5 - 95} = 1.6002 \cdot e^{-(3.0788)^2 F_o}$$

which gives $F_o$ = 0.208, greater than 0.2. Thus one-term solution is applicable.

$$F_o = 0.208 = \frac{\alpha t}{r_o^2}$$

$$\therefore \qquad t = \frac{0.208 \times (0.025)^2}{0.14 \times 10^{-6}} = 15.5 \text{ min.}$$

So it will take about 15 minutes for the centre of the egg to be heated from 5ºC to 70ºC. *Ans.*

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**Example 3.9** Large brass plates initially at 20°C are heated in an oven. The surface temperature of plates leaving the oven maintained at 500°C is to be determined after a lapse of 7 min.

*Assumptions:* 1. Heat conduction in the plate is one-dmensional since the plate is large relative to the thickness and there is thermal symmetry about the central plane. 2. The thermal properties of the plate and the heat transfer coefficient are constant. 3. The properties of brass at room temperature (20°C) are $k$ = 110 W/mK, $\rho$ = 8530 kg/m$^3$, $c$ = 380 J/kgK and $\alpha = 33.9 \times 10^{-6}$ m$^2$/s.

**Solution** The temperature at a specified location at a given time can be determined from the Heisler charts or one-term solutions. Here we will use the charts to demonstrate their use. The half-thickness of the plate is $L = 0.02$ m. We have

$$\frac{1}{\text{Bi}} = \frac{k}{h_L} = \frac{100}{120 \times 0.02} = 45.8$$

$$\text{Fo} = \frac{\alpha t}{L^2} = \frac{33.9 \times 10^{-6} \times 7 \times 60}{(0.02)^2} = 35.6$$

From Fig. 3.7(a), when 1/Bi = 45.8 and Fo = 35.6

$$\frac{T_c - T_\infty}{T_i - T_\infty} = 0.46$$

Also, $\frac{1}{\text{Bi}} = 45.8$ and $\xi = \frac{x}{L} = 1$

$$\frac{T_1 - T_\infty}{T_c - T_\infty} = 0.99 \text{ [Fig. 3.7 (b)]}$$

$$\frac{T_1 - T_\infty}{T_c - T_\infty} \times \frac{T_c - T_i}{T_i - T_\infty} = 0.46 \times 0.99 = 0.455$$

$$\therefore \quad T_1 = 0.455\,(20 - 500) + 500 = 281.6\text{ºC}$$

which is the surface temperature of the plates after a lapse of 7 min. *Ans.*

*Discussion:* We observe that Bi = 1/45.8 = 0.022 which is much less than 0.1. Therefore, we expect that the lumped system analysis is applicable. This is also evident from $(T_1 - T_\infty)/(T_c - T_\infty) = 0.99$, which indicates that the temperatures at the centre and the surface relative to the surrounding temperature are within 1%. Though the error involved in reading Heisler charts might be a few percent, the lumped system analysis may yield as accurate results with less effort, as shown below:

$$\frac{hA}{\rho c V} = \frac{h}{\rho c L} = \frac{120}{8530 \times 380 \times 0.02}$$

$$= 0.00185 \text{ s}^{-1}$$

$$\therefore \quad \frac{T - T_\infty}{T_i - T_\infty} = \frac{T - 500}{20 - 500} = e^{-0.00185 \times 420}$$

$$\therefore \quad T = 279\text{ºC}$$

which is very close to the value obtained from Heisler charts.

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**Example 3.10** *Given:* A steel cylinder 0.35 m diameter and 0.70 m long at 20°C is heated in an oven maintained at 1050°C. The temperatures at the centre and surface of the cylinder after an hour are to be determined. Take $k$ = 34.9 W/mK, $c$ = 0.7 kJ/kgK, $\rho$ = 7800 kg/m$^3$ and $h$ = 232.5 W/m$^2$K.

*Assumptions:* 1. Heat conduction in the short cylinder has thermal symmetry about the centreline. 2. The thermal properties of the cylinder and the heat transfer coefficient are constant.

**Solution** $$\alpha = k/\rho c = \frac{34.9}{7800 \times 700} = 6.39 \times 10^{-6} \text{ m}^2/\text{s}$$

Now, $$\frac{\theta_c}{\theta_i} = \left(\frac{\theta_c}{\theta_i}\right)_{2l} \times \left(\frac{\theta_c}{\theta_i}\right)_r$$

Fourier number, $$\text{Fo} = \frac{\alpha t}{r^2} = \frac{6.39 \times 10^{-6}}{(0.35/2)^2} = 0.75$$

$$\frac{1}{\text{Bi}} = \frac{k}{hr} = \frac{34.9 \times 2}{232.6 \times 0.35} = 0.857$$

From Heisler chart for infinite cylinder

$$\left(\frac{\theta_c}{\theta_i}\right)_r = 0.27, \left(\frac{\theta_s}{\theta_i}\right)_r = 0.183$$

Again, $$\text{Fo} = \frac{\alpha t}{l^2} = \frac{6.39 \times 10^{-6} \times 3600}{(0.35)^2} = 0.188$$

$$\frac{1}{\text{Bi}} = \frac{k}{hl} \frac{34.9}{232.6 \times 0.35} = 0.429$$

From Heisler chart for infinite plate,

$$\left(\frac{\theta_c}{\theta_i}\right)_{2l} = 0.964, \left(\frac{\theta_s}{\theta_i}\right)_{2l} = 0.86$$

$$\frac{\theta_c}{\theta_i} = \frac{T_c - T_\infty}{T_i - T_\infty} = 0.964 \times 0.27 = 0.26$$

$$\frac{T_c - 1050}{20 - 1050} = 0.26$$

$\therefore$ $$T_c = 1050 - 267.8 = 782.2\text{°C} \quad \textit{Ans.}$$

$$\frac{\theta_s}{\theta_i} = \frac{T_s - 1050}{20 - 1050} = 0.183 \times 0.86 = 0.157$$

$$T_s = 1050 - 161.7 = 888.3\text{°C} \quad \textit{Ans.}$$

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**Example 3.11** A billet of steel of the form of a parallelepiped with dimensions 2 m × 2 m × 5 m originally at 300°C is placed in a radiant furnace, where the furnace temperatue is held at 1200°C. Determine the temperature at the centre after 25 min. Take $\alpha$ = 6.39 × 10$^{-6}$ m$^2$/s, $k$ = 34.9 W/m K and $h$ = 232.6 W/m$^2$ K.

**Solution**

$$\frac{\theta_c}{\theta_i} = \left(\frac{\theta_c}{\theta_i}\right)_{2a} \times \left(\frac{\theta_c}{\theta_i}\right)_{2b} \times \left(\frac{\theta_c}{\theta_i}\right)_{2c}$$

Since, $$a = b, \frac{\theta_c}{\theta_i} = \left(\frac{\theta_c}{\theta_i}\right)_{2a}^2 \times \left(\frac{\theta_c}{\theta_i}\right)_{2c}$$

$$\text{Fo}_1 = \frac{\alpha t}{l^2} = \frac{6.39 \times 10^{-6} \times 25 \times 60}{l^2} = 0.009585$$

$$\text{Fo}_2 = \frac{6.39 \times 10^{-6} \times 25 \times 60}{(2.5)^2} = 0.001534$$

$$\frac{1}{\text{Bi}_1} = \frac{k}{hl} = \frac{34.9}{232.6 \times l}\ 0.15$$

$$\frac{1}{\text{Bi}_2} = \frac{34.9}{232.6 \times 2.5} = 0.06$$

From Heisler's charts,

$$\left(\frac{\theta_c}{\theta_i}\right)_{2a} = 0.865 \left(\frac{\theta_c}{\theta_i}\right)_{2c} = 0.81$$

$$\frac{\theta_c}{\theta_i} = (0.865)^2 \times 0.81 = 0.606 = \frac{T_c - T_\infty}{T_i - T_\infty}$$

$$T_c = 1200 + 0.606\ (300 - 1200) = 654.5°\text{C} \quad \textit{Ans.}$$

**Example 3.12** A semi-infinite aluminium cylinder ($k$ = 237 W/m K, $\alpha$ = 9.71 × 10$^{-5}$ m$^2$/s) of diameter 20 cm is initially at a uniform temperature of 200°C. The cylinder is now placed in water at 15°C where heat transfer takes place by convection with $h$ = 120 W/m$^2$ K. Determine the temperature at the centre of the cylinder 15 cm from the end surface 5 min after the start of the cooling.

**Solution** We will solve the problem using the one-term solution for the cylinder and the analytic solution for the semi-infinite medium (Fig. Ex. 3.12).

$$\text{Bi} = \frac{hr_o}{k} = \frac{120 \times 0.1}{237} = 0.05$$

$$\text{Fo} = \frac{\alpha t}{r^2} = \frac{9.71 \times 10^{-5} \times 5 \times 60}{(0.1)^2} = 2.913 > 0.2$$

Thus, the one-term solution is applicable. From Table 3.2 for cylinder, and Bi = 0.05, $A_1$ = 1.0124 and $\delta_1$ = 0.3126

$$\theta_c = A_1\, e^{-\delta_1^2\, \text{Fo}} = 1.0124\ e^{-(0.3126)^2 \times (2.913)}$$
$$= 0.7616$$

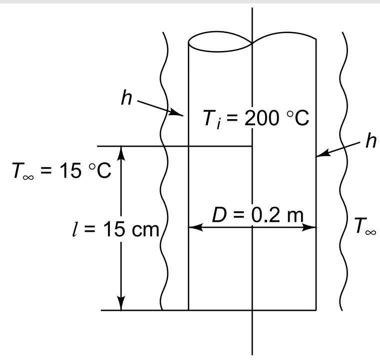


**Fig.** *Ex. 3.12*

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The solution for the semi-infinite solid can be determined from Eq. (3.52)

$$1-\theta_{\text{semi-inf}}(x,t)=\operatorname{erf}c\left[\frac{x}{2(\alpha t)^{1/2}}\right]-\exp\left(\frac{hx}{k}+\frac{h^2\alpha t}{k^2}\right)\left\{\operatorname{erf}c\left[\frac{x}{2(\alpha t)^{1/2}}+\frac{h(\alpha t)^{1/2}}{k}\right]\right\}$$

where $$\xi=\frac{x}{2(\alpha t)^{1/2}}=\frac{0.15\text{ m}}{2(9.71\times10^{-5}\times5\times60)^{1/2}}=0.88$$

$$\frac{h(\alpha t)^{1/2}}{k}=\frac{120\left[9.71\times10^{-5}\times5\times60\right]^{1/2}}{237}=0.086$$

$$\frac{hx}{k}=\frac{120\times0.15}{237}=0.0759$$

$$\frac{h^2\alpha t}{k^2}=(0.086)^2=0.0074$$

$$(\theta)_{\text{semi-inf}}=1-\operatorname{erf}c\,(0.88)+\exp\,(0.0759+0.0074)\operatorname{erf}c\,(0.88+0.086)$$

$$=1-0.2133+\exp\,(0.0833)\times0.170=0.974$$

$$\frac{T_c-T_\infty}{T_i-T_\infty}=\theta_{\text{semi-inf}}\times\theta_{\text{cyl}}=0.974\times0.762=0.742$$

$$T_c=15+(200-15)\times0.742=152.3°\text{C}\quad \textit{Ans.}$$

**Example 3.13** An infinite plate 30 cm thick and having thermal diffusivity $\alpha = 1.39\times10^{-5}$ m$^2$/s is initially at a temperature of 0°C. At time $t$ = 0, its surface temperatures at $x$ = 0 and $x$ = $l$ are suddenly raised to and maintained at 400°C for all $t$ > 0. Determine the instantaneous temperature distribution inside the plate after 15 min, taking $M$ = 2 and $M$ = 3, and the mid-plane temperature.

**Solution** Infinite plate, $l$ = 30 cm, $T_i$ = 0°C

$(T_w)_{t=0} = T_o = T_6 = 400$°C (Fig. Ex. 3.13)

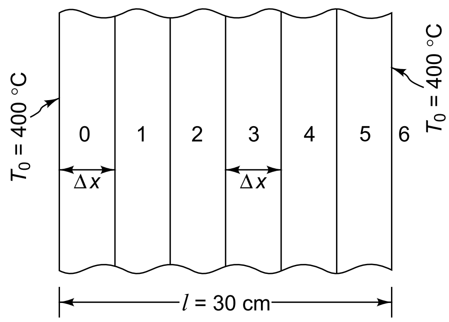


**Fig.** Ex. 3.13

(a) With $M$ = 2

Let $\Delta x$ = 5 cm

∴ No. of slices $n_s$ = 6

$$M=\frac{1}{\text{Fo}}=\frac{(\Delta x)^2}{\alpha\Delta t}=\frac{5^2\times10^{-4}}{1.39\times10^{-5}\,\Delta t}=2$$

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$$\Delta t = 90 \text{ s} = 1.5 \text{ min}$$

$$\text{Number of time steps} = \frac{15 \text{ min}}{1.5 \text{ min}} = 10$$

For $M = 2$

| $t$ | $T_0$ | $T_1$ | $T_2$ | $T_3$ | $T_4$ | $T_5$ | $T_6$ |
|---|---|---|---|---|---|---|---|
| 0 | 400 | 0 | 0 | 0 | 0 | 0 | 400 |
| 1 | 400 | 200 | 0 | 0 | 0 | 200 | 400 |
| 2 | 400 | 200 | 100 | 0 | 100 | 200 | 400 |
| 3 | 400 | 250 | 100 | 100 | 100 | 250 | 400 |
| 4 | 400 | 250 | 175 | 100 | 175 | 250 | 400 |
| 5 | 400 | 287.5 | 175 | 175 | 175 | 287.5 | 400 |
| 6 | 400 | 287.5 | 231.25 | 175 | 231.25 | 287.5 | 400 |
| 7 | 400 | 315.625 | 231.25 | 231.25 | 231.25 | 315.625 | 400 |
| 8 | 400 | 315.625 | 273.44 | 231.25 | 273.44 | 315.625 | 400 |
| 9 | 400 | 336.714 | 273.44 | 273.44 | 273.44 | 336.714 | 400 |
| 10 | 400 | 336.714 | 305.08 | 273.44 | 305.08 | 336.714 | 400 |

Mid-plane temperature, $T_3 = 273.44°\text{C}$ *Ans.*

(b) With $M = 3$

$$M = \frac{1}{\text{Fo}} = \frac{(\Delta x)^2}{\alpha \Delta t} = \frac{5^2 \times 10^{-4}}{1.39 \times 10^{-5} \Delta t} = 3$$

$$\Delta t = \frac{250}{1.39 \times 3} = 59.95 \text{ s} = 1 \text{ min}$$

$$\text{Number of time steps} = \frac{15 \text{ min}}{1 \text{ min}} = 15$$

Mid-plane temperature, $T_3 = 278°\text{C}$ *Ans.*

For $M = 3$

| $\Delta t$ | $T_0$ | $T_1$ | $T_2$ | $T_3$ | $T_4$ | $T_5$ | $T_6$ |
|---|---|---|---|---|---|---|---|
| 0 | 400 | 0 | 0 | 0 | 0 | 0 | 400 |
| 1 | 400 | 133 | 0 | 0 | 0 | 133 | 400 |
| 2 | 400 | 178 | 44 | 0 | 44 | 178 | 400 |
| 3 | 400 | 207 | 74 | 29 | 74 | 207 | 400 |
| 4 | 400 | 227 | 103 | 59 | 103 | 227 | 400 |
| 5 | 400 | 243 | 130 | 88 | 130 | 243 | 400 |
| 6 | 400 | 258 | 154 | 116 | 154 | 258 | 400 |
| 7 | 400 | 271 | 176 | 141 | 176 | 271 | 400 |
| 8 | 400 | 282 | 196 | 164 | 196 | 282 | 400 |
| 9 | 400 | 293 | 214 | 185 | 214 | 293 | 400 |
| 10 | 400 | 302 | 231 | 204 | 231 | 302 | 400 |
| 11 | 400 | 311 | 246 | 222 | 246 | 311 | 400 |
| 12 | 400 | 319 | 260 | 238 | 260 | 319 | 400 |
| 13 | 400 | 326 | 372 | 253 | 272 | 326 | 400 |
| 14 | 400 | 333 | 284 | 266 | 284 | 333 | 400 |
| 15 | 400 | 339 | 294 | 278 | 294 | 339 | 400 |

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**Example 3.14** A 25 cm thick wall of common brick is initially at 80°C, and suddenly its surfaces are reduced to 15°C. Find the temperature at a point 10 cm from the surface and at the mid-plane after 2 h. How much heat has been conducted out of the wall per unit surface area during that time?

*Given:* $\rho = 1.6 \times 10^3$ kg/m$^3$, $c = 0.84$ kJ/kg K,
$\alpha = 5.2 \times 10^{-7}$ m$^2$/s $k = 0.7$ W/m K.

**Solution** We have from Eq. (3.13),

$$\frac{T - T_1}{T_i - T_1} = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}e^{-\left(\frac{n\pi}{L}\right)^2 \alpha t}\sin\frac{n\pi}{L}x$$

Given: $T_i = 80°\text{C}$, $T_1 = 15°\text{C}$, $L = 25$ cm $= 0.25$ m
$t = 2$ h $= 7200$ s

When, $x = 10$ cm $= 0.1$ m, taking $n = 1$ and 3,

$$\frac{T - T_1}{T_i - T_1} = \frac{4}{\pi}\left[e^{-\left(\frac{n\pi}{0.25}\right)^2 \times 5.2\times10^{-7}\times7200}\sin\frac{\pi\times0.1}{0.25} + \frac{1}{3}e^{-\left(\frac{3\pi}{0.25}\right)^2\times5.2\times10^{-7}\times7200}\sin\frac{30\pi}{25}\right]$$

$$= \frac{4}{\pi}\left(e^{-0.591}\times0.951 + \frac{1}{3}e^{-5.321}\times0.588\right)$$

$$= \frac{4}{\pi}(0.527 + 9.58\times10^{-4}) = 0.671$$

$T = 15 + 0.671\,(80 - 15) = 58.6°\text{C}$ *Ans.*

At mid-plane, i.e. $x = 12.5 \text{ cm} = \frac{l}{2}$

$$\frac{T - T_1}{T_i - T_1} = \frac{4}{\pi}\left[e^{-\left(\frac{\pi}{0.25}\right)^2\times5.2\times10^{-7}\times7200}\sin\frac{\pi}{L}\frac{L}{2} + \frac{1}{3}e^{-\left(\frac{\pi}{0.25}\right)^2\times5.2\times10^{-7}\times7200}\sin\frac{3\pi}{2}\right]$$

$$= \frac{4}{\pi}\left(e^{-0.591}\times1 + \frac{1}{3}e^{-5.321}\right) = 0.705$$

$T = 15 + 0.705\,(80 - 15) = 60.82°\text{C}$ *Ans.*

Heat conducted per unit area at $x = 0.1$ m

$$q = \frac{Q}{A} = \frac{4}{\pi}\frac{kL}{\alpha}(T_1 - T_i)\sum_{n=1}^{\infty}\frac{1}{n^2}\left[1 - e^{-\left(\frac{n\pi}{L}\right)^2\alpha t}\right]\cos\frac{n\pi}{L}x$$

$$= -\frac{4}{\pi}\times65\times\frac{0.7\times0.25}{5.2\times10^{-7}}\left[(1 - e^{-0.591})\cos\frac{10\pi}{25} - \frac{1}{9}(1 - e^{-5.321})\cos\frac{30\pi}{25}\right]$$

$$= -2.785\times10^7\left[(1 - 0.554)\,0.31 - \frac{1}{9}(1 - 4.88\times10^{-3})\ 0.81\right]$$

$= -0.385\times10^7$ J/m$^2$ $= 3850$ kJ/m$^2$ *Ans.*

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At mid-plane i.e., $x = 12.5 \text{ cm} = \frac{1}{2}$

$$Q/A = 0$$

The mid-plane is an adiabatic plane.

**Example 3.15** An iron plate ($k$ = 60 W/m K, $c$ = 0.46 kJ/kg K, $\rho$ = 7850 kg/m$^3$ and $\alpha = 1.6 \times 10^{-5}$ m$^2$/s) of 50 mm thickness is initially at 225°C. Suddenly, both surfaces are exposed to an ambient temperature of 25°C with a heat transfer coefficient of 500 W/m$^2$ K. Calculate (a) the centre temperature at 2 min after the start of cooling (b) the temperature at a depth 1 cm from the surface at 2 min after the start of cooling and (c) the energy removed from the plate per sq. m during this time.

**Solution** Transient temperature charts may be used to solve this problem, since the lumped system analysis is not applicable. We have

$$2L = 50 \times 10^{-3} = 0.05 \text{ m or } L = 0.025 \text{ m}$$

$$\text{Fo} = \frac{\alpha t}{L^2} = \frac{(1.6 \times 10^{-5})(2 \times 60)}{(0.025)^2} = 3.1$$

$$\frac{1}{\text{Bi}} = \frac{k}{hL} = \frac{60}{500 \times 0.025} = 4.8$$

$$\text{Bi} = 0.21$$

From Fig. 3.7(a) for Fo = 3.1 and 1/Bi = 4.8, the centre temperature $T_c$ is

$$\frac{T_c - T_\infty}{T_i - T_\infty} = 0.58$$

$$T_c = T_\infty + (T_i - T_\infty)(0.58) = 25 + 200 \times 0.58$$

$$= 141°\text{C} \quad \textit{Ans.} \text{ (a)}$$

The temperature 1 cm from the surface is determined as

$$\xi = \frac{x}{L} = \frac{2.5 - 1}{2.5} = 0.6$$

For 1/Bi = 4.8 and $x/L$ = 0.6, from Fig. 3.7(b) we have

$$\frac{T - T_\infty}{T_c - T_\infty} = 0.95$$

$$T = T_\infty + 0.95\,(T_c - T_\infty) = 25 + 0.95\,(141 - 25)$$

$$= 135.2°\text{C} \quad \textit{Ans.} \text{ (b)}$$

The heat loss from the plate per sq. m. (including both sides) during the time of 2 min is determined as given below. From Fig. 3.7(c) for Bi = 0.21 and Bi$^2$ Fo = $(0.21)^2 \times 3.1 = 0.137$, we have

$$\frac{Q}{Q_i} = 0.45$$

where $Q_i = \rho\,(2L)\,A\,c_p\,(T_i - T_\infty)$

$$= 7850 \times 0.05 \times 1 \times 460 \times (225 - 25) = 35.33 \times 10^6 \text{ J}$$

$$Q = 0.45 \times 35.33 \times 10^6 = 15.9 \times 10^6 \text{ J} \quad \textit{Ans.} \text{ (c)}$$

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**Example 3.16** An iron sphere of diameter 5 cm, initially at a uniform temperature of 225°C, has its surface suddenly exposed to an ambient temperature of 25°C with a heat transfer coefficient of 500 W/m$^2$ K. Calculate (a) the centre temperature 2 min after the start of cooling, (b) the temperature at a depth 1 cm from the surface 2 min after the start of cooling and (c) the energy removed from the sphere during this time. For iron, take $k$ = 60 W/mK, $\rho$ = 7850 kg/m$^3$, $c$ = 460 J/kg K and $\alpha = 1.6 \times 10^{-5}$ m$^2$/s.

**Solution**

$$\text{Fo} = \frac{\alpha t}{r^2} = \frac{1.6 \times 10^{-5} \times 2 \times 60}{(2.5)^2 \times 10^{-4}} = 3.1$$

$$\frac{1}{\text{Bi}} = \frac{k}{hr} = \frac{60}{500 \times 2.5 \times 10^{-2}} = 4.8$$

$$\text{Bi} = 0.21$$

From Fig. 3.9(a) for Fo = 3.1 and 1/Bi = 4.8, the centre temperature $T_c$ is found as

$$\frac{T_c - T_\infty}{T_i - T_\infty} = 0.18$$

$$T_c = T_\infty + 0.18\,(T_i - T_\infty) = 25 + 0.18 \times 200$$

$$= 61°\text{C} \quad \textit{Ans.}\ \text{(a)}$$

The temperature 1 cm from the surface is obtained as

$$\frac{x}{r} = \frac{2.5 - 1}{2.5} = 0.6$$

From Fig. 3.9(b) for 1/Bi = 4.8 and $x/r$ = 0.6, we have

$$\frac{T - T_\infty}{T_o - T_\infty} = 0.95$$

$$T = 25 + 0.95(61 - 25) = 59.2°\text{C} \quad \textit{Ans.}\ \text{(b)}$$

From Fig. 3.9 (c) for Bi = 0.21 and Bi$^2$Fo = 0.137, we find

$$Q/Q_i = 0.8$$

where
$$Q_i = \rho\left(\frac{4}{3}\pi r^3\right) c\,(T_i - T_\infty)$$

$$= 7850\left(\frac{4}{3}\pi \times 2.5^3 \times 10^{-6}\right)(460)\,(225 - 25)$$

$$= 47{,}268 \text{ J}$$

Then, the heat loss from the sphere is

$$Q = 0.8 \times 47{,}268 = 37{,}814 \text{ J} \quad \textit{Ans.}$$

The boundary and initial conditions and the physical properties of the slab and the sphere in the examples 3.15 and 3.16 are the same. We note that after 2 min the centre temperature $T_o$ of the slab is 141°C, whereas for the sphere it is 61°C. The sphere loses heat at a much faster rate than the slab. This conclusion is also drawn from the ratio of fractional heat loss $Q/Q_i$ which is 0.46 for the slab and 0.8 for the sphere.

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**Example 3.17** A water pipe is to be buried in soil at a sufficient depth from the surface to prevent freezing in winter. When the soil is at a uniform temperature of 10°C the surface is subjected to a uniform temperature of –15°C continuously for 50 days. What minimum burial depth is needed to prevent the freezing of the pipe? Assume that $\alpha = 0.2 \times 10^{-6}$ m²/s for the soil and that the pipe surface temperature should not fall below 0°C.

**Solution** Figure 3.17 for semi-infinite solid may be used to determine the temperature distribution in the soil. For $\alpha = 0.2 \times 10^{-6}$ and $t = 50 \times 24 \times 3600$ s, the parameter $\xi$ becomes

$$\xi = \frac{x}{2(\alpha t)^{1/2}} = \frac{x}{2(0.2\times 10^{-6} \times 50 \times 24 \times 3600)^{1/2}}$$
$$= 0.538\, x$$

Taking, $T_i = 10°\text{C}$, $T_o = -15°\text{C}$ and $T(x, t) \geq 0°\text{C}$,

we obtain, $q(x, t) = \dfrac{T(x,t) - T_o}{T_i - T_o} = \dfrac{0+15}{10+15} = \dfrac{15}{25} = 0.6$

From Fig. 3.17(a) for $\theta(x, t) = 0.6$, we find $\xi = 0.6$

$$0.538\, x = 0.6$$
$$x = 1.12 \text{ m}$$

Therefore, the pipe should be buried at least to a depth of 1.12 m. *Ans.*

**Example 3.18** A rectangular iron bar 5 cm × 4 cm having $k$ = 60 W/m K, and $\alpha = 1.6 \times 10^{-5}$ m²/s is initially at a uniform temperature of 225°C. Suddenly the surfaces of the bar are subjected to convective cooling with a heat transfer coefficient of 500 W/m² K into an ambient fluid at 25°C. Calculate the centre temperature $T_o$ of the bar 2 min after the start of the cooling.

**Solution** The dimensionless temperature $\theta(x, y, t)$ for this problem is defined as

$$\frac{\theta(x,y,t)}{\theta_i} = \frac{T(x,y,t) - T_\infty}{T_i - T_\infty}$$

The solution for $\theta$ can be made as a product of the solution of two slab problems: $\theta_1(x, t)$, the solution for a slab of thickness $2L_1$ = 5 cm and $\theta_2(y, t)$, the solution for a slab of thickness $2L_2$ = 4 cm.

For the slab $2L_1$ = 5 cm, we have

$$\text{Fo} = \frac{\alpha t}{L_1^2} = \frac{(1.6\times 10^{-5})(2\times 60)}{(2.5)^2 \times 10^{-4}} = 3.1$$

and $$\frac{1}{\text{Bi}} = \frac{k}{hL_1} = \frac{60}{500 \times 2.5 \times 10^{-2}} = 4.8$$

From Fig. 3.7(a) for Fo = 3.1 and 1/Bi = 4.8, we obtain

$$\left(\frac{\theta_c}{\theta_i}\right)_{2L_1} = 0.58$$

For the slab $2L_2$ = 4 cm, we have

$$\text{Fo} = \frac{\alpha t}{L_2^2} = \frac{(1.6\times 10^{-5})(2\times 60)}{2^2 \times 10^{-4}} = 4.8$$

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$$\frac{1}{\text{Bi}} = \frac{k}{hL_2} = \frac{60}{500 \times 2 \times 10^{-2}} = 6.0$$

The centre temperature for this problem is obtained from Fig. 3.7(a) for Fo = 4.8 and 1/Bi = 6,

$$\left(\frac{\theta_c}{\theta_i}\right)_{2L_2} = 0.45$$

Then the dimensionless centre temperature $\theta_c$ for the two-dimensional problem is determined from the product solution

$$\frac{\theta_c}{\theta_i} = \left(\frac{\theta_c}{\theta_i}\right)_{2L_1} \times \left(\frac{\theta_c}{\theta_i}\right)_{2L_2} = 0.58 \times 0.45 = 0.26$$

$$\frac{\theta_c}{\theta_i} = \frac{T_c - T_\infty}{T_i - T_\infty} = 0.26$$

$$T_c = 25 + 200 \times 0.26 = 77°\text{C} \quad \textit{Ans.}$$

**Example 3.19** A short iron cylinder ($k$ = 60 W/m K, $\alpha = 1.6 \times 10^{-3}$ m$^2$/s) of diameter 5 cm and height 4 cm is initially at a uniform temperature of 225°C. Suddenly the boundary surfaces are exposed to an ambient fluid at 25°C with a heat transfer coefficient of 500 W/m$^2$ K. Calculate the centre temperature at 2 min after the start of cooling.

**Solution** The dimensionless temperature $\theta\,(r, z, t)$ for this problem is defined as

$$\frac{\theta\,(r,z,t)}{\theta_i} = \frac{T\,(r,z,t) - T_\infty}{T_o - T_\infty}$$

The solution for $\theta$ can be made as a product of the solutions of the following two problems. $\theta_1\,(r, t)$, the solution for a long cylinder of diameter 5 cm, and $\theta_2\,(z, t)$, the solution for a slab of thickness $2L_2$ = 4 cm.

For the cylinder with $D$ = 5 cm, we have

$$\text{Fo} = \frac{\alpha t}{r^2} = \frac{(1.6 \times 10^{-5})\,(2 \times 60)}{2.5^2 \times 10^{-4}} = 3.1$$

and
$$\frac{1}{\text{Bi}} = \frac{k}{hr} = \frac{60}{500 \times 2.5 \times 10^{-2}} = 4.8$$

For Fo = 3.1 and 1/Bi = 4.8, from Fig. 3.9 (a) the centre temperature is obtained

$$\left(\frac{\theta_c}{\theta_i}\right)_P = 0.31$$

For the slab with $2L_2$ = 4 cm, we have

$$\text{Fo} = \frac{\alpha t}{L_2^2} = \frac{(1.6 \times 10^{-5})\,(2 \times 60)}{(2 \times 10^{-2})^2} = 4.8$$

$$\frac{1}{\text{Bi}} = \frac{k}{hL_2} = \frac{60}{500 \times 2 \times 10^{-2}} = 6.0$$

Then, from Fig. 3.7(a) the centre temperature of this slab for Fo = 4.8 and 1/Bi = 6 is obtained as

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$$\left(\frac{\theta_c}{\theta_i}\right)_{2L} = 0.45$$

Then the dimensionless centre temperature $\theta_c$ for the two-dimensional short cylinder is determined by product solution

$$\frac{\theta_c}{\theta_i} = \left(\frac{\theta_c}{\theta_i}\right)_R \times \left(\frac{\theta_c}{\theta_i}\right)_{2L} = 0.31 \times 0.45 = 0.14$$

$$\frac{T_c - T_\infty}{T_i - T_\infty} = 0.14$$

$$T_c = 25 + 200\ (0.14) = 53°\text{C} \quad \textit{Ans.}$$

**Example 3.20** A 16 cm long cylinder of 10 cm diameter, with properties $k = 0.5$ W/m K and $\alpha = 5 \times 10^{-7}$ m²/s is initially at a uniform temperature of 20°C. The cylinder is placed in an oven where the ambient air temperature is 500°C and $h_c = 30$ W/m² K. Determine the minimum temperatures in the cylinder 30 min after it has been placed in the oven.

**Solution** Biot number based on the cylinder radius

$$\text{Bi} = \frac{h_c\, r_o}{k} = \frac{30 \times 0.05}{0.5} = 3.0$$

Simplified approach neglecting internal resistance cannot be used. Heisler's chart solution will be used. At any time, the minimum temperature is at the geometric centre of the cylinder and the maximum temperature is at the outer surface of the cylinder. These temperatures will be obtained by the product of the solution for an infinite plate and an infinite cylinder.

Infinite plate:

$$\text{Fo} = \frac{\alpha t}{L^2} = \frac{5 \times 10^{-7} \times 1800}{(0.08)^2} = 0.14$$

$$\frac{1}{\text{Bi}} = \frac{k}{h_c\, L} = \frac{0.5}{30 \times 0.08} = 0.21$$

$$\left(\frac{\theta_c}{\theta_i}\right)_{2L} = 0.90 \quad \left(\frac{\theta_L}{\theta_i}\right)_{2L} = 0.249$$

Infinite cylinder:

$$\text{Fo} = \frac{\alpha t}{r_o^2} = \frac{5 \times 10^{-7} \times 1800}{(0.05)^2} = 0.36$$

$$\frac{1}{\text{Bi}} = \frac{k}{h_c\, r_o} = \frac{0.5}{30 \times 0.05} = 0.33$$

$$\left(\frac{\theta_c}{\theta_i}\right)_R = 0.47 \quad \left(\frac{\theta_L}{\theta_i}\right)_{2L} = 0.155$$

Therefore, $$\frac{\theta_c}{\theta_i} = \frac{\theta_{\min}}{\theta_i} = 0.90 \times 0.47 = 0.423 = \frac{T_c - T_\infty}{T_i - T_\infty}$$

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$$T_c = T_{\min} = 500 + 0.423\ (20 - 500) = 297°\text{C} \quad \textit{Ans.}$$

and
$$\frac{\theta_1}{\theta_i} = \frac{\theta_{\max}}{\theta_i} = 0.249 \times 0.155 = 0.39 = \frac{T_l - T_\infty}{T_i - T_\infty}$$

$$T_1 = T_{\max} = 500 + 0.039\ (20 - 500) = 481°\text{C} \quad \textit{Ans.}$$

**Example 3.21** A cylindrical steel ingot (diameter 100 mm, length 300 mm, $k$ = 40 W/mK, $\rho$ = 7600 kg/m$^3$ and $c$ = 600 J/kg K) is to be heated in a furnace from 50°C to 850°C. The temperature inside the furnace is 1300°C and the surface heat transfer coefficient is 100 W/m$^2$ K. Calculate the time required for heating.

**Solution** Characteristic length, $L = \dfrac{V}{A} = \dfrac{\pi r^2 L}{2\pi r^2 + 2\pi r L}$

$$\therefore \qquad L = \frac{r L}{2(r + L)} = \frac{0.05 \times 0.3}{2(0.05 + 0.3)} = 0.02143 \text{ m}$$

$$\text{Bi} = \frac{hL}{k} = \frac{100 \times 0.02143}{40} = 0.0536$$

Since Bi << 0.1, lumped capacitance method is applicable

$$\text{Fo} = \frac{\alpha t}{L^2} = \frac{k}{\rho c} \times \frac{t}{L^2} = \frac{40\, t}{7600 \times 600 \times (0.02143)^2}$$
$$= 0.0191\, t$$

Now,
$$\frac{T - T_\infty}{T_0 - T_\infty} = \exp(-\text{Bi} \cdot \text{Fo})$$

$$\frac{850 - 1300}{50 - 1300} = 0.36 = \exp(-\text{Bi} \cdot \text{Fo})$$

$$\therefore \qquad \text{Bi} \cdot \text{Fo} = 1.02$$

$$0.0536 \times 0.0191\, t = 1.02$$

$$\therefore \qquad t = 996.327\, s = 16.6 \text{ min} \quad \textit{Ans.}$$

It is the required time for heating. So the ingot should be moved at 0.3 m/16.6 = 0.018 m/min or 1.8 cm/min.

**Example 3.22** A solid copper ball of 100 mm diameter and $\rho$ = 8954 kg/m$^3$, $c_p$ = 383 J/kg K, $k$ = 386 W/mK is at a uniform temperature of 250°C. It is suddenly immersed in a well-stirred fluid which is maintained at a uniform temperature of 50°C. The heat transfer coefficient between the ball and the fluid is $h$ = 200 W/m$^2$ K. Estimate the temperature of the copper ball after a lapse of 5 minutes of immersion.

**Solution** Given: $d$ = 100 mm = 0.1 m, $c_p$ = 383 J/kg K,
$r$ = 8954 kg/m$^3$, $k$ = 386 W/mK, $T_i$ = 250°C, $T_\infty$ = 50°C,
$h$ = 200 W/m$^2$ K, $t$ = 5 min = 300 $s$.

Characteristic length, $L = \dfrac{V}{A} = \dfrac{\frac{4}{3}\pi r^3}{4\pi r^2} = \dfrac{r}{3}$

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$$= \frac{d}{6} = \frac{0.1}{6} = 0.0167 \text{ m}$$

Biot number, $$\text{Bi} = \frac{hL}{k} = \frac{200 \times 0.0167}{386}$$

$$= 0.00864$$

Since Bi << 0.1, lump capacitance method may be used.

$$\frac{T - T_\infty}{T_i - T_\infty} = e^{-\text{Bi} \cdot \text{Fo}}$$

where Fo, Fourier number $= \dfrac{\alpha t}{L^2} = \dfrac{k}{\rho c}\dfrac{t}{L^2}$

$$= \frac{386}{8954 \times 383} \times \frac{300}{(0.0167)^2}$$

$\therefore$ $$\text{Fo} = 121.076$$

$\therefore$ $$\frac{T - 50}{250 - 50} = e^{-0.00864 \times 121.076}$$

$$= 0.3513$$

$\therefore$ $$T = 120.26°\text{C} \quad \textit{Ans.}$$

**Example 3.23** An egg with a mean diameter of 40 mm and initially at 20°C is placed in a boiling water pan for 4 minutes and found to be boiled to consumer's taste. For how long should a similar egg for the same consumer be boiled when taken from a refrigerator at 5°C. Take the following properties for egg:

$$\rho = 1200 \text{ kg/m}^3, c = 2 \text{ kJ/kg K}, k = 10 \text{ W/m K and } h = 100 \text{ W/m}^2\text{K}.$$

Use lump capacity method.

**Solution**

$$L = \text{characteristic length} = \frac{r}{3} = \frac{0.02}{3} = 0.0067 \text{ m}$$

$$\text{Bi, Biot number} = \frac{hL}{k} = \frac{100 \times 0.0067}{10} = 0.067$$

As Bi < 0.1, lump theory can be used.

Let $T$ be the temperature to which the egg should be boiled to satisfy the consumer's taste. Therefore,

$$\frac{T - T_\infty}{T_i - T_\infty} = e^{-\frac{hAt}{\rho c V}} = e^{-\frac{100t}{1200 \times 2 \times 0.0067 \times 1000}}$$

$$= e^{-6.219t/1000}$$

$$\frac{T - 100}{20 - 100} = e^{-6.219 \times 240/1000} = e^{-1.493}$$

$$T = -18 + 100 = 82°\text{C}.$$

We are to find the time taken for the egg taken from refrigerator at 5°C to be boiled to 82°C.

$$\frac{82-100}{5-100} = e^{-\frac{100\times t}{1200\times 2000\times 0.0067}} = e^{-6.22\times 10^{-3}t}$$

or
$$e^{-6.22\times 10^{-3}t} = \frac{-18}{-95} = 0.1895$$

$$e^{6.22\times 10^{-3}t} = 5.278$$

$$6.22\times 10^{-3}t = 1.6635$$

$$t = 267.44s$$

$$= 4.45 \text{ min.} \quad \textit{Ans.}$$

**Example 3.24** A hot cylinder ingot of 50 mm diameter and 200 mm length is taken out from the furnace at 800°C and then dipped in water till its temperature falls to 500°C ($h_\omega$ = 200 W/m$^2$K). Then it is directly exposed to air till its temperature falls to 100°C ($h_a$ = 20 W/m$^2$K). The temperature of air and water is 30°C. Taking the properties of ingot as $\rho$ = 800 kg/m$^3$, $c$ = 0.2 kJ/kgK, $k$ = 60 W/mK, find the total time required for the ingot to reach the temperature from 800°C to 100°C.

**Solution**

$$r = 25 \text{ mm} = 0.025 \text{ m}, L = 200 \text{ mm} = 0.2 \text{ m}$$

$$L = \frac{V}{A} = \frac{\pi r^2 L}{2\pi r L} = \frac{r}{2} = \frac{0.025}{2} \text{ m}$$

Cooling in water $\text{Bi} = \dfrac{hL}{k} = \dfrac{200\times 0.025}{60\times 2} = 0.04166$

As Bi < 0.1, internal thermal resistance can be neglected and lump theory can be used.

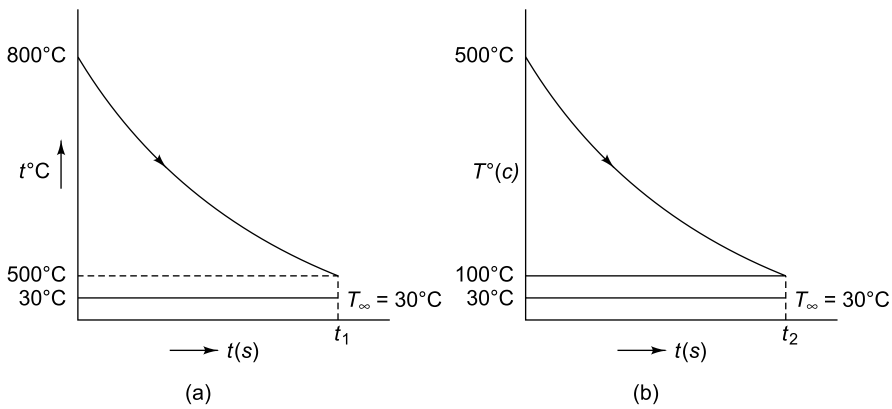


**Fig.** *Ex. 3.24*

$$\text{Bi}\cdot\text{Fo} = \frac{hAt_1}{\rho c V} = \frac{ht_1}{\rho c L} = \frac{200\,t_1\times 2}{800\times 200\times 0.025}$$

$$= 0.1\,t_1$$

$$\frac{T-T_\infty}{T_i-T_\infty} = e^{-\text{Bi}\cdot\text{Fo}}$$

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$$\frac{500-30}{800-30} = \frac{470}{770} = e^{-0.1t_1} = 0.61$$

$$\therefore \qquad t_1 = 4.94\,s$$

Cooling in air

$$\text{Bi} = \frac{hL}{k} = \frac{20 \times 0.025}{2 \times 60} = 0.00417$$

$$\text{Bi} \cdot \text{Fo} = \frac{hAt_2}{\rho c V} = \frac{20\,t_2 \times 2}{800 \times 200 \times 0.025} = 0.01\,t_2$$

$$\frac{100-30}{500-30} = e^{-0.01\,t_2} = \frac{70}{470}$$

$$t_2 = 190.42s$$

$\therefore$ Total time required $= t_1 + t_2$

$= 4.94 + 190.42$

$= 195.36\,s = 3.256$ min *Ans.*

**Example 3.25** Calculate the junction diameter of a copper thermocouple, initially at 25°C, which when placed in a gas stream at 200°C measures a temperature of 198°C in 5 seconds. For copper, $\rho$ = 8940 kg/m$^3$, $C$ = 384 J/kgK, $k$ = 390 W/mK and the convective heat transfer coefficient = 400 W/m$^2$K.

**Solution** Using lump capacitance method

$$\frac{T - T_\infty}{T_i - T_\infty} = e^{-\frac{hAt}{\rho c V}}$$

$$\frac{198-200}{25-200} = e^{-\frac{400 \times 5\,A}{8940 \times 384\,V}} = \frac{-2}{-175}$$

$$e^{-\frac{A}{1716.48\,\text{V}}} = 0.0114286$$

$$\therefore \qquad e^{-\frac{A}{1716.48\,\text{V}}} = 87.5$$

$$\frac{A}{1716.48\,\text{V}} = 4.47164$$

$$\therefore \qquad \frac{V}{A} = 1.30285 \times 10^{-4} = \frac{r}{3} \qquad \therefore r = 3.909 \times 10^{-4}\text{ m}$$

$$\therefore \qquad d = 7.817 \times 10^{-4}\text{ m}$$

$\therefore$ Diameter of the thermocouple

$$d = 7.817 \times 10^{-4}\text{ m} = 0.7817\text{ mm} \quad \textit{Ans.}$$

Check: $$\text{Bi} = \frac{hL}{k} = \frac{hr}{k \times 3} = \frac{400 \times 7.817 \times 10^{-4}}{390 \times 3}$$

$$= 2.672 \times 10^{-4}$$

Since Bi << 0.1, lumped capacitance method used is free from error.

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**Example 3.26** Determine the minimum depth at which one must place a water main below the soil surface to avoid freezing. The soil is initially at a uniform temperature of 20°C. In severe winter condition it is subjected to a surface temperature of – 15°C for a period of 60 days. Use the following properties of soil: $\rho$ = 2050 kg/m$^3$, $c$ = 1840 J/kgK, $k$ = 0.52 W/mK.

**Solution**

$$\alpha = \frac{k}{\rho c} = \frac{0.52\ \text{W/mK}}{2050\ \text{kg/m}^3 \times 1840\ \text{J/kg K}} = 0.138 \times 10^{-6}\ \text{m}^2/\text{s}$$

A sketch of the system is shown in Fig. Ex. 3.26.

Assuming the soil to be a semi-infinite medium, the transient temperature response of the soil is given by Eq. (3.50),

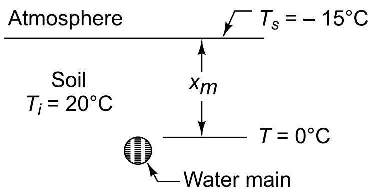


**Fig.** *Ex. 3.26*

$$\frac{T(x_m, t) - T_s}{T_i - T_s} = \text{erf}\left(\frac{x}{2(\alpha t)^{1/2}}\right)$$

$$\frac{0-(-15)}{20-(-15)} = 0.43 = \text{erf}\left(\frac{x_m}{2(\alpha t)^{1/2}}\right)$$

From error function table (Appendix B2),

$$\text{erf}(0.4) = 0.43$$

$$\therefore \quad \frac{x_m}{2\sqrt{\alpha t}} = 0.4$$

$$\therefore \quad x_m = 0.8\ (0.138 \times 10^{-6} \times 60 \times 24 \times 3600)1/2$$

$$= 0.68\ \text{m} \quad \textit{Ans.}$$

**Example 3.27** A large block of steel ($k$ = 45 W/mK, $\alpha$ = 1.4 × 10$^{-5}$ m$^2$/s) is initially at a uniform temperature of 35°C. The surface is exposed to a heat flux (a) by suddenly raising the surface temperature to 250°C and (b) through a constant surface heat flux of 3.2 × 10$^5$ W/m$^2$. Calculate the temperature at a depth of 25 mm after a time of 0.5 min for both these cases.

**Solution** Using the solutions for the semi-infinite solid as given in Eq. (3.50),

$$\frac{x}{2\sqrt{\alpha t}} = \frac{0.025}{2(1.4 \times 10^{-5} \times 30)^{1/2}} = 0.61$$

From error function table,

$$\text{erf}\,\frac{x}{2\sqrt{\alpha t}} = \text{erf}(0.61) = 0.61164$$

$$\frac{T(x,t) - T_s}{T_i - T_s} = 0.61164 = \frac{T(x,t) - 250}{35 - 250}$$

$$\therefore \quad T(x, t) = 118.5°\text{C} \quad \textit{Ans.}\ \text{(a)}$$

(b) Using Eq. (3.51), where $q''_o$ = 3.2 × 10$^5$ W/m$^2$,

$$T(x, t) - T_i = \frac{2q''_o}{k}(\alpha t/\pi)^{1/2} \exp\left(-\frac{x^2}{4\alpha t}\right) - (q''_o x/k)\ \text{erf}\,c\left[\frac{x}{2\sqrt{\alpha t}}\right]$$

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$$\therefore \quad T(x, t) = 35 + \frac{2\,(3.2\times 10^5)\,(1.4\times 10^{-5}\times 30/\pi)^{1/2}}{45}$$

$$e^{-(0.61)^2} - \frac{0.025\times 3.2\times 10^5}{45}[1 - 0.61164]$$

$$= 79.3°\text{C} \quad \textit{Ans.}\ \text{(b)}$$

(at $x$ = 25 mm, $t$ = 30 s)

**Example 3.28** The daily variation in the temperature of a certain place on the earth is from 20°C to 45°C. If the average thermal diffusivity of the earth is 0.006 $m^2$/hr, calculate the amplitude of the temperature variation at a depth of 0.15 m. What would be the time lag of the temperature wave at this depth?

**Solution** Amplitude of temperature variation

$$\theta_0 = \frac{T_{max} - T_{min}}{2} = \frac{45 - 20}{2} = 12.5°\text{C}$$

Frequency, $f = \frac{1}{24} h^{-1} = \frac{1}{P}$ where $P$ = time period of oscillation

At $x$ = 0.15 m, the amplitude is

$$\theta_x = \theta_o \cdot e^{-\left(\frac{\omega}{2\alpha}\right)^{1/2} x}$$

where $$\frac{\omega}{2\alpha} = \frac{2\pi f}{2\,\alpha} = \frac{\pi\times 1}{24h\times 0.006\ \text{m}^2/\text{h}} = \frac{\pi}{0.144}\ \text{m}^{-2}$$

$$= 21.871\ \text{m}^{-2}$$

$$\therefore \quad \theta_x = 12.5\times \exp\,(-\sqrt{21.817}\times 0.15)$$

$$= 6.2°\text{C} \quad \textit{Ans.}$$

Time lag, $$\Delta t = \frac{1}{2}\left(\frac{P}{\pi\alpha}\right)^{1/2}\cdot x$$

$$= \frac{1}{2}\left(\frac{24}{\pi\times 0.006}\right)^{1/2}\cdot 0.15$$

$$= 2.6762\ h = 2\,h\ 40.57\ \text{min.} \quad \textit{Ans.}$$

**Example 3.29** A single cylinder 2-stroke engine runs at 4000 rpm. Calculate the depth where the temperature variation is 2% of its surface value. Given: $\alpha$ = 0.045 $m^2$/h.

**Solution**

$$\frac{\theta_x}{\theta_o} = 0.02 = e^{\left(-\frac{\omega}{2\alpha}\right)^{1/2} x}$$

Frequency for the 2-stroke engine = $4000\times 60\ h^{-1}$

$$\therefore \quad \sqrt{\frac{\omega}{2\alpha}} = \sqrt{\frac{\pi f}{\alpha}} = \sqrt{\frac{\pi\,4000\times 60}{0.045}} = 4093.3\ \text{m}^{-1}$$

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$$\therefore \qquad e^{4093.3x} = 50$$

$$\therefore \qquad 4093.3x = 3.912$$

or, $$x = \frac{3912}{4093.3} \text{ mm} = 0.9557 \text{ mm} \quad \textit{Ans.}$$

**Example 3.30** The inner surface temperature of an annealing oven varies according to a sine function from 800°C to 200°C. Each cycle is completed in 12 hours. Estimate (a) the time lag of temperature wave at a depth of 100 mm from the inner surface, (b) the heat flow through a surface located at a distance of 100 mm from the surface during the first six hour interval while the temperature is above the mean value. Take $\alpha$ = 0.02 m$^2$/h and $k$ = 1.8 W/mK.

**Solution**

Here, $$w = 2\pi f = 2\pi \frac{1}{12} \text{ rad/h; } x = 0.1 \text{ m}$$

Time lag is given by Eq. (3.68),

$$\Delta t = \left(\frac{1}{2\alpha\omega}\right)^{1/2} \cdot x$$

$$= \frac{1 \times 6}{2 \times 0.02 \times \pi} \times 0.1 = 4.77 \text{ h} \quad \textit{Ans.} \text{ (a)}$$

From Eq. (3.71), the heat flow per unit area

$$\frac{Q}{A} = \frac{2}{\sqrt{\omega\alpha}} k\theta_o$$

$$= \frac{2}{\sqrt{\frac{\pi}{6} \times 0.02}} \times 1.8 \times \frac{800 - 200}{2} \times \frac{3600}{1000}$$

$$= 37993.7 \text{ kJ/m}^2 \quad \textit{Ans.} \text{ (b)}$$

**Example 3.31** A steel ball 100 mm in diameter and initially at 900°C is placed in air at 30°C. Taking for steel, $k$ = 40 W/mK, $\rho$ = 7800 kg/m$^3$, and $c$ = 460 J/kgK and if $h$ = 20 W/m$^2$K, find (a) the temperature of the ball after 30 seconds, and (b) the rate of cooling (°C/min) after 30 seconds.

**Solution**

$$r = \frac{100}{2} = 50 \text{ mm} = 0.05 \text{ m}, \ T_1 = 900°\text{C},$$

$$T_\infty = 30°\text{C}, \ \alpha = \frac{k}{\rho c} = \frac{40}{7800 \times 460} \frac{\text{W} \times \text{m}^3 \times \text{kg K}}{\text{mK} \times \text{kg} \times \text{J}}$$

$$= 11.15 \times 10^{-6} \text{ m}^2\text{/s}$$

$$L = \frac{r}{3} = \frac{0.05}{3} = 0.01667 \text{ m}$$

$$\text{Bi} = \frac{hL}{k} = \frac{20 \times 0.01667}{40} = 0.008335$$

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Since Bi << 0.1, lumped capacitance method can be used.

$$\text{Fo} = \frac{\alpha t}{L^2} = \frac{11.15 \times t \times 10^{-6}}{(0.01667)^2} = 0.04t$$

$$\frac{T - T_\infty}{T_1 - T_\infty} = e^{-\text{Bi}\,.\,\text{Fo}}$$

or $$e^{-0.008335 \times 0.04 \times 30} = \frac{T - 30}{900 - 30} = \frac{T - 30}{870}$$

$$e^{-0.01} = \frac{T - 30}{870} = \frac{1}{e^{0.01}} = 0.99$$

$$T = 891.15°\text{C} \quad \textit{Ans.}$$

On differentiating,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{T - T_\infty}{T_i - T_\infty}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(e - \frac{hAt}{\rho c V}\right)$$

$$\frac{1}{T_i - T_\infty}\frac{\mathrm{d}T}{\mathrm{d}t} = \left(-\frac{hA}{\rho c V}\right)\exp\left(-\frac{hAt}{\rho c V}\right)$$

$$\frac{1}{870}\frac{\mathrm{d}T}{\mathrm{d}t} = -\frac{20 \times 1}{7800 \times 460 \times 0.01667} \times 0.99$$

$$= -3.31 \times 10^{-4}$$

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -0.288°\text{C/s} \times 60$$

$$= -17.28°\text{C/min} \quad \textit{Ans.}$$

**Example 3.32** The temperatures of the soil on earth are recorded at various intervals and it is found that the surface reaches a mean temperature at 6 pm and the soil at a depth of 250 mm reaches its mean temperature at 9 pm. Determine the thermal diffusivity of the soil.

**Solution** Time lag between $x = 0$ and $x = 0.25$ m is 3 hours. The periodic time $\tau_1$ is 24 hours.

$$\Delta\tau = \frac{x}{\sqrt{2\alpha\,\omega}}$$

where $$\omega = 2\pi f = \frac{2\pi}{\tau_1} = \frac{2}{24 \times 3600}$$

∴ $$3 \times 3600 = \frac{0.25}{\sqrt{2\alpha \dfrac{2\pi}{24 \times 3600}}}$$

∴ $$\alpha = 3.684 \times 10^{-6}\ \text{m}^2/\text{s} \quad \textit{Ans.}$$

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## Summary

Transient heat conduction is first studied in solids with no internal thermal resistance having infinite thermal conductivity. The lumped capacitance method with its electrical analogy is explained and the significance of Biot number is brought out. The response time of a thermocouple is derived. Analytical solutions for transient conduction in bodies of simple geometries like plane walls, infinite cylinders and spheres are obtained and Heisler's charts to solve single- and multi-dimensional problems are explained. Semi-infinite solids are analyzed to derive the temperature distribution and heat transfer in transient conduction for different initial and boundary conditions. Numerical and graphical methods for conduction in simple regular shaped bodies are introduced explaining Schmidt's method and Dusinberre's method of analysis. The periodic flow of heat in one dimension for semi-infinite solids where the surface temperature varies sinusoidally or cosinusoidally with time is discussed.

## Important Formulae and Equations

| *Equation number* | *Equation* | *Remarks* |
|---|---|---|
| (3.3) | $\frac{T-T_\infty}{T_i-T_\infty} = e^{-\frac{hAt}{\rho CV}}$ | Temperature variation of a solid having infinite $k$ with time |
| (3.4) | $\frac{\theta}{\theta_i} = e^{-\text{Bi.Fo}}$<br>where $\theta = T - T_\infty, \text{Bi} = \frac{hL}{k}$<br>and $\text{Fo} = \frac{\alpha t}{L^2}$ | Temperature-time history of a solid having infinite $k$ in terms of Biot number and Fourier number |
| (3.5) | $\Delta U = \int_0^t Q dt = \rho c V \theta_i (1 - e^{-\text{BiFo}})$ | Amount of heat transfer during a time interval $\Delta t$ = change in internal energy of the solid. |
| (3.8) | $\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \theta}{\partial t}$ | Temperature distribution in an infinite plate $t(x, t)$ with no external resistance ($h = \infty$) |
| (3.13) | $\frac{T-T_1}{T_i-T_1} = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}e^{-\left(\frac{n\pi}{L}\right)^2} dt \sin\frac{n\pi}{L}x$ | Temperature distribution in a slab initially heated to a uniform temperature $T_i$ and then dropped to a certain fluid with its temperature at $x = 0$ and $x_L$ fixed at $T_1$. |

## Objective Type Questions

3.1 In transient heat conduction, the two significant dimensionless numbers are
(a) Fourier number and Reynolds number
(b) Reynolds number and Prandtl number
(c) Biot number and Fourier number
(d) Reynolds number and Biot number

3.2 What does transient conduction mean?
(a) Heat transfer for a short time
(b) Conduction when the temperature at a point varies with time
(c) Very little heat transfer
(d) Heat transfer with a very small temperature difference

3.3 **Assertion (A):** Lumped capacity analyss of transient heat conduction assumes an infinite or very large thermal conductivity of the solid.

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**Reasoning (R):** When the surface convective resistance is very large compared to the internal conduction resistance.

***Codes:***

(a) Both A and R are true.
(b) Both A and R are false.
(c) A is true, R is false.
(d) A is false, R is true.

3.4 Biot number is defined as

(a) $k/hL$ (b) $kL/h$
(c) $hL/k$ (d) $h/kL$

3.5 The quantity having the dimension of time is often called the time constant of a thermocouple

(a) $hA/\rho CV$ (b) $hV/\rho CA$
(c) $\rho CV/hA$ (d) $\rho CA/hV$

3.6 The sensitivity of a thermocouple is the time required for a thermocouple to reach $x$% of the initial temperature difference where

(a) $x = 0.3$ (b) $x = 0.368$
(c) 0.5 (d) 0.638

3.7 The internal thermal resistance of a solid can be ignored if the Biot number is less than

(a) 1.0 (b) 0.5
(c) 0.1 (d) Fourier number

3.8 Heisler's charts show the temperature-time history of a solid in transient heat conduction as a function of

(a) Fourier number and Biot number
(b) Fourier number and reciprocal of Biot number
(c) Reciprocal of Fourier number and Biot number
(d) Reciprocal of Fourier number and reciprocal of Biot number

3.9 **Assertion (A):** Thermocouples are preferred over mercury-in-glass thermometers when high sensitivity and fast response are desired.

**Reasoning (R):** Because mercury-in-glass thermometers have large thermal capacity

***Codes:***

(a) Both A and R are true and R is the only explanation of A.
(b) Both A and R are true and R is not the only explanation of A.
(c) A is true, but R is false.
(d) Both A and R are false.

3.10 The dimensionless time, called the Fourier number, is defined as

(a) $L^2/\alpha\tau$ (b) $\alpha\tau/L^2$
(c) $\alpha t/L$ (d) $L^2 t/\alpha$

3.11 **Assertion (A):** The temperature response of a thin hot copper wire is more in water than in air.

**Reasoning (R):** Because the specific heat of water is more than that of air.

***Codes:***

(a) Both A and R are true.
(b) Both A and R are true, but R is not the correct explanation of A.
(c) Both A and R are false.
(d) A is false, but R is true.

3.12 The lumped capacitance method for transient heat conduction problem is possible if

(a) the physical size of the body is very small
(b) $k$ of the material is very large
(c) $h$ is very small
(d) all of the above

3.13 The general classes of transient problems that can be solved by using Heisler's charts are

(a) The time is known while the local temperature at that time is unknown.
(b) The local temperature is known and the time required to reach that temperature is unknown.
(c) The instantaneous rate of heat transfer to or from the surface of the solid.
(d) All of the above

3.14 Match List I with List II and select the correct answer using the codes given below:

| | *List I* | *List II* |
|---|---|---|
| A. | Time constant of a thermocouple | 1. $hr_0/3k$ |
| B. | Biot numbr for a sphere of radius $r_0$. | 2. $k/hL$ |
| C. | Heisler chart variable | 3. $\dfrac{x}{2\sqrt{\alpha\tau}}$ |
| D. | Transient condition in a semi-infinite solid | 4. $2\pi r_0 Lh/\rho cV$ |

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| *Codes:* | A | B | C | D |
|---|---|---|---|---|
| (a) | 4 | 1 | 3 | 2 |
| (b) | 4 | 3 | 2 | 1 |
| (c) | 4 | 1 | 2 | 3 |
| (d) | 1 | 2 | 3 | 4 |

3.15 In one-dimensional transient numerical method, the choice of the time interval $\Delta t$ in the finite difference equations is limited by the condition that $\Delta x^2/\alpha\Delta t$

(a) is less than 2 (b) is equal to 2
(c) is greater than 2 (d) either (b) or (c)

3.16 **Assertion (A):** Electrical analogs of thermal systems have found wide use.
**Reasoning (R):** Because accurate measurements of heat transfer rates present great instrumental difficulties.
***Codes:***
(a) Both A and R are true and R is the correct explanation of A.
(b) Both A and R are true and R is not the only reason for A.
(c) A is false, R is true.
(d) A is true, R is false.

3.17 **Assertion (A):** The solution of 2-dimensional steady state heat conduction equation is obtained by assuming the temperature distribution to be expressed variables.
**Reasoning (R):** Because the governing equation is a linear differential equation.
***Codes:***
(a) Both A and R are true
(b) Both A and R are false
(c) A is true, R is false
(d) A is false, R is true

## Answers

| | | | | |
|---|---|---|---|---|
| 3.1 (c) | 3.2 (b) | 3.3 (a) | 3.4 (a) | 3.5 (c) |
| 3.6 (d) | 3.7 (c) | 3.8 (b) | 3.9 (b) | 3.10 (b) |
| 3.11 (b) | 3.12 (d) | 3.13 (d) | 3.14 (c) | 3.15 (d) |
| 3.16 (b) | 3.17 (a) | | | |

## Open Book Problems

3.1 Cylindrical pieces of size 60 mm dia and 60 mm height with density = 7800 kg/m$^3$, specific heat = 486 J/kgK and conductivity = 43 W/mK are to be heat treated. The pieces initially at 35ºC are placed in a furnace at 800ºC with convection coefficient at the surface of 85 W/m$^2$K. Determine the time required to heat the pieces to 650ºC. If by mistake the pieces were taken out of the furnace after 300 seconds, determine the shortfall in the temperature.

*Hints:* First it is necessary to check for the use of lumped parameter model by calculating the Biot number.

$$\text{Bi} = \frac{hL}{k}, L = \text{characteristic length}$$
$$= \text{volume/surface area}$$

$$\frac{V}{S} = \frac{\pi r^2 h}{2\pi r^2 + 2\pi rh}$$

If Bi < 0.1, the lumped parameter model is applicable.
Use Eq. (3.3),

$$\frac{650-800}{35-800} = \exp\left[-\frac{85\times t\times s}{7800\times 486\times V}\right]$$

Find $t$. If the piece is taken out after 300 seconds, then

$$\frac{T-800}{35-800} = \exp\left[\frac{-85\times 300}{7800\times 486\times V/S}\right]$$

Find $T$ and the shortfall in temperature.

3.2 A bearing piece in the form of half of a hollow cylinder of 60 mm ID, 90 mm OD and 100 cm long is to be cooled to –100ºC

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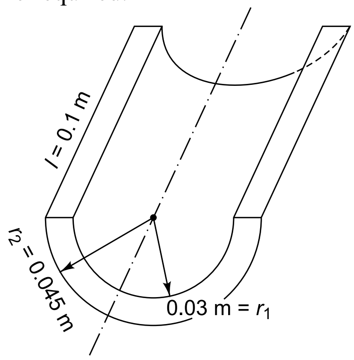
from 30ºC using a cryogenic gas at –150ºC with a heat transfer coefficient of 70 W/m$^2$K, density = 8900 kg/m$^3$, specific heat = 444 J/kgK, conductivity = 17.2 W/mK. Determine the time required.

*Hints:*




Find the volume of the piece

$$V = \frac{\pi(r_2^2 - r_1^2)l}{2}\,\text{m}^3$$

Surface area, $A = \pi r_1 l + \pi r_2 l + 2 \times (r_2 - r_1) \times$

$$l + \frac{2\pi(r_2^2 - r_1^2)}{2}\,\text{m}^2$$

$\therefore \quad L = \dfrac{V}{A}$ = characteristic length.

Establish Bi $= \dfrac{hL}{k}$ less than 0.1 and then use the lumped parameter model from Eq. (3.3).

$$\frac{-100-(-150)}{30-(-150)} = \exp\left[\frac{-70 \times t \times A}{8900 \times 444 \times V}\right]$$

Find $t$.

3.3 A thermocouple in the form of a long cylinder of 2 mm dia initially at 30ºC is used to measure the temperature of a cold gas at –160ºC. The convection coefficient over the surface is 60 W/m$^2$K. The material properties are: density = 8922 kg/m$^3$, specific heat = 410 J/kgK, conductivity = 22.7 W/mK. Determine the time it will take to indicate –150ºC and also the time constant of the thermocouple.

*Hints:* $L = \dfrac{V}{A} = \dfrac{R}{2}$ and establish $\text{Bi} = \dfrac{hL}{k} < 0.1$.

Then use Eq. (3.3),

$$\frac{-150-(-160)}{30-(-160)} = \exp\left[\frac{-60 \times t \times A}{8922 \times 410 \times V}\right]$$

Find $t$.

This can be reduced by using smaller wire diameter or higher value of $h$. The time constant $\tau^* = \dfrac{\rho c V}{hA}$ can then be found.

3.4 A copper sphere of 10 mm dia at 80ºC is placed in an air stream at 30ºC. The temperature dropped to 65ºC after 61 seconds. Calculate the value of convection coefficient. Assume property values as follows: density = 8925 kg/m$^3$, specific heat = 397 J/kgK, conductivity = 393 W/mK.

*Hints:* $L = \dfrac{V}{A} = \dfrac{R}{3}$. Establish Bi < 0.1 and use Eq. (3.3).

This method can be used for estimating convection coefficient.

3.5 A coal pellet of 1 mm dia sphere is to be heated by radiation with a source at 1200 K from 300 K to 900 K. Determine the time required. Take density = 1350 kg/m$^3$, specific heat = 1260 J/kgK, conductivity = 0.26 W/mK.

*Hints:* By energy balance: $\sigma A(T_\infty^4 - T^4)\, dt \;\; \rho c V dT$

or $$\frac{dT}{T_\infty^4 - T^4} = \frac{\rho A dt}{\rho c V}$$

A standard form available is used:

$$\int \frac{dx}{a^4 - x^4} = \frac{1}{4a^3}\ln\left[\frac{a+x}{a-x}\right] + \frac{1}{2a^3}\tan^{-1}\frac{x}{a}$$

Noting that $x \to T$ and $a \to T_\infty$.

$$\frac{\sigma A t}{\rho c V} = \left[\frac{1}{4T_\infty^3}\ln\frac{T_\infty + T}{T_\infty - T} + \frac{1}{2T_\infty^3}\tan^{-1}\frac{T}{T_\infty}\right]_{T_2}^{T_1}$$

Take $\tan^{-1}$ $(T/T_\infty)$ in radian mode and solve for $t$.

3.6 A metal sphere 20 mm radius at 90 K is allowed to cool in a room at 310 K by (a) convection only, (b) radiation only. Determine in each case the time required for the sphere to reach 450 K. Take density

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= 2700 kg/m$^3$, specific heat = 1110 J/kgK, conductivity = 218 W/mK, heat transfer coefficient 18 W/m$^2$K.

*Hints:* (a) Solve by lumped parameter model. (b) Cooling by radiation, as in the previous problem

$$t = -\frac{\rho c V}{\sigma A}\left[\frac{1}{4T_\infty^3}\ln\frac{T_\infty + T}{T_\infty - T} + \frac{1}{2T_\infty^3}\tan^{-1}\frac{T}{T_\infty}\right]_{T_2}^{T_1}$$

Negative sign is being used due to cooling.

3.7 On a hot day, the wood surface gets heated to 50ºC to a considerable depth. Sudden sharp showers cool the surface to 20ºC and maintain the surface at this temperature level. Determine the temperature at 2 cm depth after 40 minutes. The material properties are: $\rho$ = 2115 kg/m$^3$, $c$ = 920 J/kgK, $k$ = 0.062 W/mK. Calculate also the heat flow from the surface upto that time and instantaneous heat flow at the surface.

*Hints:* Use semi-infinite solid model. Here $T_A$ = 20ºC, $T_i$ = 50ºC

From Eq. (3.50), $\dfrac{T(x,t) - T_s}{T_i - T_s} = \text{erf}\,\dfrac{x}{2(\alpha t)^{1/2}}$

where, $x$ = 0.02 m, $t$ = 40 × 60s.

$\alpha = k/\rho c A$. Find $x/2(\alpha t)^{½}$. from error function table given in Appendix B–2, Find $T(x, t)$

[*Ans.* Total heat flow upto the time

$$q_s'''(t) = \int_0^t \frac{k(T_s - T_i)}{(\pi\alpha t)^{1/2}}\,dt = 2k(T_s - T_i)\sqrt{t/\pi\alpha}$$

Instantaneous heat flow from the surface

$q_s''' = k(T_s - T_i)/\sqrt{\pi\alpha t}$ ]

3.8 Calculate the depth of penetration of the temperature oscillation into the cylinder wall of a single cylinder 2-stroke IC engine operating at 2000 rpm. Take the thermal diffusivity of the wall material as 0.06 m$^2$h.

*Hints:* Period of oscillation is $\tau_0 = \dfrac{1}{60 \times 2000}$ h

From Fig. 3.28, the oscillations are seen to die out when $x/2\sqrt{\pi\alpha\tau_0} = 0.8$. Find $x$, to which the temperature fluctuations penetrate.

## Review Questions

3.1 How does transient heat conduction differ from steady conduction?

3.2 What is lumped system analysis? When is it applicable?

3.3 Consider heat transfer between two identical hot solid bodies and the air surrounding them. The first solid is being cooled by a fan while the second one is allowed to cool naturally. For which solid is the lumped system analysis more likely to be applicable? Why?

3.4 Consider heat transfer between two identical hot solid bodies and their environments. The first solid is dropped in a large container filled with water, while the second one is allowed to cool naturally in the air. For which solid is the lumped system analysis more likely to be applicable? Why?

3.5 Consider a hot baked potato in a plate. The temperature of the potato is observed to drop by 4°C during the first minute. Will the temperature drop during the second minute be less than, equal to, or more than 4°C? Why?

3.6 What is the physical significance of Biot number? Is the Biot number more likely to be large for highly conducting solids or poorly conducting ones?

3.7 Consider a sphere and a cylinder of equal volume made of copper. Both the sphere and the cylinder are initially at the same temperature, and are exposed to convection in the same environment. Which do you think will cool faster, the cylinder or the sphere? Why?

3.8 In what medium is the lumped system analysis more likely to be applicable: in water or in air? Why?

3.9 For which solid is the lumped system analysis more likely to be applicable: an actual apple or a gold apple of the same size? Why?

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3.10 Obtain a relation for the time required for a lumped system to reach the average temperature 1/2 ($T_i$ + $T_\infty$) where $T_i$ is the initial temperature and $T_\infty$ is the temperature of the environment.

3.11 Can the transient temperature charts in Fig. 3.7 for a plane wall exposed to convection on both sides be used for a plane wall whose one side is exposed to convection while the other side is insulated? Explain.

3.12 Why are the transient temperature charts prepared using non-dimensionalised quantities such as the Biot and Fourier numbers instead of the actual variables like thermal conductivity and time?

3.13 The Biot number during a heat transfer process between a sphere and its surroundings is determined to be 0.02. Would you use lumped system analysis or the transient temperature charts when determining the centre temperature of the sphere? Why?

3.14 What is a semi-infinite medium? Give examples of solid bodies that can be treated as semi-infinite mediums for heat transfer purposes.

3.15 Under what conditions can a plane wall be treated as a semi-infinite medium?

3.16 What is the product solution method? How is it used to determine the transient temperature distribution in a two-dimensional system?

3.17 How is the product solution used to determine the variation of temperature with time and position in three-dimensional systems?

3.18 A short cylinder initially at a uniform temperature $T_i$ is subjected to convection from all of its surfaces to a medium at temperature $T_\infty$. Explain how you can determine the temperature of the midpoint of the cylinder at a specified time.

3.19 Under what conditions analytical treatment of transient heat conduction problems becomes difficult?

3.20 Explain Schmidt's graphical method of transient temperature distribution in a solid without and with convective resistance. What are its merits and demerits?

3.21 How does Dusinberre's method differ from Schmidt's method of analysing transient heat conduction problems?

3.22 What do you understand by periodic flow of heat? Give some examples of this phenomenon.

3.23 Give the equation for periodic heat flow that expresses the temperature at any time $t$ and distance $x$ from the surface. What is the velocity of propagation of the thermal wave into the solid?

3.24 Explain why with higher frequency, the penetration of thermal wave decreases.

## Problems for practice

3.1 An aluminium plate ($k$ = 160 W/m K, $\rho$ = 2790 kg/m$^3$, $c_p$ = 0.88 kJ/kg K) of thickness 30 mm and at a uniform temperature of 225°C is suddenly immersed at time $t$ = 0 in a well-stirred fluid at a constant temperature of 25°C. The heat transfer coefficient between the plate and the fluid is 320 W/m$^2$ K. Determine the time required for the centre of the plate to reach 50°C.

(*Ans.* 4 min)

3.2 A cubical piece of aluminium with the same properties, as given in P.3.1, is 10 mm on a side and is heated from 50°C to 300°C by a direct flame. How long should the aluminium remain in the flame if the flame temperature is 800°C and the convective heat transfer coefficient between the flame and aluminium is 190 W/m$^2$ K?

3.3 A billet of steel of the form of a parallelepiped with dimensions 2 m × 2 m × 5 m, originally at 300°C, is placed in a radiant furnace, where the furnace temperature is held at 1000°C. Determine the temperature at the centre after 30 minutes. For steel, take $k$ = 35 W/m K, $\rho$ = 7800 kg/m$^3$, $c_p$ = 0.83 kJ/kg K, while $h$ = 233 W/m$^2$ K.

(*Ans.* 580°C)

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3.4 A steel cylinder of diameter 0.25 m and length 0.8 m initially at 25°C is placed in a furnace, where $T_\infty$ = 1000°C. Determine the temperature at the centre and on the surface of the cylinder after a lapse of 1 h. Assume $k$, $\rho$, $c_p$ and $h$ as in P.3.3.

3.5 The temperature of a gas stream is measured with a thermocouple. The junction may be approximated as a sphere of diameter 1 mm, $k$ = 25 W/m K, $\rho$ = 8400 kg/m$^3$ and $c_p$ = 0.4 kJ/kg K. The heat transfer coefficient between the junction and the gas stream is 500 W/m$^2$ K. How long will it take for the thermocouple to record 99% of the applied temperature difference? (*Ans.* 4.6 s)

3.6 A household electric iron has a steel base ($k$ = 70 W/m K, $\rho$ = 7840 kg/m$^3$, $c_p$ = 0.45 kJ/kg K) which weighs 1 kg. The base has an ironing surface of 0.025 m$^2$ and is heated from the other surface with a 250 W heating element. Initially the iron is at a uniform temperature of 20°C. Suddenly the heating starts, and the iron dissipates heat by convection from the ironing surface into an ambient at 20°C with a heat transfer coefficient 50 W/m$^2$ K. Calculate the temperature of the iron 5 min after the start of heating. What would the equilibrium temperature of the iron be if the control did not switch off the current?

(*Ans.* 133°C, 220°C)

3.7 A 20 mm diameter stainless steel ball ($\rho$ = 7865 kg/m$^3$, $c_p$ = 0.46 kJ/kg K and $k$ = 61 W/m K) is uniformly heated to 800°C. It is to be hardened by suddenly dropping it into an oil bath at 50°C. If the quenching occurs when the ball reaches 100°C and the heat transfer coefficient between the oil and the sphere is 300 W/m$^2$ K, how long should the ball be kept in the oil bath? If 100 balls are to be quenched per minute, determine the rate of heat removal from the oil bath per minute needed to maintain its temperature at 40°C. (*Ans.* 66 s, 3.58 kJ/min)

3.8 A 3 cm diameter aluminium sphere ($k$ = 204 W/m K, $\rho$ = 2700 kg/m$^3$ and $c_p$ = 0.896 kJ/kg K) is initially at 175°C. It is suddenly immersed in a well stirred fluid at 25°C. The temperature of the sphere is lowered to 100°C in 42 s. Calculate the heat transfer coefficient. (*Ans.* 200 W/m$^2$ K)

3.9 An orange of diameter 10 cm is initially at a uniform temperature of 30°C. It is placed in a refrigerator in which the air temperature is 2°C. If the heat transfer coefficient between the air and the orange surface is 50 W/m$^2$ K, determine the time required for the centre of the orange to reach 10°C. Assume the thermal properties of the orange are the same as those of water at the same temperature ($\alpha = 1.4 \times 10^{-7}$ m$^2$/s and $k$ = 0.59 W/m K).

(*Ans.* 1 h 32 min)

3.10 A solid iron rod ($\alpha = 2 \times 10^{-5}$ m$^2$/s, $k$ = 60 W/m K) of diameter 6 cm, initially at 800°C is suddenly dropped into an oil both at 50°C. The heat transfer coefficient between the fluid and the surface is 400 W/m$^2$ K. (a) Using Heisler's charts, determine the centreline temperature 10 min after immersion in the fluid. (b) How long will it take the centreline temperature to reach 100°C?

(*Ans.* (a) 54.5°C, (b) 5 min 47 s)

3.11 A 6 cm diameter potato initially at a uniform temperature of 20°C is suddenly dropped into boiling water at 100°C. The heat transfer coefficient between the water and the surface is 6000 W/m$^2$ K. The thermophysical properties of potato can be taken the same as those of water ($\alpha = 1.6 \times 10^{-7}$ m$^2$/s and $k$ = 0.68 W/m K). Determine the time required for the centre temperature of the potato to reach 95°C and the energy transferred to the potato during this time.

(*Ans.* 33 min, 37.8 kJ)

3.12 A slab of thickness 10 cm, a cylinder of diameter 10 cm and a sphere of diameter 10 cm, each made of steel ($\alpha = 1.6 \times 10^{-5}$ m$^2$/s and $k$ = 61 W/m K) were initially at 300°C,

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and then suddenly immersed into a well-stirred fluid at 50°C. The heat transfer coefficient between the surface and fluid is 1000 W/m$^2$ K. Calculate the time required for the centres of slab, cylinder and sphere to cool to 80°C. (*Ans.* 547, 266 and 188 s)

3.13 A thick bronze plate ($\alpha$ = 0.86 × 10$^{-5}$ m$^2$/s, $k$ = 26 W/m K) is initially at a uniform temperature of 250°C. Suddenly the surface is exposed to a coolant at 25°C. Assuming $h$ = 150 W/m$^2$ K, determine the temperature 5 cm from the surface 10 min after the exposure. (*Ans.* 205°C)

3.14 A rectangular bar 6 cm × 3 cm in cross-section is made of aluminium ($\rho$ = 2700 kg/m$^3$, $c$ = 890 J/kg K, $k$ = 200 W/m K and $\alpha$ = 8.4 × 10$^{-5}$ m$^2$/s). It is initially at a uniform temperature of 175°C. Suddenly the surfaces are subjected to convective cooling with a heat transfer coefficient 250 W/m$^2$ K into an ambient at 25°C. Determine the centre temperature of the bar 1 min after the start of cooling. (*Ans.* 107.5°C)

3.15 A short cylindrical aluminium bar (with the same thermophysical properties as in the Prob. 3.14) of diameter 6 cm and height 3 cm is initially at a uniform temperature of 175°C. Suddenly the surfaces are subjected to convective cooling with a $h$ = 250 W/m$^2$ K into an ambient at 25°C. Calculate the centre temperature of the cylinder 1 min after the start of cooling. (*Ans.* 93.3°C)

3.16 The ground at a particular location is covered with snow pack at –10°C for a continuous period of 3 months, and the average soil properties at that location are $k$ = 0.4 W/mK and $\alpha$ = 0.15 × 10$^{-6}$ m$^2$/s. Assuming an initial uniform temperature of 15°C for the ground, determine the minimum burial depth to prevent the water pipes from freezing. (*Ans.* 0.80 m)

3.17 A short brass cylinder ($k$ = 110 W/m K and $\alpha$ = 3.39 × 10$^{-5}$ m$^2$/s) of diameter 10 cm and height 12 cm is initially at a uniform temperature of 120°C. The cylinder is now placed in atmospheric air at 25°C, where heat transfer takes place by convection with $h$ = 60 W/m$^2$ K. Calculate (a) the temperature at the centre of the cylinder 15 min after the start of cooling, and (b) the total heat transfer from the brass cylinder ($\rho$ = 8530 kg/m$^3$, $c$ = 380 J/kg K) [*Ans.* (a) 63°C, (b) 85.9 kJ]

3.18 A semi-infinite aluminium cylinder ($k$ = 237 W/m K, $\alpha$ = 9.71 × 10$^{-6}$ m$^2$/s) of diameter 20 cm is initially at a uniform temperature of 200°C. The cylinder is now placed in water at 15°C where heat transfer takes place by convection with $h$ = 120 W/m$^2$ K. Determine the temperature at the centre of the cylinder 15 cm from the end surface 5 min after the start of cooling. [*Ans.* 152°C]

3.19 Estimate the minimum depth at which one must place a water main below the surface to avoid freezing. The soil is initially at a uniform temperature of 20°C. Assume that under the worst conditions anticipated it is subjected to a surface temperature of –15°C for a period of 60 days, and use the following properties for soil (300 K): $\rho$ = 2050 kg/m$^3$, $k$ = 0.52 W/m K, $c$ = 1840 J/kg K, $\alpha$ = 0.138 × 10$^{-6}$ m$^2$/s. [*Ans.* 0.68 m]

3.20 A large concrete wall 50 cm thick is initially at 60°C. One side of the wall is insulated. The other side is suddenly exposed to hot combustion gases at 900°C with a heat transfer coefficient of 25 W/m$^2$ K. Determine (a) the time required for the insulated surface to reach 600°C, (b) the temperature distribution in the wall at that instant and (c) the heat transferred during the process. Take $k$ = 1.25 W/m K, $c$ = 837 J/kg K, $\rho$ = 500 kg/m$^3$, $\alpha$ = 0.30 × 10$^{-5}$ m$^2$/s. [*Ans.* (a) 16.2 $h$, (c) – 1.758 × 10$^8$ J/m$^2$]

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# Convection Heat Transfer: Forced Convection

# 4

When energy transfer takes place between a solid surface and a fluid system in motion, the process is known as *convection*. If the fluid motion is impressed by a pump or compressor, it is *forced convection*. If it is caused by density difference, it is *natural* or *free convection*.

It is not possible to separate the problem of heat transfer from that of the motion of the fluid, and so a study of the hydrodynamic behaviour of the fluid is very much necessary in order to gain an understanding of the heat transfer phenomena within a moving fluid.

A fluid may be defined as a material that supports no shearing stress when at rest or in a state of uniform motion. Fluids exhibiting a linear relation between the rate of strain and the applied shear stress are called *Newtonian fluids*. Common substances such as the gases, water and oils are of this type. Certain suspensions that do not conform to a linear stress rate and strain law are called *non-Newtonian fluids*. We will, however, be concerned with the former type. A fluid will be treated as a continuum i.e., we will ignore the fact that the fluid is made up of discrete particles (atoms, molecules, ions or electrons) and consider that the smallest volume of fluid with which we are concerned is sufficiently large so that macroscopic properties such as density, pressure and temperature have their usual meanings and the motion of the fluid in contact with a solid surface is identical with that of the surface (no slip motion).

In the case of gases, it is known that the ratio of the mean free path of the molecules to some characteristic dimension of the flow field, called the *Knudsen number*, is an important parameter when this number is sufficiently large. This may occur in high vacuum systems and high altitude flight. An important characteristic of such rarefied gas flows is the slipping of the gas at the solid boundary.

In general, the behaviour of the flow depends on the properties of the fluid and on the boundary conditions imposed. To analyse this behaviour requires the application of the principle of conservation of mass (continuity equation), Newton's laws of motion (momentum equations) and the laws of thermodynamics (energy equation) along with the phenomenological laws like Fourier's law, Fick's law and Newton's law of viscosity.

Fluids include both liquids and gases. While liquids are incompressible, gases are compressible, having their densities varying with pressure greatly, and also with temperature.

## 4.1 BOUNDARY LAYER THEORY

Let us consider two plates a distance $S$ apart (Fig. 4.1). The lower plate is at rest, while the upper plate moves with a constant velocity $U_0$. The space between the plates is filled with a fluid.

Experience shows that in order to maintain the velocity $U_0$ of the upper plate a force is necessary, and this force is directly proportional to velocity $U_0$ and inversely proportional to the distance $S$. The force per unit area of the plate is equal to the shear stress $\tau$. Therefore,

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$$\tau \propto \frac{U_0}{S} \quad \text{or} \quad \tau = \mu \frac{U_0}{S} \tag{4.1}$$

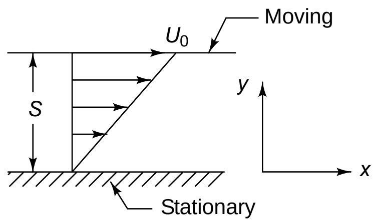


**Fig. 4.1** *Couette flow*

where $\mu$ is the constant of proportionality.

The fluid layers immediately adjacent to the two plates possess velocities equal to those of the plates, namely $U_0$ and zero, respectively, while in the rest of the fluid the velocity varies in a linear manner. Here $\mu$ is a property of the fluid, called *dynamic or absolute viscosity*. This is *Newton's law of viscosity*. The fluids obeying this law are known as Newtonian fluids, as mentioned earlier. Nonviscous or inviscid fluids are known as perfect or ideal fluids.

A more general form of Newton's law is

$$\tau = \mu \frac{du}{dy} \tag{4.2}$$

where $du/dy$ is the velocity gradient at the wall ($y = 0$) as shown in Fig. 4.2. When a fluid flows over a solid surface, there is a stagnant film immediately adjacent to the wall where the fluid velocity is zero and through which heat is conducted.

In an elastic solid, shear stress is proportional to the shear strain. In a viscous fluid, shear stress is proportional to the rate of shear strain.

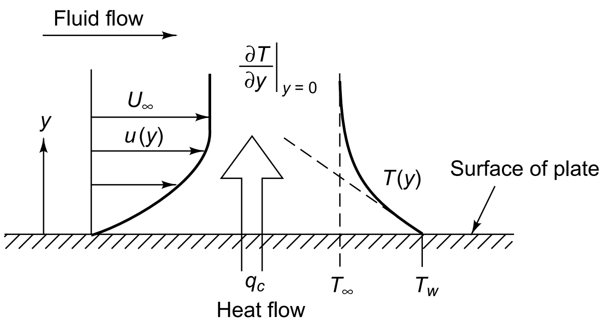


**Fig. 4.2** *Velocity and temperature distributions in laminar forced convection flow over a flat plate (heated) at temperature $T_w$*

The dimension of $\mu$ is $\boxed{\frac{\text{N}}{\text{m}^2} \cdot \frac{\text{ms}}{\text{m}}}$ or $\frac{\text{Ns}}{\text{m}^2}$. It is also $\boxed{\frac{\text{kgm}}{\text{s}^2} \cdot \frac{\text{s}}{\text{m}^2}}$ or $\frac{\text{kg}}{\text{ms}}$. In cgs units, it is $\frac{\text{dyne s}}{\text{cm}^2}$, which was called "poise" after the French physicist L. J. M. Poiseuille. However, it used to be expressed in centipoises (cp).

The viscosity of a liquid is much larger than that of a gas, i.e. $\mu_{\text{liq}} \gg \mu_{\text{gas}}$. As temperature increases, the viscosity of a liquid decreases, because of the decrease of cohesive forces between the molecules as the liquid becomes lighter. But as temperature increases, the viscosity of a gas increases, because the molecules travel faster as a result of which there is increase in the transfer of molecular momentum.

There is another frequently used property, the *kinematic viscosity*, $\nu$, defined as

$$\nu = \frac{\mu}{\rho}$$

The dimension of $\nu$ is $\frac{\text{kg m}^3}{\text{ms kg}}$ or $\text{m}^2/\text{s}$. It is also called *momentum diffusivity*. In cgs units, it used to be in

$$\frac{\text{dyne-sec cm}^3}{\text{cm}^2\ \text{gm}} \quad \text{or} \quad \frac{\text{gm cm}}{\text{sec}^2} \cdot \frac{\text{sec}}{\text{cm}^2} \cdot \frac{\text{cm}^3}{\text{gm}} \quad \text{or} \quad \frac{\text{cm}^2}{\text{s}}$$

which used to be referred as "stoke" after the British physicist G.G. Stoke. However, it used to be expressed in centistokes (cs).

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Let us assume that some fluid is flowing over a solid surface [Fig. 4.3(a)]. If we imagine a curve in the fluid, the tangent at every point of which indicates the direction of the velocity of the fluid particle, then the curve is known as a *streamline*. When one streamline slides smoothly over another streamline, and if this is maintained in the entire flow, the flow is known as *laminar*. When there is transverse flow of fluid particles and streamlines are interwoven with one another, the flow is known as *turbulent* [Fig. 4.3(b)].

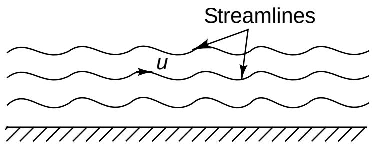


**Fig. 4.3 (a)** *Laminar flow*

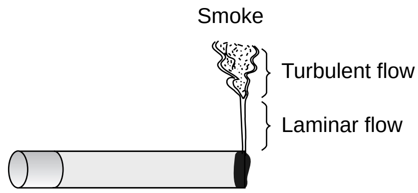


**Fig. 4.3(b)** *Laminar and turbulent flow regimes of cigarette smoke*

It was the British scientist Osborne Reynolds who first differentiated these two types of flow in a series of experiments conducted in 1883. He injected dye into a stream of water flowing inside a pipe. When the velocity of water was low, no mixing was observed between dye and water. Water was moving in parallel layers or laminae, a movement he called laminar flow. However, when the velocity of water was increased to a certain magnitude, the water particles began to move transversely, eddies appeared and mixing occurred between dye and water. This movement he called turbulent flow.

From various experiments on the flows of fluids through pipes, Reynolds discovered that the absolute viscosity $\mu$, the mass density of the fluid $\rho$, and the diameter of the pipe $D$ are the three other factors besides the velocity of the fluid, controlling the transition from laminar to turbulent flow. The results were confirmed by various experiments of French scientist M. Couette in 1890. Reynolds and Couette arranged the four quantities in a dimensionless form, which is called, in honour of Osborne Reynolds, the *Reynolds number* (Re), defined as

$$\mathrm{Re} = \frac{\rho u_m D}{\mu} = \frac{u_m D}{\nu} \tag{4.3}$$

where $u_m$ is the mean velocity of the fluid.

The value of Reynolds number at which the flow pattern changes from laminar to turbulent motion is called *critical Reynolds number* $\mathrm{Re}_c$.

For smooth circular pipes, $\mathrm{Re}_c$ is usually taken as 2300. If $\mathrm{Re} > 2300$, the flow is turbulent. If $\mathrm{Re} < 2300$, the flow is laminar.

### 4.1.1 Flow Over a Flat Plate

In a transfer process it is the phenomena at the boundary between a fluid and a solid surface, which is usually of the greatest importance in flow and heat transfer calculations.

When a fluid flows over a flat plate and its velocity is measured at various points normal to the surface in the immediate vicinity of the wall, a velocity profile is obtained, as shown in Fig. 4.4. The velocity begins with the value zero at the wall and increases within a thin layer of thickness $\delta$ to the value of free-stream velocity $u_\infty$. This distance from the wall $\delta$ is called the *boundary layer thickness* where there is velocity gradient and above which velocity is uniform and there is no viscous effect. Viscosity effect is thus confined only in the boundary layer, and the main flow outside the boundary layer, called the *potential flow*, is considered frictionless, where for each streamline *Bernoulli equation* applies.

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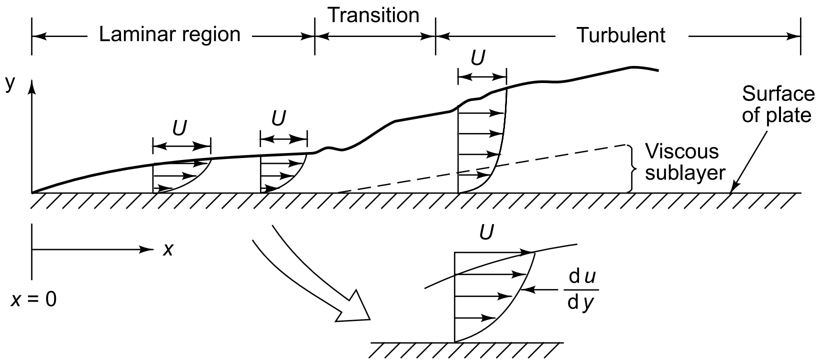


**Fig. 4.4** *Velocity profiles in laminar, transition and turbulent boundary layers in flow over a flat plate*

In the flow of a fluid over a flat plate held parallel to the direction of flow (Fig. 4.4), the vertical scale is purposely enlarged in order to show the detail of the flow pattern. When the fluid passes the leading edge of the plate, the velocity gradient and the viscous boundary shear are high. The fluid is moving in the laminar regime, and the boundary layer is thin. This is called *laminar boundary layer*. As the fluid travels further down stream along the plate, the retardation of fluid flow progresses due to viscous shear, and the boundary layer grows in thickness. As a result the velocity gradient gradually decreases. Meanwhile the boundary shear is reduced as the thickness increases. When the boundary layer becomes thick enough, the particles begin to move out of the smooth layers or laminae, the laminar motion becomes unstable, and finally the flow becomes turbulent. However, under the turbulent boundary layer, there is still a thin layer of fluid immediately next to the solid boundary, and this is still flowing in the laminar pattern. This is called the *laminar sublayer*. The layer of transition from the laminar sublayer to the turbulent layer is called the *buffer layer*. Since a laminar boundary layer cannot suddenly change into a turbulent one, a transition zone exists between them.

The behaviour of flow in the boundary layer with the distance $x$ from the leading edge is governed by the magnitude of Reynolds number given by

$$\overline{\mathrm{Re}}_x = \frac{u_\infty x}{\nu}$$

where $u_\infty$ is the free-stream velocity, $x$ is the distance from leading edge and $\nu$ is the kinematic viscosity.

The orderly motion of fluid continues along the plate until a critical distance is reached or Reynolds number attains a critical value (Fig. 4.5), $\mathrm{Re}_{x_c}$, when fluid eddies begin to form characterizing the end of the

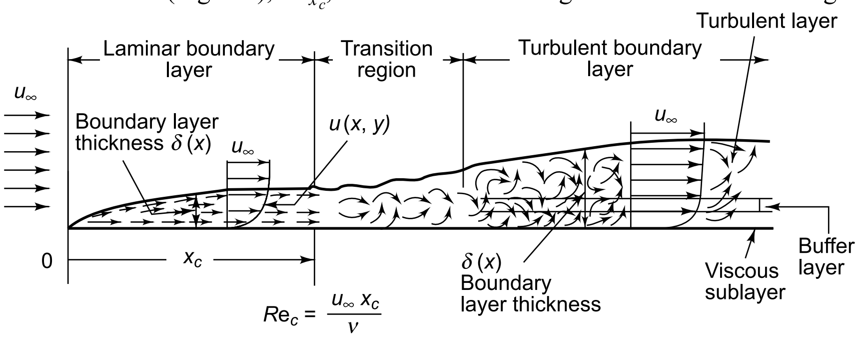


**Fig. 4.5** *Boundary layer growth for flow over a flat plate* $Re_c = \frac{u_\infty x_c}{\nu}$

laminar boundary layer and the beginning of transition from the laminar to turbulent boundary layer. This value of critical Reynolds number for flow over a flat plate is

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$$\overline{\text{Re}}_{x_c} = \frac{u_\infty \, x_c}{\nu} = 5 \times 10^5 \tag{4.4}$$

This critical value, however, strongly depends on the surface roughness and the turbulence level of the free stream. For example, with large disturbances in the free stream, the transition may begin at a Reynolds number as low as $10^5$, and for flows which are free from disturbances, it may not start until a Reynolds number of $10^6$ or more.

The boundary layer concept for flow over a curved body is illustrated in Fig. 4.6. Here, the $x$-coordinate is measured along the curved surface of the body. By starting from the stagnation point and at each $x$-location, the $y$-coordinate is measured normal to the surface of the body. The free-stream velocity $u_\infty(x)$ is not constant, but varies with distance $x$ along the curved surface. The boundary layer thickness $\delta(x)$ increases with distance $x$ along the surface. However, because of the curvature of the surface, after some distance $x$, the velocity profile $u\,(x, y)$ exhibits a point of inflection, i.e. $\partial u/\partial y$ becomes zero at the wall surface. Beyond the point of inflection, the flow reversal takes place, and the boundary layer is said to be detached from the surface. Beyond the point of flow reversal, the flow patterns are complicated with vortices and the boundary layer analysis does not hold good.

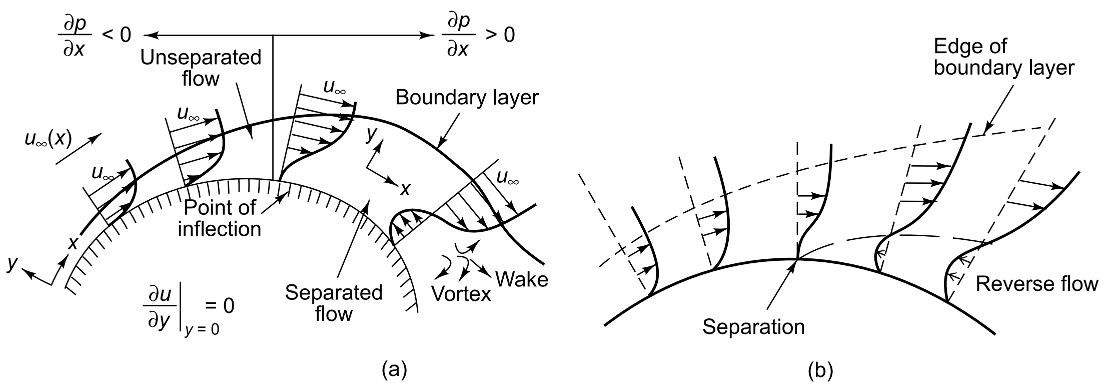


**Fig. 4.6** *Boundary layer growth for flow along (a) a curved body and (b) circular cylinder*

It is being elucidated further. From Euler's equation for an inviscid flow, $u_\infty(x)$ exhibits behaviour opposite to that of $p(x)$. That is, from $u_\infty = 0$ at the stagnation point, the fluid accelerates because of the favourable pressure gradient ($\mathrm{d}u_\infty/\mathrm{d}x > 0$ when $\mathrm{d}p/\mathrm{d}x < 0$), reaches a maximum velocity when $\mathrm{d}p/\mathrm{d}x = 0$, and decelerates because of the adverse pressure gradient ($\mathrm{d}u_\infty/\mathrm{d}x < 0$ when $\mathrm{d}p/\mathrm{d}x > 0$). As the fluid decelerates, the velocity gradient at the surface, $(\mathrm{d}u/\mathrm{d}y)_{y=0}$ eventually becomes zero. At this location, termed the *separation point*, fluid near the surface lacks sufficient momentum to overcome the pressure gradient, and continued downstream movement is impossible. Since the oncoming fluid also precludes flow back upstream, *boundary layer separation* must occur (Fig. 4.6). This is a condition for which the boundary layer detaches from the surface, and a *wake* is formed in the downstream region. Flow in this region is characterized by vortex formation and is highly irregular. The *separation point* is the location for which $\left(\frac{\partial u}{\partial y}\right)_s = 0$. The occurrence of *boundary layer transition* depends on the Reynolds number which strongly influences the position of the separation point and is defined by

$$\text{Re}_d = \frac{\rho u_m d}{\mu} = \frac{u_m d}{\nu}$$

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where $d$ is the diameter of the cylinder.

If a line 1–1 is drawn in the boundary layer and parallel to the boundary surface so that

the area 234 = area 256

the distance between the line 1–1 and the boundary line is called the *displacement thickness* ($\delta^*$) (Fig. 4.7). Here $\delta^*$ represents the distance by which an equivalent uniform stream would have to be displaced from the surface to give the same volume flow of the fluid.

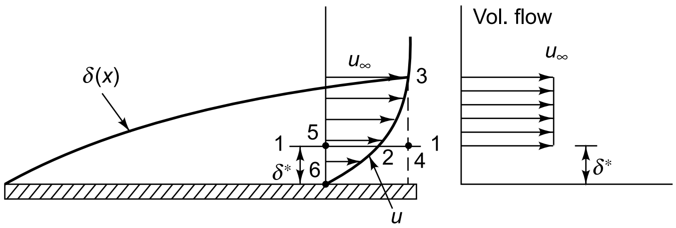


**Fig. 4.7** *Displacement thickness $\delta^*$*

The volume flow in the boundary layer

$$Q = \int_0^\delta u \, dy = u_\infty (\delta - \delta^*)$$

$$\delta^* = \frac{1}{u_\infty} \int_0^\delta (u_\infty - u) \, dy = \int_0^\delta \left(1 - \frac{\mu}{u_\infty}\right) dy \tag{4.5}$$

Thus *displacement thickness* can be difined as the distance measured perpendicular to the solid boundary by which the free stream is displaced on account of the formation of the boundary layer. Consequently, it is an additional 'wall thickness' that would have to be added to compensate for the reduction in flow rate on account of boundary layer formation.

In this regard, there are two similar terms called momentum thickness and energy thickness. The *momentum thickness* is defined as the distance from the solid boundary wall by which the boundary should be displaced to compensate for reduction in momentum of the flowing fluid on account of boundary layer formation.

Mass flow rate per second through an elementary strip $dy$ in the boundary layer = $\rho u \, dy$

Momentum of the same mass of fluid = $\rho u^2 \, dy$

Momentum of the same fluid entering the boundary layer = $(\rho \, u dy) \, u_\infty$

Thus loss of momentum per second

$$= \rho u u_\infty \, dy - \rho u^2 \, dy = du \, (u_\infty - u) \, dy \ldots.$$

Let $\theta$ be the distance by which the plate is displaced had the fluid been flowing with a constant velocity $u_\infty$.

Then less of momentum of the fluid per second = $\rho \theta u_\infty^2$

Equating the two equations

$$\rho \theta u_\infty^2 = \int_0^\delta \rho u (u_\infty - u) \, dy$$

$$\therefore \qquad \boldsymbol{\theta} = \int_0^\delta \frac{\mathbf{u}}{\mathbf{u}_\infty}\left(1 - \frac{\mathbf{u}}{\mathbf{u}_\infty}\right) \mathbf{dy}.$$

where $\theta$ is the momentum thickness.

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Similarly, the *energy thickness* is difined as the distance normal to the solid surface by which the boundary should be displaced to compensate for the reduction in KE of the flowing fluid on account of boundary layer formation. It is denoted by $\delta_e$.

Mass flow per second through the elementary strip = $\rho u \mathrm{d}y$.

KE of this fluid = $\frac{1}{2} mu^2 = \frac{1}{2}(\rho u \mathrm{d}y)u^2$

KE of the same mass of fluid before entering the solid plate = $\frac{1}{2}(\rho u \mathrm{d}y)u_\infty^2$

Loss of KE through the strip = $\frac{1}{2}(u\rho \mathrm{d}u)u_\infty^2 - \frac{1}{2}(\rho u \mathrm{d}y)u^2$

$$= \frac{1}{2}\rho u(u_\infty^2 - u^2)\,\mathrm{d}y$$

Total loss of KE of the fluid $= \int_0^\delta \frac{1}{2}\rho u(u_\infty^2 - u^2)$.

This is also equal to = $\frac{1}{2}(\rho u_\infty \delta_e)u_\infty^2$

$$\therefore \qquad \delta_e = \frac{1}{u_\infty^3}\int_0^\delta u(u_\infty^2 - u^2)\mathrm{d}y$$

$$= \int_0^\delta \frac{\mathbf{u}}{\mathbf{u}_\infty}\left(1 - \frac{\mathbf{u}^2}{\mathbf{u}_\infty^2}\right)\mathbf{dy}$$

A complete discussion of boundary layer theory is beyond the scope of this book. An excellent analytic description of boundary layer theory is available in the classic book "Boundary Layer Theory" by Schlitchting [1].

### 4.1.2 Drag Coefficient and Drag Force

Let the velocity profile $u(x, y)$ in the boundary layer is known. The viscous shear stress $\tau_x$ acting on the wall at any location $x$ is defined by

$$\tau_x = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} \tag{4.6}$$

and it can be determined from the known velocity profile. In practice, shear stress, or the local drag force per unit area $\tau_x$, is related to local drag coefficient $C_{f_x}$ by the relation

$$\boldsymbol{\tau_x = C_{f_x}\frac{\rho u_\infty^2}{2}} \tag{4.7}$$

Thus knowing the drag coefficient, one can calculate the drag force exerted by the fluid flowing over the flat plate. From Eqs (4.6) and (4.7), we have

$$C_{f_x} = \frac{2\nu}{u_\infty^2}\left(\frac{\partial u}{\partial y}\right)_{y=0} \tag{4.8}$$

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Thus the local drag coefficient $C_{f_x}$ can be determined from Eq. (4.8) if the velocity profile $u(x, y)$ in the boundary layer is known.

The mean value of the drag coefficient over the plate length $L$ is

$$\bar{C}_f = \frac{1}{L}\int_0^L C_{f_x}\,\mathrm{d}x \tag{4.9}$$

Then the drag force $F$ acting on the plate of length $L$ and width $b$ is equal to

$$\mathbf{F = bL\ \bar{C}_f\ \frac{\rho u_\infty^2}{2}} \text{ newtons} \tag{4.10}$$

### 4.1.3 Thermal Boundary Layer

Analogous to the concept of velocity boundary layer, one can visualise the development of a thermal boundary layer with temperature varying from $T_w$ to $T_\infty$ in the boundary layer thickness $\delta_t$ (Fig. 4.8). Let us consider that a fluid at a uniform temperature $T_\infty$ flows along a flat plate maintained at a constant temperature $T_w$. If we define the dimensionless temperature $\theta(x, y)$ as

$$\theta(x, y) = \frac{T_w - T}{T_w - T_\infty} \tag{4.11}$$

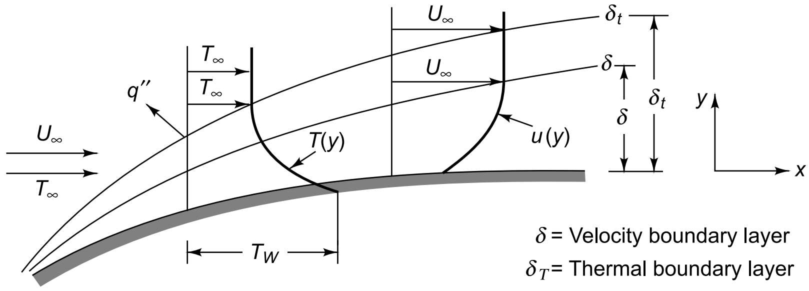


**Fig. 4.8** *Growth of velocity and thermal boundary layers in flow over a flat surface of arbitrary shape*

where $T(x, y)$ is the local temperature in the fluid, then at $y = 0$, $\theta = 0$ and at $y = \infty$, $\theta = 1$. Therefore, at each location $x$ along the plate one finds a location $y = \delta_t(x)$ in the fluid where $\theta$ equals 0.99. The locus of such points where $\theta = 0.99$ is called the *thermal boundary layer* $\delta_t(x)$.

The relative thickness of the thermal boundary layer $\delta_t(x)$ and the velocity boundary layer $\delta(x)$ depend on the Prandtl number of the fluid. For fluids having Pr = 1, $\delta_t(x) = \delta(x)$. For fluids having Pr << 1, such as liquid metals, $\delta_t >> \delta$, whereas for fluids having Pr >> 1, $\delta_t << \delta$.

### 4.1.4 Heat Transfer Coefficient

If the temperature profile $T(x, y)$ in the thermal boundary layer is known, then the heat flux

$$q(x) = -k\left(\frac{\partial T}{\partial y}\right)_{y=0} \tag{4.12}$$

where $k$ is the thermal conductivity of the fluid. In practice, a local heat transfer coefficient $h(x)$ is defined as

$$q(x) = h_x(T_w - T_\infty) \tag{4.13}$$

which, as we saw before, is the *Newton's law of cooling*.

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From Eqs (4.12) and (4.13),

$$\mathbf{h_x = k \frac{(\partial T/\partial y)_{y=0}}{T_\infty - T_w} = k\left(\frac{\partial \theta}{\partial y}\right)_{y=0}} \tag{4.14}$$

The local heat transfer coefficient $h_x$ can be determined by knowing the dimensionless temperature distribution $\theta(x, y)$ in the thermal boundary layer. It decreases along the length (Fig. 4.9).

The mean heat transfer coefficient over the length $L$ of the plate is

$$h_m = \bar{h}_c = \frac{1}{L}\int_0^L h_x \, dx \tag{4.15}$$

The total heat transfer rate $Q$ from the plate of length $L$ and width $b$ is

$$Q = b\,L\,h_m\,(T_w - T_\infty) \tag{4.16}$$

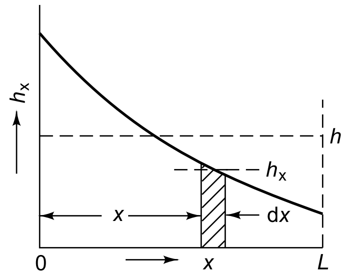


**Fig. 4.9** *Mean heat transfer coefficinet,* $h_m = \frac{1}{L}\int_0^L h_x \, dx$

The transfer of heat from the solid wall to the fluid takes place by a combination of conduction and mass transport. In laminar flow, heat is transferred by molecular conduction from streamline to streamline. In turbulent flow, the conduction mechanism is aided by innumerable eddies which carry lumps of fluid across the streamlines. These fluid particles act as carriers of energy and transfer energy by mixing with other fluid particles

$$\mathbf{Q = -k_f A \frac{T_\infty - T_w}{\delta_t} = hA(T_w - T_\infty)} \tag{4.17}$$

Since $k_f$ of fluids is generally small (except liquid metals), the rate of heat transfer very much depends on the rate of mixing of fluid particles. Higher the Reynolds number, higher will be the rate of mixing, lower the value of $\delta_t$ and higher the values of $h$ and $Q$.

## 4.2 CONSERVATION EQUATIONS OF MASS, MOMENTUM AND ENERGY FOR LAMINAR FLOW OVER A FLAT PLATE

To derive the conservation of mass or continuity equation, let us consider a control volume within the boundary layer as shown in Fig. 4.10 and assume that steady-state conditions prevail. The rates of mass flow into and out of the control volume in the $x$-direction are

$$M'_x = (\rho u)\, dy\, dz$$

and

$$M''_x = [\rho u + \frac{\partial}{\partial x}(\rho u)\, dx]\, dy\, dz$$

Net mass flow in $x$-direction,

$$M'_x - M''_x = -\frac{\partial}{\partial x}(\rho u)\, dx\, dy\, dz$$

Similarly, net mass flow in $y$- and $z$-directions

$$M'_y - M''_y = -\frac{\partial}{\partial y}(\rho v)\, dx\, dy\, dz$$

and

$$M'_z - M''_z = -\frac{\partial}{\partial z}(\rho w)\, dx\, dy\, dz$$

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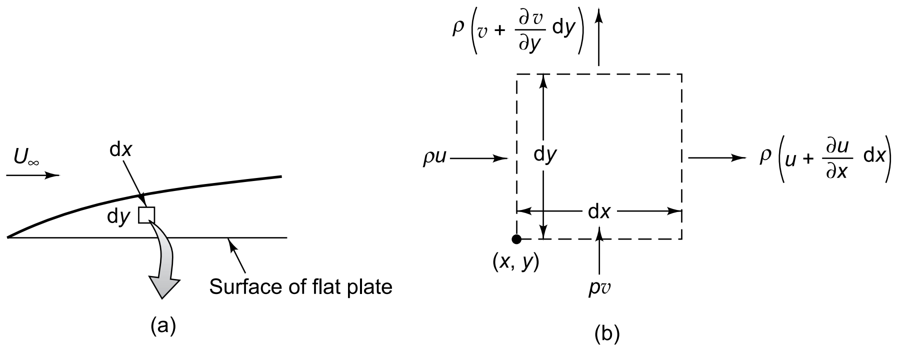


**Fig. 4.10** *Control volume (dx dy.1) for conservation of mass in an incompressible boundary layer in flow over a flat plate*

Net rate of mass accumulation within the control volume

$$= \frac{\partial}{\partial t}(\rho\, dx\, dy\, dz)$$

$$= -\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] dx\, dy\, dz$$

Therefore,

$$\frac{\partial \rho}{\partial t} + \text{div}\, \rho \bar{V} = 0 \tag{4.18}$$

For steady state,

$$\frac{\partial \rho}{\partial t} = 0$$

$\therefore$

$$\text{div}\, \rho \bar{V} = 0$$

For an incompressible fluid,

$$\text{div}\, \bar{V} = 0$$

i.e.,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{4.19}$$

For a two-dimensional flow,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

The same equation can also be derived vectorially. Let us consider a fixed control surface $S$ enclosing a volume $V$ (Fig. 4.11). The rate of accumulation of mass inside $S$ must be equal to the rate of inflow across the control surface minus the rate of outflow.

Rate of accumulation of mass in the control volume

$$= \frac{\partial}{\partial t} \oint_V \rho\, dV$$

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Net rate of outflow across the control surface

$$\oint_S \rho \bar{v}\, \mathrm{d}S$$

The surface integral is transformed to volume integral as

$$\oint_V \mathrm{div}\,(\rho \bar{v})\, \mathrm{d}V$$

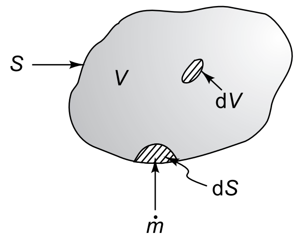


**Fig. 4.11** *Control volume*

By mass balance,

$$\frac{\partial}{\partial t} \oint_V \rho \mathrm{d}V = -\oint_V \mathrm{div}\,(\rho \bar{V})\, \mathrm{d}V$$

Writing the conservation of mass equation for the differential volume d$V$ with the integral sign removed

$$\frac{\partial}{\partial t}(\rho \mathrm{d}V) = -\,\mathrm{div}\,(\rho \bar{V})\, \mathrm{d}V$$

Since d$V$ is now independent, it is struck off from both sides of the above equation, and we obtain

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho \bar{V}) = 0$$

This is the same equation as Eq. (4.18).

By expansion of Eq. (4.18)

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

or

$$\frac{\partial \rho}{\partial t} + \rho \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} = 0 \tag{4.20}$$

The expressions $u \dfrac{\partial \rho}{\partial x}$, etc. describe the changes in density suffered by the differential element as a result of displacements of the type d$x = u$ d$t$, etc. to a new position, at which the density has a different local value. The expressions are said to denote the components of the convective rate of change of density, while the term $\partial \rho / \partial t$ denotes the local time rate of change of density. The sum of local and convective components gives the total or "substantial" rate of change of density, for which the notation $D\rho/Dt$ is usually employed. Thus

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} \tag{4.21}$$

The continuity equation may be written as

$$\frac{D\rho}{Dt} + \rho \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0$$

or

$$\frac{D\rho}{Dt} = -\rho\, \mathrm{div}\, V \tag{4.22}$$

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Similarly, if the velocity $u = u\,(x, y, z, t)$, then the acceleration of the fluid particle is

$$\frac{Du}{Dt} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial u}{\partial t}$$

$$= \underbrace{\mathbf{u}\frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \mathbf{v}\frac{\partial \mathbf{u}}{\partial \mathbf{y}} + \mathbf{w}\frac{\partial \mathbf{u}}{\partial \mathbf{z}}}_{\textbf{Convective differential}} \quad + \quad \underbrace{\frac{\partial \mathbf{u}}{\partial \mathbf{t}}}_{\textbf{Local differential}}$$

The *conservation of momentum equation* is obtained from application of Newton's second law of motion to the element. Let us consider in the flow of a fluid within the laminar boundary layer an elementary parallelepiped of sides d$x$, d$y$ and d$z$ (Fig. 4.12).

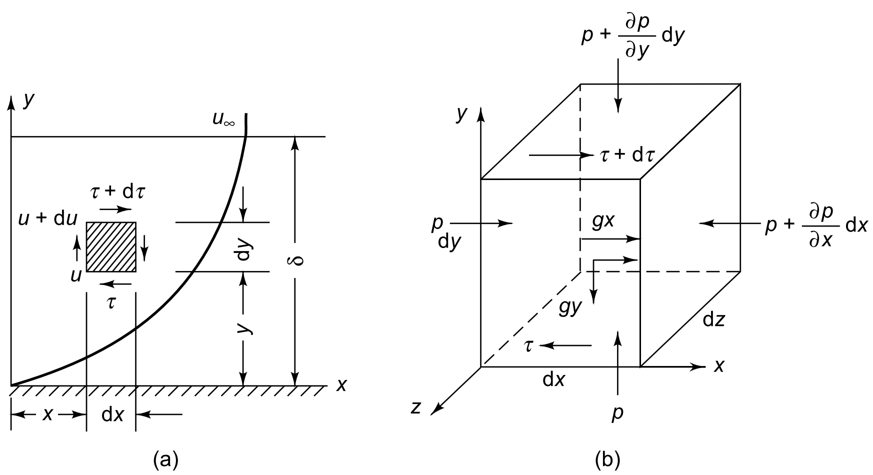


**Fig. 4.12** *Forces acting on a fluid element*

Three forces act on the element: inertia force, pressure force and viscous or friction force.

Let us consider forces only in $x$-direction.

*Inertia force:*

$$F_g = g_x\,\rho\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{4.23}$$

*Pressure force:*

$$F_p = p\,\mathrm{d}y\,\mathrm{d}z - \left(p + \frac{\partial p}{\partial x}\mathrm{d}x\right)\mathrm{d}y\,\mathrm{d}z$$

$$= -\frac{\partial p}{\partial x}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{4.24}$$

*Viscous force:*

The velocity of fluid particles at the bottom surface of the element is less than that of the particles within the element. Therefore, the shear stress will develop which would tend to oppose the flow and the shear force is $-\,\tau \mathrm{d}x\,\mathrm{d}z$. On the top surface of the element the particles above ($y$ + d$y$) move at a velocity exceeding that of the particles within the element and hence would tend to accelerate and the shear force would be $\left(\tau + \frac{\partial \tau}{\partial y}\mathrm{d}y\right)\mathrm{d}x\,\mathrm{d}z$

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Therefore, the net shear force is $\frac{\partial \tau}{\partial y}$ dx dy dz
or

$$F_\tau = \frac{\partial \tau}{\partial y}\,dx\,dy\,dz = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right)dx\,dy\,dz$$

$$= \mu \frac{\partial^2 u}{\partial y^2}\,dx\,dy\,dz$$

This applies only to one-dimensional flow. In three dimensions,

$$F_\tau = \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)dx\,dy\,dz$$

$$= \mu \nabla^2 u\,dx\,dy\,dz \tag{4.25}$$

Adding Eqs (4.23), (4.24) and (4.25), the $x$-axis component of the resultant of all the forces acting upon the considered elemental volume.

$$F_x = F_g + F_p + F_\tau = g_x\,\rho\,dv - \frac{\partial p}{\partial V}\,dv + \mu\nabla^2 u\,dv$$

where $dv = dx\,dy\,dz$
or

$$F_x = \left(g_x\rho - \frac{\partial p}{\partial x} + \mu\nabla^2 u\right)dv \tag{4.26}$$

By Newton's second law of motion

$$F_x = (\rho dv)\frac{Du}{Dt} = \left(g_x\rho - \frac{\partial p}{\partial x} + \mu\nabla^2 u\right)dv$$

or

$$\boldsymbol{\rho \frac{Du}{Dt} = \rho g_x - \frac{\partial p}{\partial x} + \mu\nabla^2 u} \tag{4.27}$$

The components of the resultant force along the $y$- and $z$-directions can similarly be obtained

$$\rho\frac{Dv}{Dt} = g_y\rho - \frac{\partial p}{\partial y} + \mu\nabla^2 v \tag{4.28}$$

$$\rho\frac{Dw}{Dt} = g_z\rho - \frac{\partial p}{\partial z} + \mu\nabla^2 w \tag{4.29}$$

This system of three Eqs (4.27), (4.28) and (4.29) is known as *Navier-Stokes differential equation* for incompressible viscous liquids. For compressible fluids it can be shown [1 – 3]

$$\rho\frac{Du}{Dt} = g_x\rho - \frac{\partial p}{\partial x} + \mu\nabla^2 u + \frac{1}{3}\mu\frac{\partial}{\partial x}\,\text{div}\,\bar{V}$$

$$= g_x\rho - \frac{\partial p}{\partial x} + \mu\left(\nabla^2 u + \frac{1}{3}\frac{\partial}{\partial x}\,\text{div}\,\bar{V}\right) \tag{4.30}$$

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$$\rho \frac{Dv}{Dt} = g_y \rho - \frac{\partial p}{\partial y} + \mu\left(\nabla^2 v + \frac{1}{3}\frac{\partial}{\partial y}\text{div } \bar{V}\right) \tag{4.31}$$

$$\rho \frac{Dw}{Dt} = g_z \rho - \frac{\partial p}{\partial z} + \mu\left(\nabla^2 w + \frac{1}{3}\frac{\partial}{\partial z}\text{div } \bar{V}\right) \tag{4.32}$$

or

$$\rho \frac{D\bar{V}}{Dt} = G - \text{grad } p + \mu\left(\nabla^2 \bar{V} + \frac{1}{3}\text{grad div } \bar{V}\right) \tag{4.33}$$

For incompressible fluids

$$\text{div } \bar{V} = 0.$$

The Navier–Stokes equations, together with the continuity equation, form the basis of the mechanics of viscous fluids. They together represent four equations for the four unknowns $u$, $v$, $w$ and $p$. In the case of compressible fluids we encounter an additional unknown $\rho$, but we also have at our disposal the equation of state.

We will now derive the *conservation of energy* applied to the fluid element. The equation governing the conduction of heat in a stationary medium, in the absence of heat sources, is

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T$$

where $\partial T/\partial t$ is the local differential or the rate of change of temperature at a point which is stationary.

If the medium is in motion, as in the case of a fluid, the total or substantial rate of change of temperature is required, which is given below.

$$\begin{aligned} \frac{DT}{Dt} &= \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} \\ &= \alpha \nabla^2 T = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) \end{aligned} \tag{4.34}$$

In presence of large pressure gradients or for fluids moving at high velocities, two additional terms must be included to account for the compression work and for the dissipation of energy due to friction.

The complete energy equation for a compressible fluid may thus be written as

$$\rho c_p \frac{DT}{Dt} - \frac{Dp}{Dt} = k\nabla^2 T + \mu\phi(V) \tag{4.35}$$

where $\phi(V)$ denotes the *dissipation function*, first used by Lord Rayleigh, and given by [1]

$$\begin{aligned} \phi(V) &= 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 \\ &\quad + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) - \frac{2}{3}(\text{div } \bar{V})^2 \end{aligned} \tag{4.36}$$

The effect of viscous dissipation can be significant if the fluid is very viscous, as in journal bearings, or if the fluid shear rate is very high [2, 3].

From Eq. (4.34), the following two-dimensional expression for the energy equation without dissipation is obtained

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$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right) \tag{4.37}$$

Since the boundary layer is quite thin, under normal conditions $\frac{\partial T}{\partial y} >> \frac{\partial T}{\partial x}$. Also the pressure term in the momentum equation, Eq. (4.27), is zero for flow over a flat plate, since $\left(\frac{\partial u_\infty}{\partial x}\right) = 0$. Eq. (4.27) thus reduces to

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \tag{4.38}$$

Then the similarity between the momentum and energy equations becomes apparent, as given below

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\left(\frac{\partial^2 u}{\partial y^2}\right) \tag{4.39a}$$

and

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\left(\frac{\partial^2 T}{\partial y^2}\right) \tag{4.39b}$$

where $\nu$ is the kinematic or momentum diffusivity and $\alpha$ is the thermal diffusivity, the dimensions of both being m$^2$/s. The ratio of these two transport properties is called the *Prandtl number*, Pr. Therefore,

$$\text{Pr}=\frac{\nu}{\alpha}=\frac{\rho\mu c_p}{\rho k}=\frac{\mu c_p}{k} \tag{4.40}$$

If $\nu = \alpha$, then Pr = 1, and the momentum and energy equations are identical. For this condition, nondimensional solutions of $u(y)$ and $T(y)$ are identical if the boundary conditions are similar. Thus, it is apparent that the Prandtl number controls the relation between the velocity and temperature distributions.

The conditions which are important to study in the analysis of a heat transfer process are

(a) *Geometric conditions:* Round, rectangular, smooth or rough
(b) *Physical conditions:* Properties of the fluid like oil, water, air etc., $c_p$, $\mu$, $k$ and $\rho$
(c) *Boundary conditions:* Velocity and temperature distribution
(d) *Time condition:* Steady, unsteady, periodic etc.

Any flow or heat transfer problem can be solved by solving mass, momentum and energy equations with appropriate boundary conditions, but in actual cases we often find analytical solutions very complex and difficult. Broad simplifying assumptions are frequently needed to arrive at a solution, and the experimental results vary widely from the theoretical data.

## 4.3 PRINCIPLE OF SIMILARITY APPLIED TO HEAT TRANSFER

The concept of similarity is derived from geometry. Two bodies are considered similar when their corresponding linear dimensions are in a constant ratio to one another (Fig. 4.13), such that in the two triangles

$$\frac{l_1}{l_1'}=\frac{l_2}{l_2'}=\frac{l_3}{l_3'}=C, \text{ similarity constant}$$

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The concept of physical similarity demands in addition to the above that all the other physical quantities involved in a given pair of systems, e.g. force, time intervals, velocities and temperatures, are respectively proportional to one another. For two similar systems

$$\frac{\rho_2}{\rho_1} = f_\rho, \quad \frac{\mu_2}{\mu_1} = f_\mu, \quad \frac{T_2}{T_1} = f_T$$

and so on,
where $f_\rho$, $f_\mu$ and $f_T$ are similarity parameters for density, viscosity and temperature respectively.

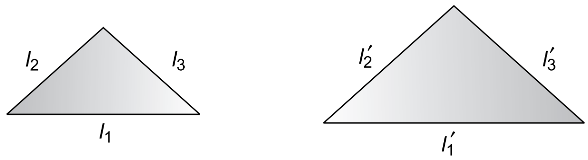


**Fig. 4.13** *Geometrical similarity of two triangles*

The physical laws discovered on the basis of the study of a model would apply not only to the original system, but also to an infinite number of other systems, provided they are physically similar to the model.

The methods of similarity allow us to generalize the experimental results with the aid of model rules [4, 7]. These model rules help to establish dimensionless parameters which must have the same values for both the model and the original system.

By Newton's laws of motion,

$$P = mf = \frac{mu}{t}$$

so that $$P_1 = \frac{m_1 u_1}{t_1} \text{ and } P_2 = \frac{m_2 u_2}{t_2}$$

$$\frac{P_2}{P_1} = \frac{m_2}{m_1}\frac{u_2}{u_1}\frac{t_1}{t_2}$$

or $$f_p = f_m \frac{f_u}{f_t}$$

so that $$\frac{f_p f_t}{f_m f_u} = \frac{P_1 t_1}{m_1 u_1} = \frac{P_2 t_2}{m_2 u_2} = \text{Ne or Newton number}$$

The dimensioned quantities are grouped together to yield meaningful dimensionless parameters.

### 4.3.1 Derivation of Dimensionless Parameters from the Differential Equations

The laws of similarity, as applied to heat transfer, were obtained for the first time by Nusselt, who derived the dimensionless parameters appropriate to forced and free convection from the differential equations together with the respective boundary conditions.

#### (a) Forced Convection

Let us consider two pipes of different diameters, each carrying a steady flow of a different fluid (Fig. 4.14). Both fluids are incompressible. Each flow is generated by pressure difference applied at the ends of the pipe. Each flow is classified as forced convection. The fluids exchange heat with the walls of the pipes, the actual direction of the heat flow being immaterial i.e., one fluid may be cooled while the other may be heated by the pipe. The physical properties of the fluid are assumed to be constant, independent of temperature. The flow of heat is considered to be steady.

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The entire flow and heat transfer processes are described by the continuity, momentum and energy equations:

*Continuity equation:*

$$\frac{\partial \rho}{\partial t} + \text{div}\ \rho \bar{V} = 0$$

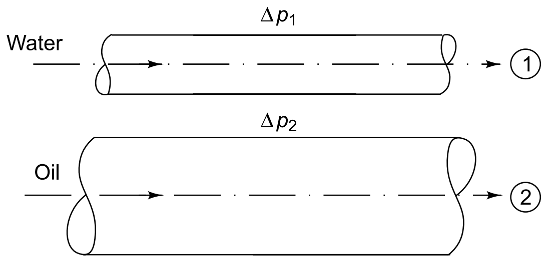


**Fig. 4.14** *Application of similarity principle*

*Momentum equation:* ($x$-component)

$$\rho \frac{Du}{\bar{D}t} = \rho g - \frac{\partial p}{\partial x} + \mu \nabla^2 u + \frac{1}{3}\mu \frac{\partial}{\partial x}\text{div}\ \bar{V}$$

*Energy equation:*

$$\frac{DT}{\bar{D}t} = \alpha \nabla^2 T$$

By denoting the quantities appropriate to each pipe by subscripts 1 and 2,

*Continuity equation:* Incompressible and steady flow, div $\bar{V} = 0$

For the first pipe,

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial v_1}{\partial y_1} + \frac{\partial w_1}{\partial z_1} = 0 \tag{4.41}$$

*Momentum equation:* ($x$-component)

Forced convection, $\rho g = 0$; Steady state, $\partial u/\partial t = 0$

$$\therefore \quad u_1 \frac{\partial u_1}{\partial x_1} + v_1 \frac{\partial u_1}{\partial y_1} + w_1 \frac{\partial u_1}{\partial z_1}$$

$$= -\frac{1}{\rho_1}\frac{\partial p_1}{\partial x_1} + \nu_1 \left( \frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial y_1^2} + \frac{\partial^2 u_1}{\partial z_1^2} \right) \tag{4.42}$$

*Energy equation:*

$$u_1 \frac{\partial T_1}{\partial x_1} + v_1 \frac{\partial T_1}{\partial y_1} + w_1 \frac{\partial T_1}{\partial z_1}$$

$$= \alpha_1 \left( \frac{\partial^2 T_1}{\partial x_1^2} + \frac{\partial^2 T_1}{\partial y_1^2} + \frac{\partial^2 T_1}{\partial z_1^2} \right) \tag{4.43}$$

We can describe the heat flow at the wall in terms of heat transfer coefficient, and a mean temperature excess of the fluid $\theta_m$ above the measured wall temperature (Fig. 4.15).

$$\bar{q} = h(T_w - T_\infty) = -k\frac{T_\infty - T_w}{\delta_t} = h\theta_m$$

$$\bar{q}_1 = h_1 \theta_1 = -k_1 \left( \frac{\partial \theta_1}{\partial n_1} \right)_w \tag{4.44}$$

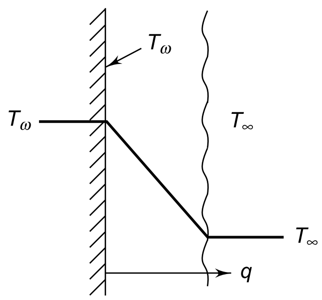


**Fig. 4.15** *Heat transfer at the wall*

Identical equations and boundary conditions apply to the second pipe except for the suffix 1 being replaced by 2.

We now postulate that the two systems are physically similar. Thus for length scale, the proportionality factor $f_L$ is defined as

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$$f_L = \frac{L_2}{L_1}$$

Similarly, the other proportionality factors are listed below:

$$\begin{array}{ll} \text{For the velocities } u, v, w & f_u = u_2/u_1 \\ \text{For the pressure } p & f_p = p_2/p_1 \\ \text{For the density} & f_\rho = \rho_2/\rho_1 \\ \text{For the kinematic viscosity} & f_\nu = \nu_2/\nu_1 \\ \text{For the temperature } T \text{ and } \theta & f_T = T_2/T_1 \\ \text{For the thermal diffusivities} & f_\alpha = \alpha_2/\alpha_1 \\ \text{For the heat transfer coefficients } h & f_h = h_2/h_1 \\ \text{For the thermal conductivities } k & f_k = k_2/k_1 \end{array} \tag{4.45}$$

If we now introduce these proportionality factors (or similarity constants) into the equations appropriate to the second pipe and take common factors outside the brackets, we obtain for the second pipe (putting $u_2 = f_u \cdot u_1$, $T_2 = f_T \cdot T_1$, $\nu_2 = f_\nu \cdot \nu_1$ and so on):

*Continuity equation:*

$$\frac{f_u}{f_L}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial v_1}{\partial y_1} + \frac{\partial w_1}{\partial z_1}\right) = 0 \tag{4.46}$$

*x-Momentum equation:*

$$\frac{f_u^2}{f_L}\left(u_1\frac{\partial u_1}{\partial x_1} + v_1\frac{\partial u_1}{\partial y_1} + w_1\frac{\partial u_1}{\partial z_1}\right)$$

$$= -\frac{f_p}{f_L f_\rho}\frac{1}{\rho_1}\frac{\partial p_1}{\partial x_1} + \frac{f_\nu f_u}{f_L^2}\nu_1\left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial y_1^2} + \frac{\partial^2 u_1}{\partial z_1^2}\right) \tag{4.47}$$

*Energy equation:*

$$\frac{f_u f_T}{f_L}\left(u_1\frac{\partial T_1}{\partial x_1} + v_1\frac{\partial T_1}{\partial y_1} + w_1\frac{\partial T_1}{\partial z_1}\right)$$

$$= \frac{f_\alpha f_T}{f_L^2}\alpha_1\left(\frac{\partial^2 T_1}{\partial x_1^2} + \frac{\partial^2 T_1}{\partial y_1^2} + \frac{\partial^2 T_1}{\partial z_1^2}\right) \tag{4.48}$$

Equation describing the heat flow:

$$f_h f_T f h_1 \, \theta m_1 = -\frac{f_k f_T}{f_L}k\left(\frac{\partial \theta_1}{\partial n_1}\right)_w \tag{4.49}$$

These Eqs (4.46 – 4.49) describe the processes in the second pipe, but they become identical with those of the first pipe [Eqs (4.41) – (4.44)] if the following conditions are satisfied:

$$\frac{f_u^2}{f_L} = \frac{f_p}{f_L f_\rho} = \frac{f_\nu f_u}{f_L^2} \tag{4.50}$$

$$\frac{f_u f_T}{f_L} = \frac{f_\alpha f_T}{f_L^2} \tag{4.51}$$

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$$f_h f_T = \frac{f_k f_T}{f_L} \tag{4.52}$$

It may be noticed that the continuity equation yields no condition for the proportionality constants, since the value of $f_u/f_L$ is arbitrary. If we now substitute for the proportionality factors from Eq. (4.45), we obtain the conditions which must be satisfied for physical similarity to exist between our two systems and to permit one system to be considered as a model of the other. From Eq. (4.50):

$$\frac{u_2^2}{u_1^2}\frac{L_1}{L_2} = \frac{p_2}{p_1}\frac{L_1}{L_2}\frac{\rho_1}{\rho_2} = \frac{\nu_2}{\nu_1}\frac{u_2}{u_1}\frac{L_1^2}{L_2^2}$$

$$\frac{u_2 L_2}{\nu_2} = \frac{u_1 L_1}{\nu_1} = \frac{uL}{\nu} \tag{4.53}$$

$$\frac{\rho_1 u_1^2}{p_1} = \frac{\rho_2 u_2^2}{p_2} = \frac{\rho u^2}{p} \tag{4.54}$$

From Eq. (4.51):

$$\frac{u_2}{u_1}\frac{T_2}{T_1}\frac{L_1}{L_2} = \frac{\alpha_2}{\alpha_1}\frac{T_2}{T_1}\frac{L_1^2}{L_2^2}$$

$$\frac{u_2 L_2}{\alpha_2} = \frac{u_1 L_1}{\alpha_1} = \frac{uL}{\alpha} \tag{4.55}$$

From Eq. (4.52):

$$\frac{h_2}{h_1}\frac{T_2}{T_1} = \frac{k_2}{k_1}\frac{T_2}{T_1}\frac{L_1}{L_2}$$

$$\frac{h_2 L_2}{k_2} = \frac{h_1 L_1}{k_1} = \frac{hL}{k} \tag{4.56}$$

Following a suggestion by Gröber [6], the resulting dimensionless parameters are designated with the names of outstanding scientists in the field

$\frac{uL}{\nu}$ = Re or $N_{Re}$, Reynolds number

$\frac{uL}{\alpha}$ = Pe or $N_{Pe}$, Peclet number

$\frac{hL}{k}$ = Nu or $N_{Nu}$, Nusselt number

The dimensionless product $(\rho u^2)/p$ does not represent a true similarity parameter, since the pressure in a channel with prescribed dimensions and velocities, adjusts itself to the correct value in accordance with Eq. (4.42). Thus the flow is determined by the Reynolds number alone, if geometrically similar boundaries are assumed.

In addition to the three similarity parameters derived above, we may choose to use any combinations of them, as long as a total number of three independent parameters is preserved. Thus, for example, the ratio

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Pe/Re = $\nu/\alpha$ = Pr or $N_{pr}$, Prandtl number is particularly useful, because it contains only the properties of the fluid.

Our original problem is thus solved, since if the three dimensionless parameters Re, Pr and Nu are equal in both the systems, then the two systems are physically similar and constant proportions exist between all quantities concerned. This may be immediately extended to include all geometrically similar systems (i.e. all circular pipes) for which the dimensionless parameters have the same values. If we can obtain the solution for any one system by any means, e.g. empirically, it must be possible to write this solution in the form

$$F\,(\text{Re, Pr, Nu}) = 0 \tag{4.57}$$

and it is valid for all the systems characterised by the same values of the dimensionless parameters.

To obtain an expression for a given variable, for instance $h$, from Eq. (4.57), we solve the equation explicitly for the parameter which contains the variable, i.e. Nu.

$$\text{Nu} = F(\text{Re, Pr}) \tag{4.57a}$$

or

$$h = \frac{k}{L}\,F(\text{Re, Pr}) \tag{4.58}$$

Hence, the heat transfer coefficient may be predicted for all similar systems from a single model experiment. Equations (4.57a) and (4.58) also express the fact that the flow of the fluid is not affected by the heat transfer. The momentum equation, Eq. (4.42) yields as a single dimensionless parameter the Reynolds number.

### (b) Free Convection

Equation (4.57) is valid for forced convection only. In the case of free convection, the buoyancy force experienced by a fluid system at a higher temperature i.e., at a lower density, than the surrounding fluid must be introduced into the momentum equation as a body force. Let us consider, as an example, a vertical wall at a temperature higher than that of the surrounding fluid (Fig. 4.16). The fluid layer heated up by the wall suffers a decrease in density and as a result experiences an upthrust relative to the surrounding fluid. The change in the specific volume $v_s = 1/\rho$ with temperature may be expressed in terms of coefficient of thermal expansion

$$\beta = \frac{1}{v_s}\left(\frac{\partial v_s}{\partial T}\right)_p$$

where $v_s$ is the specific volume.

We will assume that only density varies with temperature, while other properties ($\mu$, $c_p$, $k$) are still assumed constant.

For an ideal gas,

$$pv_s = RT \text{ or } v_s = \frac{RT}{p}$$

$$\left(\frac{\partial v_s}{\partial T}\right)_p = \frac{R}{T} = \frac{v_s}{T}$$

$$\therefore \qquad \beta = \frac{1}{T}$$

Buoyancy force per unit mass is $(\rho_o - \rho)\,g/\rho$.

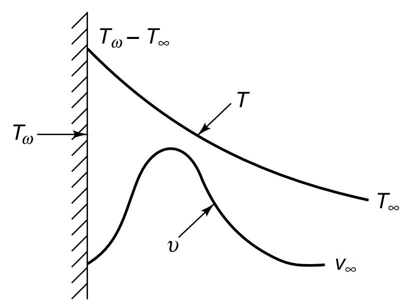


**Fig. 4.16** Temperature and velocity distribution in natural convection

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Now, $\nu = \nu_o (1 + \beta\theta)$

where $\theta = T - T_o$ = excess temperature

$$\frac{1}{\rho} = \frac{1}{\rho_o}(1+\beta\theta)$$

or $\rho_o = \rho + \rho\beta\theta$

$$\rho_o - \rho = \rho\beta\theta$$

Buoyancy force per unit mass is $\rho\beta\theta g/\rho = \beta\theta g$.

Buoyancy force per unit volume, $G = \beta\theta g\rho$

This is directed upward. By introducing this body force in the momentum equation, Eq. (4.42),

$$u_1 \frac{\partial u_1}{\partial x_1} + v_1 \frac{\partial u_1}{\partial y_1} + w_1 \frac{\partial u_1}{\partial z_1} = -\frac{1}{\rho_1}\frac{\partial p_1}{\partial x_1} + g_1\beta_1\theta_1 + \nu_1\left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial y_1^2} + \frac{\partial^2 u_1}{\partial z_1^2}\right) \tag{4.59}$$

The subscript 1 refers to the model of a system, while subscript 2 refers to the system itself. To describe the system itself, we require, in addition to the proportionality factors of Eq. (4.45), two more such factors defined by $f_\beta = \beta_2/\beta_1$ and $f_g = g_2/g_1$.

In accordance with the procedure adopted for forced convection earlier, we obtain the following condition

$$\frac{f_u^2}{f_L} = \frac{f_p}{f_L f_\rho} = f_g f_T f_\beta = \frac{f_\nu f_u}{f_L^2} \tag{4.60}$$

Since the pressure differences in a free convection flow are generally negligible, the second term of Eq. (4.60) does not yield any dimensionless parameter. Unlike forced convection there exists no prescribed velocity. The fluid velocity is zero both at the plate and at a large distance from it, outside the boundary layer. The ratio $f_u$ is thus largely meaningless and may be eliminated. For Eq. (4.60),

$$\frac{f_u^2}{f_L} = f_g f_T f_\beta = \frac{f_\nu\ f_u}{f_L^2}$$

Equating the first and third terms, $f_u = f_\nu/f_L$, and substituting

$$f_g f_T f_\beta = \frac{f_\nu^2}{f_L^3}$$

$$\frac{g_2}{g_1}\frac{\theta_2}{\theta_1}\frac{\beta_2}{\beta_1} = \frac{\nu_2^2}{\nu_1^2}\frac{L_1^3}{L_2^3}$$

$$\frac{g_2\beta_2\theta_2 L_2^3}{\nu_2^2} = \frac{g_1\beta_1\theta_1 L_1^3}{\nu_1^2} = \frac{g\beta\theta L^3}{\nu^2} = \mathrm{Gr}$$

where Gr = Grashof number = $\dfrac{g\beta\theta L^3}{\nu^2}$

Equations (4.51) and (4.52) representing energy and heat transfer equations are equally valid for free convection.

$$\frac{f_u f_T}{f_L} = \frac{f_\alpha f_T}{f_L^2} \text{ and } f_h f_T = \frac{f_k f_T}{f_L}$$

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Substituting, $f_u = \dfrac{f_v}{f_L}$ in the first equation i.e., Eq (4.51)

$$\frac{f_v f_T}{f_L^2} = \frac{f_\alpha f_T}{f_L^2}$$

$$f_v = f_\alpha \quad \text{or} \quad v_2/v_1 = \alpha_2/\alpha_1$$

$$\frac{v_2}{\alpha_2} = \frac{v_1}{\alpha_1} = \frac{v}{\alpha} \text{ Pr, the Prandtl number}$$

Again, $$f_h f_T = \frac{f_k f_T}{f_L}$$

which yields $hL/k$ = Nu, the Nusselt number.

Thus, the heat transfer in free convection can be described by the equation

$$\text{Nu} = F(\text{Gr, Pr}) \tag{4.61}$$

### (c) Unsteady Heat Conduction

Fourier's differential equation for unsteady heat conduction can be treated in the same manner in which we have handled the equations of forced and free convection. For one-dimensional heat flow, we have

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

For physical similarity between a system and its model, we have to introduce an additional proportionality factor for time $f_t = t_2/t_1$. Following the same procedure, we obtain

$$\frac{f_T}{f_t} = \frac{f_\alpha f_T}{f_L^2}$$

$$\frac{\alpha t}{L^2} = \text{constant} \tag{4.62}$$

It is the Fourier number, Fo, which is often referred to as *dimensionless time*.

## 4.4 EVALUATION OF CONVECTION HEAT TRANSFER COEFFICIENTS

In convection heat transfer the key unknown is the heat transfer coefficient. Five general methods are available for its evaluation:

1. Dimensional analysis combined with experiments.
2. Exact mathematical solutions of the boundary layer equations.
3. Approximate analysis of the boundary layer equations by integral methods.
4. Analogy between heat and momentum transfer.
5. Numerical analysis.

All five of these techniques have contributed to our understanding of convection heat transfer. Yet no single method can solve all the problems because each one has limitations that restrict its scope of application.

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1. *Dimensional analysis:* It is mathematically simple and has found a wide range of applications [4, 7]. The chief limitation of this method is that the results obtained are incomplete and quite useless without experimental data. Dimensional analysis makes little contribution to our understanding of the transfer process, but facilitates the interpretation and extends the range of experimental data by correlating them in terms of dimensionless groups.

   To correlate the experimental data there are two methods for determining dimensionless groups:
   (a) Enlisting the variables pertinent to a phenomenon and rationally grouping them. This technique is simple to use, but if a pertinent variable is omitted, erroneous results ensue.
   (b) Dimensionless groups and similarity conditions are deduced from the differential equations describing the phenomenon. This method, as presented in the previous section, is preferable if the phenomenon can be described mathematically.
2. Exact mathematical analyses require simultaneous solution of the equations describing the fluid motion and the transfer of energy in the moving fluid [5]. The physics of the problem must be well understood to describe it mathematically. Complete mathematical equations can be written only for laminar flow under simple boundary conditions. Exact solutions are, however, important because the assumptions made can be specified precisely and their validity can be verified experimentally. Availability of high speed computers has increased the range of problems amenable to mathematical solution.
3. *Approximate analysis of boundary layer:* It avoids the detailed mathematical description of the flow in the boundary layer. Instead, a plausible but simple equation is used to describe the velocity and temperature profiles in the boundary layer. The problem is then analysed on a macroscopic basis by applying the equation of motion and the energy equation to the fluid in the boundary layer. This method is relatively simple and yields solutions within engineering accuracy to problems that cannot be treated by an exact mathematical analysis.
4. *Analogy between heat and momentum transfer:* It is a useful tool for analysing turbulent transfer processes. Our knowledge of turbulent-exchange mechanisms is not good enough to write mathematical equations describing the temperature distribution directly, but the transfer mechanism can be explained in terms of a simplified model. One such model explains a mixing motion in a direction perpendicular to the mean flow accounting for the transfer of momentum as well as energy, similar to that used to picture the motion of gas molecules in the kinetic theory. Experimental results are substantially in agreement with analytical predictions based on the model.
5. *Numerical methods:* They can solve approximately the equations of motion and energy [8, 9]. The approximation results from the need to express the field variables (velocity, temperature and pressure) at discrete points in time and space rather than continuously. However, the solution can be made sufficiently accurate if care is taken in discretising the exact solutions. One of the most important features of numerical methods is that once the solution procedure has been programmed, solutions for different boundary conditions, property variables and so on can be easily computed. Generally, numerical methods can handle complex boundary conditions easily [9].

## 4.5 DIMENSIONAL ANALYSIS

In engineering, we represent physical concepts by symbols or dimensions, as for instance length by $L$ and velocity by $V$. Through experience, we have learned that we can select a certain number of dimensions as fundamental, and express all other dimensions in terms of products of powers of these fundamental dimensions. The dimensions of commonly used quantities in heat transfer analysis are listed in Table 4.1 with four fundamental dimensions: mass $M$, length $L$, time $t$ and temperature $T$. For example, the dimension

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of heat transfer coefficient is W/m$^2$ K. In MLtT system, it is (Nm/s) (1/m$^2$K) or, (kg m/s$^2$) (1/s m K) or kg/s$^3$ K which is M/t$^3$T (or Mt$^{-3}$ T$^{-1}$). Similarly, all other quantities can be expressed in fundamental dimensions, as given in Table 4.1.

**Table 4.1** *Important heat transfer physical quantities and their dimensions*

| *Quantity* | *Symbol* | *Dimensions in MLtT system* |
|---|---|---|
| Length | $L, x, y$ | L |
| Time | $t$ | t |
| Mass | $M$ | M |
| Force | $F$ | MLt$^{-2}$ |
| Temperature | $T$ | T |
| Heat transfer | $Q$ | ML$^2$t$^{-2}$ |
| Velocity | $u, \bar{v}, u_\infty$ | Lt$^{-1}$ |
| Acceleration | $a, g$ | Lt$^{-2}$ |
| Work | $W$ | ML$^2$t$^{-2}$ |
| Pressure | $p$ | Mt$^{-2}$ L$^{-1}$ |
| Density | $\rho$ | ML$^{-3}$ |
| Internal energy | $e$ | L$^2$t$^{-2}$ |
| Enthalpy | $h$ | L$^2$t$^{-2}$ |
| Specific heat | $c$ | L$^2$t$^{-2}$T$^{-1}$ |
| Absolute viscosity | $\mu$ | ML$^{-1}$t$^{-1}$ |
| Kinematic viscosity | $\nu$ | L$^2$t$^{-1}$ |
| Thermal conductivity | $k$ | MLt$^{-3}$ T$^{-1}$ |
| Thermal diffusivity | $\alpha$ | L$^2$t$^{-1}$ |
| Thermal resistance | $R$ | Tt$^3$ M$^{-1}$L$^{-2}$ |
| Coefficient of expansion | $\beta$ | T$^{-1}$ |
| Surface tension | $\sigma$ | Mt$^{-2}$ |
| Shear stress | $\tau$ | ML$^{-1}$t$^{-2}$ |
| Heat transfer coefficient | $h$ | Mt$^{-3}$ T$^{-1}$ |
| Mass flow rate | $m$ | Mt$^{-1}$ |

The methods of dimensional analysis are founded upon the principle of dimensional homogeneity, which states that all equations describing the behaviour of physical systems, must be dimensionally consistent i.e., each term with reference to a given set of fundamental dimensions must have the same dimensions. When the equations governing a process are known and solvable, dimensional analysis suggests logical grouping of quantities for presenting the results. When the mathematical equations governing certain processes are unknown or too complex, dimensional analysis lays the foundation of an efficient experimental program for obtaining the results, by reducing the number of variables requiring investigation and by indicating a possible form of the semi-empirical correlations that may be formulated. It should be borne in mind that dimensional analysis by itself cannot provide quantitative answers, and thus cannot be a substitute for the exact or the approximate mathematical solutions. It is nevertheless an important tool to learn to use, especially in instances when mathematical analysis is impractical or when some rapid, qualitative answers are needed.

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The application of similarity principle to the continuity, momentum and energy equations for identifying the dimensionless parameters that govern the concerned process, which was discussed earlier, is also based on the principle of dimensional homogeneity. However, when the governing equations of a problem are unknown, an alternate approach in the application of dimensional analysis is necessary. At the very start, it is required to know, or more typically to guess, the independent variables that determine the behaviour of a particular dependent variable of interest. These can usually be found by logic or intuition developed from previous experiences with problems of a similar nature, but there is no way to ensure that all essential quantities have been included or not. Rayleigh first used this method and the rules of algebra to combine the many variables of a problem into dimensionless groups. We are providing in the next section two examples of application of Rayleigh's method.

### 4.5.1 Rayleigh's Methods

(a) Let us consider the frictional resistance of fluid flow per unit area of the inside surface of the pipe. A reasonable assumption can be made that the resistance which causes pressure drop of the fluid ($\Delta p$) is a function of tube diameter ($D$), fluid density ($\rho$), fluid velocity ($u$) and fluid viscosity ($\mu$), or

$$\Delta p = f[u, D, \rho, \mu]$$

Let $\Delta p = Cu^a D^b \rho^c \mu^d$

where $C$ is a diamensionless constant.

The dimensional equation of the above expression in fundamental dimensions $M$, $L$ and $t$ are

$$\frac{\mathrm{MLt}^{-2}}{\angle^2} = (\mathrm{Lt}^{-1})^a \, \mathrm{L}^b \, (\mathrm{ML}^{-3})^c \, (\mathrm{ML}^{-1}\,\mathrm{t}^{-1})^d$$

$$\mathrm{ML}^{-1}\,\mathrm{t}^{-2} = \mathrm{L}^{a+b-3c-d}\,\mathrm{t}^{-a-d}\,\mathrm{M}^{c+d}$$

For the homogeneity of M : $1 = c + d$,

L : $-1 = a + b - 3c - d$ and,

t : $-2 = -a - d$.

On solving these equations we have $b = -d$, $c = 1 - d$ and $a = 2 - d$.

$$\Delta p = Cu^{2-d} D^{-d} \rho^{1-d} \mu^d$$

$$= C\rho u^2 \left(\frac{\mu}{\rho u D}\right)^d = C\frac{\rho u^2}{\mathrm{Re}_d^d} \tag{4.63}$$

where $\mathrm{Re}_d = \rho u D/\mu$ = Reynolds number.

The values of constants $C$ and $d$ have to be determined by experiments.

(b) Let us consider forced convection heat transfer between a fluid flowing through a pipe and its wall. We enlist the variables pertinent to the phenomenon by logic or intuition and group them into dimensionless parameters. Let us assume that the heat transfer coefficient $h$ is a function of pipe diameter ($D$), fluid velocity ($u$), and the fluid properties of density ($\rho$), viscosity ($\mu$), thermal conductivity ($k$) and specific heat ($c_p$), or

$$h = f[D, u, \rho, \mu, k, c_p]$$

$$h = BD^a \, u^b \rho^c \mu^d k^e c_p^f$$

where $B$ is a constant.

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Expressing the quantities in terms of fundamental dimensions M, L, t and T,

$$\mathrm{Mt^{-3}\,T^{-1}} = \mathrm{BL}^a\,(\mathrm{Lt^{-1}})^b\,(\mathrm{ML^{-3}})^c\,(\mathrm{ML^{-1}t^{-1}})^d\,(\mathrm{MLt^{-3}T^{-1}})^e\,(\mathrm{L^2t^{-2}T^{-1}})^f$$

or,
$$\mathrm{Mt^{-3}T^{-1}} = \mathrm{BL}^{a+b-3c-d+e+2f}\,\mathrm{t}^{-b-d-3e-2f}\,\mathrm{M}^{c+d+e}\,\mathrm{T}^{-e-f}$$

For dimensional homogeneity of

$$\begin{aligned} \mathrm{M}&: 1 = c + d + e \\ \mathrm{L}&: 0 = a + b - 3c - d + e + 2f \\ \mathrm{t}&: -3 = -b - d - 3e - 2f \\ \mathrm{T}&: -1 = -e - f \end{aligned}$$

On solving the above equations, $a = c - 1$, $b = c$, $d = -c + f$ and $e = 1 - f$. Therefore,

$$h = BD^{c-1}\,u^c\,\rho^c\,\mu^{-c+f}\,k^{1-f}\,c_p^f$$

$$\frac{hD}{k} = B\left(\frac{uD\rho}{\mu}\right)^c\left(\frac{c_p\mu}{k}\right)^f$$

or,
$$\mathrm{Nu}_d = B\mathrm{Re}_d^c\,\mathrm{Pr} \tag{4.64}$$

where $\mathrm{Re}_d = uD\rho/\mu$ = Reynolds number, $\mathrm{Pr} = c_p\mu/k$ = Prandtl number and $\mathrm{Nu}_d = hD/k$ = Nusselt number. Thus, in forced convection heat transfer, Nusselt number is a function of Reynolds number and Prandtl number. The constants $B$, $c$ and $f$ have to be evaluated from the experimental data.

### 4.5.2 Buckingham $\pi$-theorem

A simple and more systematic way of determining the dimensionless groups was suggested by Buckingham and has come to be known as "pi-theorem". If a small number of physical quantities is involved, the Rayleigh method is simpler. But if the number of physical quantities increases beyond a given limit, the procedure is tedious, and the pi-theorem may be used advantageously. More physical quantities simply mean a few more $\pi$-terms. Each $\pi$-term can be solved exactly the same way as in the case of fewer physical quantities.

According to the Buckingham $\pi$-theorem, any physical equation may be described by

$$\phi(Q_1, Q_2, Q_3, \cdots, Q_m) = 0 \tag{4.65}$$

which is a function of $m$ common quantities $Q_1, Q_2, \cdots, Q_m$. If $n$ fundamental dimensions M, L, t, T etc. are chosen, then the equation may be transformed into a new equation containing $(m - n)$ dimensionless terms represented by $\pi$ as follows

$$\psi(\pi_1, \pi_2, \pi_3 \cdots, \pi_{m-n}) \tag{4.66}$$

where each $\pi$-term consists of $(n + 1)$ quantities of $Q$s. To determine the $\pi$-terms, $n$ of the $Q$ quantities have to be so chosen that they contain all the fundamental dimensions M, L, t and T, and these are taken as repeating variables which form a dimensionless number with each of the remaining variables.

We are giving below a few examples illustrating the application of Buckingham $\pi$-theorem.

(a) We want to find out the relationship of pressure drop of a fluid per unit length of a pipe through which it is flowing. We know or guess that $\Delta p$ depends upon the tube diameter ($D$), velocity ($u$), density ($\rho$) and viscosity ($\mu$) of the fluid. The physical equation may be written as

$$\phi(\underline{u, D, \rho}, \mu, \Delta p) = 0 \tag{4.67}$$

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There are five variables, or $m = 5$. Let us suppose that the three fundamental dimensions are M, L and t, or $n = 3$. Therefore, the number of dimensionless $\pi$-terms $= m - n = 5 - 3 = 2$. Therefore, Eq. (4.67) can be written as

$$\psi(\pi_1, \pi_2) = 0 \tag{4.68}$$

where each $\pi$-term consists of $n + 1 = 3 + 1 = 4$ common quantities. Let the repeating variables ($n$) be $u$, $D$ and $\rho$ (which contain all the fundamental dimensions M, L and t), and these will form a dimensionless number with each of the remaining variables, i.e. $\Delta p$ and $\mu$.

Let $$\pi_1 = \rho^a u^b D^c \Delta p,$$

and $$\pi_2 = \rho^e u^f D^g \mu$$

Therefore,

$$\pi_1 = (\mathrm{ML}^{-3})^a (\mathrm{Lt}^{-1})^b (\mathrm{L})^c (\mathrm{Mt}^{-2}\mathrm{L}^{-2})$$

$$\mathrm{M}^0 \mathrm{L}^0 \mathrm{t}^0 = \mathrm{M}^{a+1}\mathrm{L}^{-3a+b+c-2}\mathrm{t}^{-b-2}$$

In order to make $\pi_1$ dimensionless, the exponents of M, L and t must all be equal to zero. For dimensional homogeneity,

$$\begin{aligned} a + 1 &= 0 \\ -3a + b + c - 2 &= 0 \\ -b - 2 &= 0 \end{aligned}$$

On solving we obtain $a = -1$, $b = -2$, $c = 1$.

$$\pi_1 = \rho^{-1} u^{-2} D\Delta p = \frac{\Delta p D}{\rho u^2} \tag{4.69}$$

Similarly,

$$\pi_2 = (\mathrm{ML}^{-3})^e (\mathrm{Lt}^{-1})^f (\mathrm{L})^g \mathrm{ML}^{-1} \mathrm{t}^{-1}$$

or $$\mathrm{M}^0 \mathrm{L}^0 \mathrm{t}^0 = \mathrm{M}^{e+1}\mathrm{L}^{-3e+f+g-1}\mathrm{t}^{-f-1}$$

$$\begin{aligned} e + 1 &= 0 \\ -3e + f + g - 1 &= 0 \\ -f - 1 &= 0 \\ e &= 1, f = -1, g = -1 \end{aligned}$$

or $$\pi_2 = \rho^{-1} u^{-1} D^{-1} \mu = \frac{\mu}{\rho D u} \tag{4.70}$$

Therefore, Eq. (4.68) can be written as

$$\psi\left(\frac{\Delta p D}{\rho u^2}, \frac{\mu}{\rho D u}\right) = 0 \tag{4.71}$$

It can be written as

$$\frac{\Delta p D}{\rho u^2} = \frac{1}{2}\zeta(\mathrm{Re}_d) \text{ where } \mathrm{Re}_d = \frac{\rho D u}{\mu}$$

or $$\Delta p = \frac{\rho u^2}{2D}\zeta(\mathrm{Re}_d)$$

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If the length of the pipe is L, the total pressure drop is

$$p_1 - p_2 = \Delta p L = \frac{fL}{D}\frac{\rho u^2}{2} \tag{4.72}$$

where $f$ is the Darcy-Weisbach friction factor for a smooth pipe, which is a function of Reynolds number, i.e. $f = \zeta(\mathrm{Re}_d)$.

### (b) Forced Convection Heat Transfer

The heat transfer coefficient can be expressed as a functional relation

$$h = f(D, \rho, u, \mu, c_p, k)$$

or, $$\phi(D, \rho, u, \mu, c_p, k, h) = 0$$

Here, the number of variables, $m = 7$. Let the number of fundamental dimensions be $n = 4$, which are M, L, t and T. The number of $\pi$-terms $= m - n = 3$.

Therefore,

$$\psi(\pi_1, \pi_2, \pi_3) = 0 \tag{4.73}$$

Each $\pi$-term is composed of $(n + 1)$ i.e., five quantities of which four are repeating variables, forming a dimensionless number with each of the remaining variables

$$\phi(\underline{\rho, D, u, c_p}, \mu, k, h) = 0$$

Let $\rho$, $D$, $u$ and $c_p$ be taken as repeating variables which contain all the four fundamental dimensions M, L, t and T and which will make a dimensionless number with each of $\mu$, $k$ and $h$.

Let $$\pi_1 = \mu\rho^a D^b u^c c_p^d$$

$$\mathrm{M}^0\,\mathrm{L}^0\,\mathrm{t}^0\,\mathrm{T}^0 = \mathrm{ML}^{-1}\,\mathrm{t}^{-1}\,(\mathrm{ML}^{-3})^a\,(\mathrm{L})^b\,(\mathrm{Lt}^{-1})^c\,(\mathrm{L}^2\mathrm{t}^{-2}\,\mathrm{T}^{-1})^d$$

$$= \mathrm{M}^{1+a}\,\mathrm{L}^{-1-3a+b+c+2d}\,\mathrm{t}^{-1-c-2d}\,\mathrm{T}^{-d}$$

Therefore,

$$0 = 1 + a$$

$$0 = -1 - 3a + b + c + 2d = 0$$

$$0 = 1 - c - 2d$$

$$0 = -d$$

Solving these, we obtain, $a = -1$, $b = -1$, $c = -1$ and $d = 0$.

$$\pi_1 = \mu\rho^{-1} D^{-1} u^{-1} c_p^0 = \frac{\mu}{\rho D u}$$

Similarly,

$$\pi_2 = k\rho^{a_1} D^{b_1} u^{c_1} c_p^{d_1}$$

$$= \mathrm{MLt}^{-3}\,\mathrm{T}^{-1}\,(\mathrm{ML}^{-3})^{a_1}\,(\mathrm{L})^{b_1}\,(\mathrm{Lt}^{-1})^{c_1}\,(\mathrm{L}^2\mathrm{t}^{-2}\mathrm{T}^{-1})^{d_1}$$

$$\mathrm{M}^0\,\mathrm{L}^0\,\mathrm{T}^0\,\mathrm{t}^0 = \mathrm{M}^{1+a_1}\,\mathrm{L}^{1-3a_1+b_1+c_1+2d_1}\,\mathrm{t}^{-3-c_1-2d_1}\,\mathrm{T}^{-1-d_1}$$

$$1 + a_1 = 0, \quad 1 - 3a_1 + b_1 + c_1 + 2d_1 = 0, \quad -3 - c_1 - 2d_1 = 0,$$

$$-1 - d_1 = 0$$

Solving these equations, we obtain

$$a_1 = -1, \quad b_1 = -1, \quad c_1 = -1, \quad d_1 = -1$$

$$\pi_2 = k\rho^{-1} D^{-1} u^{-1} c_p^{-1} = \frac{k}{\rho D u c_p}$$

$$\pi_3 = h\rho^{a_2} D^{b_2} u^{c_2} c_p^{d_2}$$

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$$= (\mathrm{Mt^{-3}\,T^{-1}})\,(\mathrm{ML^{-3}})^{a_2}\,(\mathrm{L})^{b_2}\,(\mathrm{Lt^{-1}})^{c_2}\,(\mathrm{L^2\,t^{-2}\,T^{-1}})^{d_2}$$

$$\mathrm{M}^0\,\mathrm{L}^0\,\mathrm{t}^0\,\mathrm{T}^0 = \mathrm{M}^{1+a_2}\,\mathrm{L}^{-3a_2+b_2+c_2+2d_2}\,\mathrm{t}^{-3-c_2-2d_2}\,\mathrm{T}^{-1-d_2}$$

Therefore,

$$1 + a_2 = 0$$
$$-3a_2 + b_2 + c_2 + 2d_2 = 0$$
$$-3 - c_2 - 2d_2 = 0$$
$$1 - d_2 = 0$$

On solving,

$$a_2 = -1,\quad b_2 = 0,\quad c_2 = -1,\quad d_2 = -1$$

$$\pi_3 = h\rho^{-1}\,D^0\,u^{-1}\,c_p^{-1} = \frac{h}{\rho c_p u}$$

$$\psi(\pi_1, \pi_2, \pi_3) = 0$$

It can be written as

$$\pi_3 = B\pi_1^m\,\pi_2^n$$

$$\frac{h}{\rho c_p u} = B\left(\frac{\mu}{\rho D u}\right)^m \left(\frac{k}{\rho d u c_p}\right)^n$$

$$\frac{hD}{k}\cdot\frac{k}{\rho c_p u D} = B\left(\frac{\mu}{\rho D u}\right)^m \left(\frac{k}{\rho D u c_p}\right)^n$$

$$\frac{hD}{k} = B\left(\frac{\mu}{\rho D u}\right)^m \left(\frac{k}{\mu c_p}\cdot\frac{\mu}{\rho D u}\right)^{n-1}$$

$$= B\left(\frac{\rho D u}{\mu}\right)^{-m-n+1} \left(\frac{\mu c_p}{k}\right)^{1-n}$$

or, $$\mathrm{Nu}_d = B\mathrm{Re}_d^a\,\mathrm{Pr}^b \tag{4.74}$$

which is the same as Eq. (4.64).

### (c) Free Convection Heat Transfer

The heat transfer coefficient depends upon the buoyancy force per unit mass ($g\beta\theta$), density ($\rho$), vertical height ($L$), viscosity ($\mu$), thermal conductivity ($k$) and specific heat ($c_p$). Thus, it can be written

$$\phi(\underline{\rho, L, \mu, k}, c_p, g\beta\theta, h) = 0 \tag{4.75}$$

The number of variables, $m = 7$. The number of fundamental dimensions, $n = 4$, as before. The number of $\pi$-terms $= 7 - 4 = 3$.

Each $\pi$-term will consist of (4 + 1) or 5 variables. Let $\rho$, $L$, $\mu$ and $k$ be the repeating variables, which contain all the fundamental dimensions. Now,

$$\psi(\pi_1, \pi_2, \pi_3) = 0 \tag{4.76}$$

Then, $$\pi_1 = \rho^a\,L^b\,\mu^c\,k^d\,g\beta\theta$$

$$= (\mathrm{ML^{-3}})^a\,(\mathrm{L})^b\,(\mathrm{ML^{-1}\,t^{-1}})^c\,(\mathrm{MLt^{-3}\,T^{-1}})^d\,(\mathrm{Lt^{-2}})$$

$$= \mathrm{M}^{a+c+d}\,\mathrm{L}^{-3a+b-c+d+1}\,\mathrm{t}^{-c-3d-2}\,\mathrm{T}^{-d}$$

$$= \mathrm{M}^0\mathrm{L}^0\mathrm{t}^0\mathrm{T}^0$$

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$$a + c + d = 0$$
$$-3a + b - c + d + 1 = 0$$
$$-c - 3d - 2 = 0$$
$$-d = 0$$

On solving we get $a = 2$, $b = 3$, $c = -2$ and $d = 0$

$$\pi_1 = \rho^2 L^3 \mu^{-2} k^0 g\beta\theta = \frac{g\beta\theta L^3 \rho^2}{\mu^2}$$

$$\pi_2 = \rho^{a_1} L^{b_1} m^{c_1} k^{d_1} c_p$$
$$= (ML^{-3})^{a_1} (L)^{b_1} (ML^{-1}t^{-1})^{c_1} (MLt^{-3}T^{-1})^{d_1} (L^2t^{-2}T^{-1})$$
$$= M^{a_1 + c_1 + d_1} L^{-3a_1 + b_1 - c_1 + d_1 + 2} t^{-c_1 - 3d_1 - 2} T^{-d - 1}$$
$$= M^0L^0t^0T^0$$

$$a_1 + c_1 + d_1 = 0$$
$$-3a_1 + b_1 - c_1 + d_1 + 2 = 0$$
$$-c_1 - 3d_1 - 2 = 0$$
$$-d_1 - 1 = 0$$

Solving $a_1 = 0$, $b_1 = 0$, $c_1 = 1$ and $d_1 = -1$.

$$\pi_2 = \rho^0 L^0 \mu^1 k^{-1} c_p = \frac{\mu C_p}{k}$$

$$\pi_3 = \rho^{a_2} L^{b_2} m^{c_2} k^{d_2} h$$
$$= (ML^{-3})^{a_2} (L)^{b_2} (ML^{-1}t^{-1})^{c_2} (MLt^{-3}T^{-1})^{d_2} (Mt^{-3}T^{-1})$$
$$= M^{a_2 + c_2 + d_2 + 1} L^{-3a_2 + b_2 - c_2 + d_2} t^{-c_2 - 3d_2 - 3} T^{-d_2 - 1}$$
$$= M^0L^0t^0T^0$$

$$a_2 + c_2 + d_2 + 1 = 0$$
$$-3a_2 + b_2 - c_2 + d_2 = 0$$
$$-c_2 - 3d_2 - 3 = 0$$
$$-d_2 - 1 = 0$$

Solving, $a_2 = 0$, $b_2 = 1$, $c_2 = 0$ and $d_2 = -1$.

$$\pi_3 = \rho^0 L^1 \mu^0 k^{-1} h = \frac{hL}{k}$$

Therefore, Eq. (4.76) becomes

$$\left(\frac{g\beta\theta L^3 \rho^2}{\mu^2}, \frac{\mu c_p}{k}, \frac{hL}{k}\right) = 0$$

or,
$$\frac{hL}{k} = B\left(\frac{g\beta\theta L^3}{\nu^2}\right)^a \left(\frac{\mu c_p}{k}\right)^b \qquad (4.77)$$

where
$$\text{Gr} = \text{Grashof number} = \frac{g\beta\theta L^3}{\nu^2}$$

$$\text{Pr} = \text{Prandtl number} = \frac{\mu c_p}{k}$$

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Nu = Nusselt number = (hL)/k

$$\text{Nu} = B\text{Gr}^a\,\text{Pr}^b \tag{4.78}$$

Dimensional analyses have been performed on many heat transfer systems, and Table 4.2 summarises the most important dimensionless groups used in design.

**Table 4.2** *Dimensionless Groups of Importance for Heat Transfer and Fluid Flow*

| *Group* | *Definition* | *Physical interpretation* |
|---|---|---|
| Biot number (Bi) | $\dfrac{hL}{k_s}$ | Ratio of internal thermal resistance of a solid body to its surface thermal resistance |
| Drag coefficient ($C_f$) | $\dfrac{\tau_s}{\rho U_x^2/2}$ | Ratio of surface shear stress to free-stream kinetic energy |
| Eckert number (Ec) | $\dfrac{U_\infty^2}{c_p(T_s - T_\infty)}$ | Kinetic energy of flow relative to boundary-layer enthalpy difference |
| Fourier number (Fo) | $\dfrac{\alpha t}{L^2}$ | Dimensionless time; ratio of rate of heat conduction to rate of internal energy storage in a solid |
| Friction factor (f) | $\dfrac{\Delta p}{(L/D)\,(\rho U_m^2/2)}$ | Dimensionless pressure drop for internal flow trough ducts |
| Grashof number ($Gr_L$) | $\dfrac{g\beta(T_s - T_\infty)L^3}{v^2}$ | Ratio of buoyancy to viscous forces |
| Colburn *j* factor ($j_H$) | $StPr^{2/3}$ | Dimensionless heat transfer coefficient |
| Nusselt number ($Nu_L$) | $\dfrac{\bar{h}_c L}{k_f}$ | Dimensionless heat transfer coefficient; ratio of convection heat transfer to conduction in a fluid layer of thickness L. |
| Peclet number ($Pe_L$) | $Re_L Pr$ | Product of Reynolds and Prandtl numbers |
| Prandtl number (Pr) | $\dfrac{c_p\mu}{k} = \dfrac{v}{\alpha}$ | Ratio of molecular momentum diffusivity to thermal diffusivity |
| Rayleigh number (Ra) | $Gr_L Pr$ | Product of Grashof and Prandtl numbers |
| Reynolds number ($Re_L$) | $\dfrac{U_x L}{v}$ | Ratio of inertia to viscous forces |
| Stanton number | $\dfrac{h_c}{\rho U_\infty c_p} = \dfrac{Nu_L}{Re_L\,Pr}$ | Dimensionless heat transfer coefficient |

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### 4.5.3 Correlation of Experimental Data

For forced convection heat transfer in a tube, Eq. (4.74) gives

$$\mathrm{Nu}_d = B\mathrm{Re}_d^a\,\mathrm{Pr}^b$$

where $B$, $a$ and $b$ are to be evaluated from the experimental data.

Let us suppose that in a series of experiments with air flowing over a 25 mm diameter tube, the heat transfer coefficient $\bar{h}_c$ has been measured experimentally at velocities ranging from 0.15 to 30 m/s. This range of velocities corresponds to Reynolds number ($u_m D\rho/\mu$) varying from 250 to 50,000. Since the velocity was the only variable in these tests, the heat transfer coefficients $\bar{h}_c$ measured are directly plotted against the velocity $u_\infty$ [Fig. 4.17(a)]. The resulting curve can be used to determine $\bar{h}_c$ at any velocity in this range

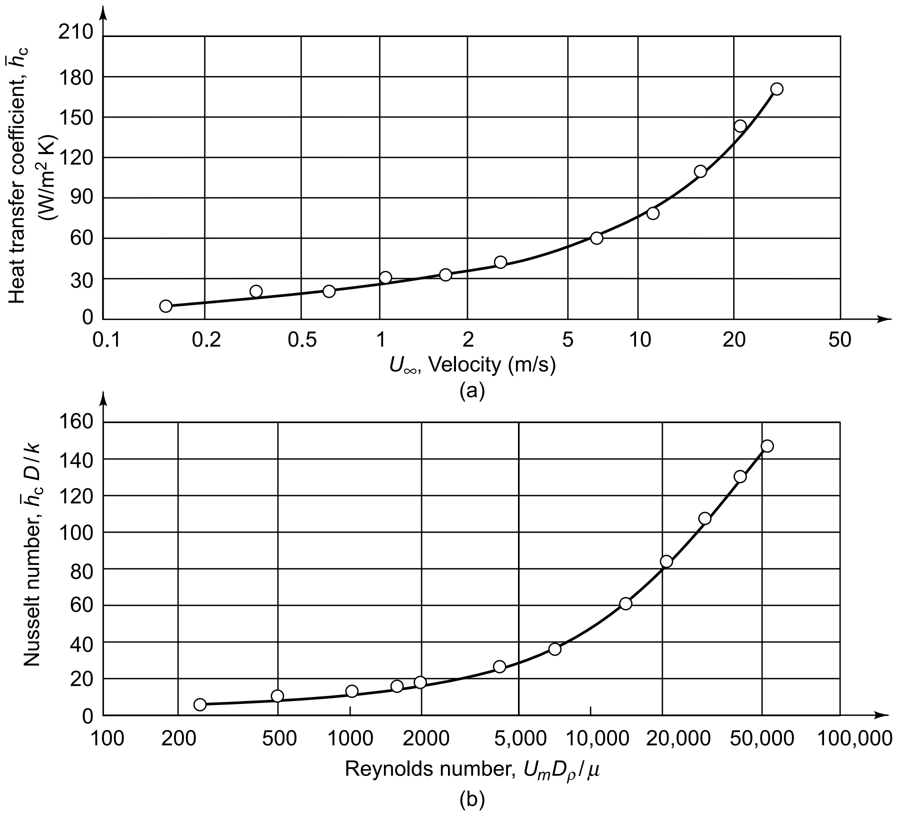


**Fig. 4.17** *Variation of Nusselt number with Reynolds number for cross-flow of air over a pipe or a long cylinder (a) Dimensional plot, (b) Dimensionless plot*

for 25 mm diameter tube only, and also only for air at that pressure or density. It cannot be used for tubes that are larger or smaller than the one used in the tests, nor can it be used if the fluid is different, say oil or water, or air is at a different pressure.

With the aid of dimensional analysis, however, the results of one series of tests can be applied to a variety of other problems, as illustrated in Fig. 4.17(b), where the data of Fig. 4.17(a) are re-plotted in terms of pertinent dimensionless groups i.e., Reynolds number ($u_m D\rho/\mu$) and Nusselt number ($h_c D/k$). This curve permits the evaluation of $\bar{h}_c$ for air only flowing over any size of pipe at any velocity in the range.

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Figure 4.18 shows the experimental results for different working fluids like air, water and oils flowing over a tube of different sizes and at different velocities, with $Nu_d/Pr^{0.3}$ as ordinate and $Re_d$ as abscissa. All the data are seen to follow a single line in the log–log plot, and they can be correlated empirically. Thus, proper nondimensionalisation of the pertinent data extends the applicability of the experimental results in a variety of operating conditions.

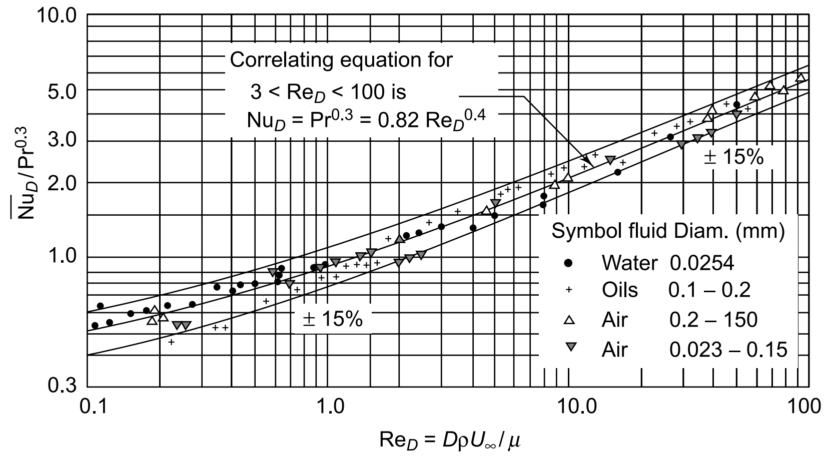


**Fig. 4.18** *Correlation of experimental heat transfer data for various fluids in cross-flow over pipes, wires and circular cylinders*

## 4.6 ANALYTIC SOLUTION FOR LAMINAR BOUNDARY LAYER FLOW OVER A FLAT PLATE

To determine the forced convection heat transfer coefficient $\bar{h}_c$ and the friction coefficient $\bar{C}_f$ for incompressible steady laminar flow over a flat surface, we must satisfy the continuity, momentum and energy equations simultaneously. These equations were derived in Section 4.2, and are given below for two-dimensional flow:

$$\textbf{Continuity: } \frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \frac{\partial \mathbf{v}}{\partial \mathbf{y}} = \mathbf{0} \tag{4.79a}$$

$$\text{Momentum: } u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \tag{4.79b}$$

$$\text{Energy: } u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{\partial^2 T}{\partial y^2} \tag{4.79c}$$

### 4.6.1 Boundary-layer Thickness and Skin Friction Coefficient

Equation (4.79b) must be solved simultaneously with the continuity equation, (Eq. 4.79a), in order to determine the velocity distribution, boundary layer thickness and skin friction coefficient. Let us first make an order-of-magnitude analysis of the differential equations to obtain the functional form of the solution. Within the boundary layer we may say that the velocity $u$ is of the order of free-stream velocity $u_\infty$. Similarly, the $y$ dimension is of the order of the boundary layer thickness $\delta$. Thus.

$$u \sim u_\infty$$
$$y \sim \delta$$

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and we might write the continuity equation in an approximate form as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{u_\infty}{x} + \frac{v}{\delta} = 0$$

$$v \sim \frac{u_\infty \delta}{x}$$

Then, by using this order of magnitude for $v$, the analysis of momentum equation would yield

$$u = \frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

$$u_\infty \frac{u_\infty}{x} + \frac{u_\infty \delta}{x}\frac{u_\infty}{\delta} = \nu\frac{u_\infty}{\delta^2}$$

$$\delta^2 \sim \frac{\nu x}{u_\infty}$$

or
$$\delta \sim \left(\frac{\nu x}{u_\infty}\right)^{1/2} \tag{4.80}$$

Dividing by $x$ to express the result in dimensionless form gives

$$\frac{\delta}{x} \sim \left(\frac{\nu}{u_\infty x}\right)^{1/2} = \frac{1}{(\mathrm{Re}_x)^{1/2}} \tag{4.81}$$

This is the functional relationship of $\delta$ with the local Reynolds number. Since the velocity profiles have similar shapes at various distances from the leading edge of the flat plate, the important variable is $y/\delta$, and we assume that the velocity may be expressed as a function of this variable.

$$\frac{u}{u_\infty} = g\left(\frac{y}{\delta}\right)$$

Introducing the order-of-magnitude estimate for $\delta$ from Eq. (4.80),

$$\frac{u}{u_\infty} = g(\eta) \tag{4.82}$$

where
$$\eta = \frac{y}{(\nu x/u_\infty)^{1/2}} = y\left(\frac{u_\infty}{\nu x}\right)^{1/2} \tag{4.83}$$

Here, $\eta$ is called the similarity variable and $g(\eta)$ is the function for which we seek a solution. In accordance with the continuity equation, a stream function $\psi$ may be defined so that

$$u = \frac{\partial \psi}{\partial y} \tag{4.84}$$

$$v = \frac{\partial \psi}{\partial x} \tag{4.85}$$

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Inserting Eq. (4.84) in Eq. (4.82) gives

$$\psi = \int u_\infty \, g(\eta)\, dy = \int u_\infty \left(\frac{\nu x}{u_\infty}\right)^{1/2} g(\eta)\, d\eta$$

or
$$\psi = u_\infty \left(\frac{\nu x}{u_\infty}\right)^{1/2} f(\eta) \tag{4.86}$$

where $f(\eta) = \int g(\eta) d\eta.$

From Eqs (4.85) and (4.86), we obtain

$$v = \frac{1}{2}\left(\frac{\nu u_\infty}{u}\right)^{1/2}\left(\eta \frac{df}{d\eta} + f\right) \tag{4.87}$$

Expressing $\partial u/\partial x$, $\partial u/\partial y$ and $\partial^2 u/\partial y^2$ in terms of $\eta$ and inserting the resulting expressions in the momentum equation, Eq. (4.79b), we obtain the ordinary, nonlinear, third-order differential equation

$$f \frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2} + 2\frac{\mathrm{d}^3 f}{\mathrm{d}\eta^3} = 0 \tag{4.88}$$

It may be solved numerically for the function $f(\eta)$ subject to the three boundary conditions:

| *Physical coordinates* | *Similarity coordinates* |
|---|---|
| 1. $u = 0$ at $y = 0$ | $\dfrac{\mathrm{d}f}{\mathrm{d}\eta} = 0$ at $\eta = 0$ |
| 2. $v = 0$ at $y = 0$ | $f = 0$ at $\eta = 0$ |
| 3. $\dfrac{\partial u}{\partial y} = 0$ at $y \to \infty$ | $\dfrac{\mathrm{d}f}{\mathrm{d}\eta} = 1.0$ at $\eta \to \infty$ |

The solution to Eq. (4.88) was first obtained by Blasius in 1908[1]. The important results are shown in Figs. 4.19 and 4.20.

Figure 4.19 shows the Blasius velocity profiles in the laminar boundary layer on a flat plate in dimensionless form together with the experimental data of Hansen [10]. The ordinate is the ratio of local and free-stream velocities, and the abscissa is a dimensionless distance parameter $(y/x)\,(\rho u_\infty\, x/\mu)^{1/2}$. The velocity $u$ reaches 99% of the free-stream velocity $u_\infty$ at $(y/x)\,(\rho u_\infty\, x/\mu)^{1/2} = 5.0$. If we define **the hydrodynamic boundary layer thickness $\delta$ as that distance from the surface at which the local velocity $u$ reaches 99% of the free-stream velocity $u_\infty$, then**

$$\frac{\delta}{x} = \frac{5}{(\mathrm{Re}_x)^{1/2}} \tag{4.89}$$

where $\mathrm{Re}_x = (\rho u_\infty x)/\mu$ or $(u_\infty x)/\nu$. Eq. (4.89) satisfies the qualitative description of the boundary layer growth, as was shown by order-of-magnitude analysis in Eq. (4.81). It explains that at $x = 0$, $\delta = 0$, and as $x$ increases, $\delta$ increases. Again, where $u_\infty$ increases, $\delta$ decreases.

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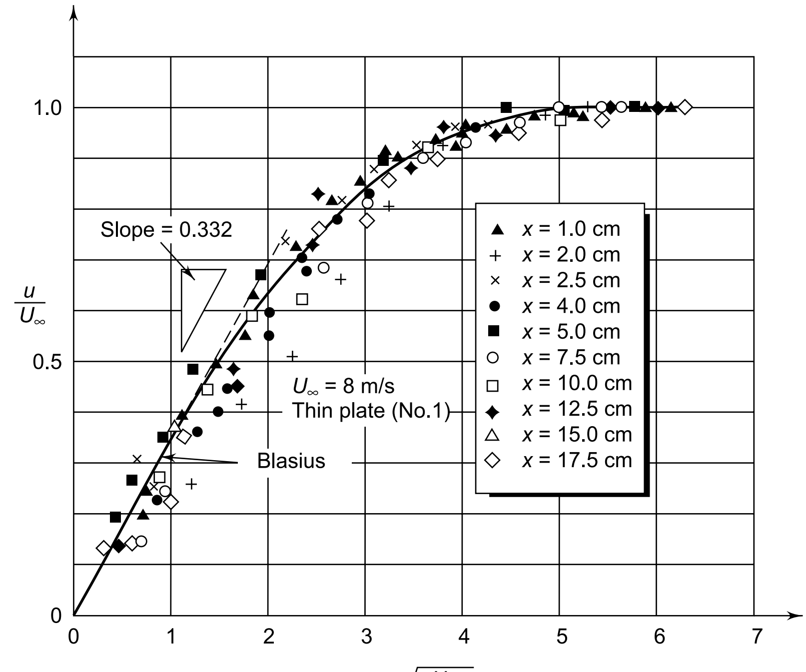
The shear stress at the wall can be obtained from the velocity gradient at $y = 0$ in Fig. 4.19. We find that

$$\left.\frac{\partial (u/u_\infty)}{\partial (y/x)(\mathrm{Re}_x)^{1/2}}\right|_{y=0} = 0.332$$




$\frac{y}{x}\sqrt{\frac{\rho U_\infty x}{\mu}}$

**Fig. 4.19** *Velocity profile in a laminar boundary layer according to Blasius, with the experimental data of Hansen [10]*

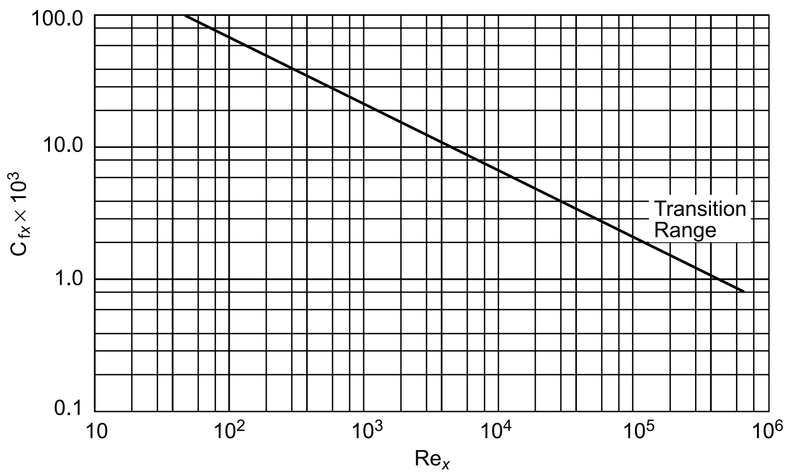



**Fig. 4.20** *Local friction coefficient varying with Reynolds number along the distance from the leading edge for laminar flow for over a flat plate*

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$$\left(\frac{\partial u}{\partial u}\right)_{y=0} = 0.332\frac{u_\infty}{x}(\text{Re}_x)^{1/2}$$

The wall shear stress becomes

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = 0.332\,\mu\frac{u_\infty}{x}(\text{Re}_x)^{1/2} \tag{4.90}$$

It may be noted that the wall shear stress at the leading edge ($x = 0$) is very large, and it decreases with increasing distance ($x$) from the leading edge.

Dividing both sides of Eq. (4.90) by the dynamic pressure of the free-stream, $\rho u_\infty^2/2$, we obtain

$$C_{f_x} = \frac{\tau_w}{\rho u_\infty^2/2} = 0.332\,\mu\frac{u_\infty}{x}(\text{Re}_x)^{1/2}\frac{2}{\rho u_\infty^2}$$

$$= \frac{\mathbf{0.664}}{(\mathbf{Re_x})^{\mathbf{1/2}}} \tag{4.91}$$

where $C_{f_x}$ is the dimensionless local drag or friction coefficient. Figure 4.20 shows the variation of $C_{f_x}$ with $\text{Re}_x$. The average friction coefficient is obtained by integrating Eq. (4.91) over the whole length of the plate, or

$$\bar{\mathbf{C}}_\mathbf{f} = \frac{\mathbf{1}}{\mathbf{L}}\int_\mathbf{o}^\mathbf{L}\mathbf{C}_{\mathbf{f_x}}\,\mathbf{dx} = \mathbf{2}(\mathbf{C}_{\mathbf{f_x}})_{\mathbf{x=L}} = \frac{\mathbf{1.328}}{(\mathbf{Re_L})^{\mathbf{1/2}}} \tag{4.92}$$

where $\quad \text{Re}_L = \dfrac{u_\infty L}{\nu}.$

### 4.6.2 Convection Heat Transfer

The velocities $u$ and $v$ in the energy conservation equation

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}$$

have the same values at any point ($x$, $y$) as in the momentum equation, Eq. (4.79b). For the case of the flat plate, Pohlhausen [1] used the velocities calculated previously by Blasius to obtain the solution for the heat transfer problem. If the momentum and energy equations are compared, we find them to be similar if $\nu = \alpha$ and the surface temperature $T_w$ is constant. A solution for $u\,(x, y)$ is also a solution for $T(x, y)$ which can easily be checked if the symbol $T$ is replaced by the symbol $u$ while the boundary conditions for both are identical. If we define a dimensionless temperature

$$\theta(\eta) = \frac{T(\eta) - T_w}{T_\infty - T_w} \tag{4.93}$$

then $\quad \theta = 0$ and $u/u_\infty = 0$ at $y = 0$

$\qquad \theta = 1$ and $u/u_\infty = 1$ at $y \to \infty$

where $T_w$ is the wall or surface temperature and $T_\infty$ is the free–stream temperature. The condition $\nu = \alpha$ corresponds to a Prandtl number ($= \nu/\alpha$) of unity. For Pr = 1, the velocity distribution is identical to the temperature distribution. The transfer of momentum is analogous to the transfer of heat when Pr = 1. For gases Prandtl number ranges from 0.6 to 1.0. The analogy is, therefore, satisfactory for gases. Liquids, however, have Prandtl numbers considerably different from unity, and the preceding analysis is not applicable directly to liquids.

Pohlhausen's results can be modified empirically to include fluids having different values of Prandtl number. In Fig. 4.21 the temperature profiles computed theoretically have been plotted for various

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Prandtl numbers. We define a thermal boundary layer thickness $\delta_t$ as the distance from the wall at which the temperature difference between the wall and the fluid $(T - T_w)$ reaches 99% of the free-stream value $(T_\infty - T_w)$. From the temperature profiles, we observe that for Pr < 1, $\delta_t > \delta$, and for fluids having Pr > 1, $\delta_t < \delta$. According to Pohlhausen,

$$\delta/\delta_t = \mathrm{Pr}^{1/3} \tag{4.94}$$

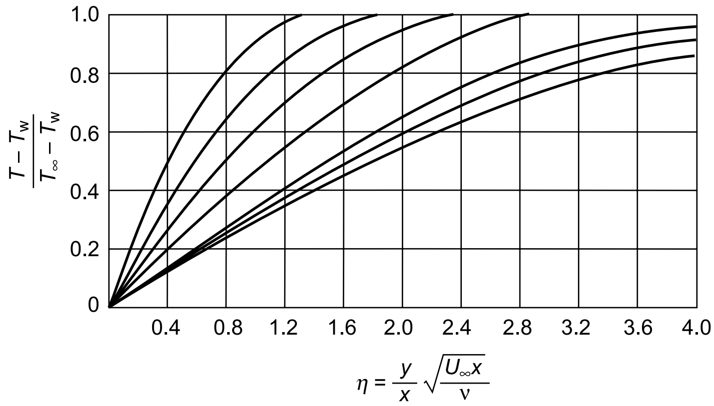


**Fig. 4.21** *Temperature distributions in a fluid flowing over a heated plate for various Prandtl numbers*

The curves of Fig. 4.21 are replotted in Fig. 4.22 using the correction factor $\mathrm{Pr}^{1/3}$ in the abscissa, which is now $(y/x)\,(\mathrm{Re}_x)^{1/2}\,\mathrm{Pr}^{1/3}$. The dimensionless temperature gradient at the surface ($y = 0$) is

$$\frac{\partial\,[(T - T_w)/(T_\infty - T_w)]}{\partial\left[(y/x)\,(\mathrm{Re}_x)^{1/2}\ \mathrm{Pr}^{1/3}\right]} = 0.332$$

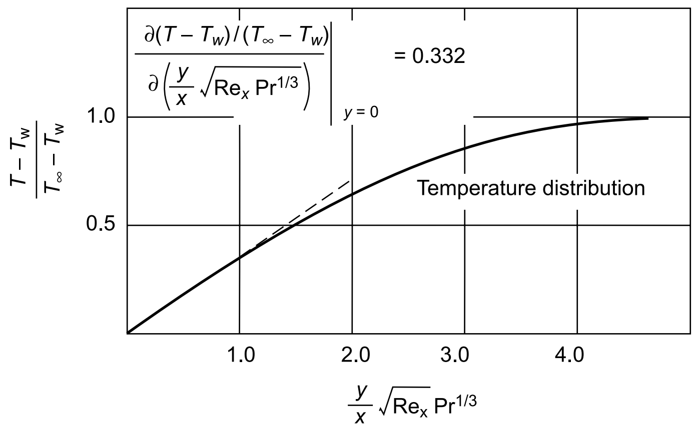


**Fig. 4.22** *Temperature distribution for laminar flow over a heated plate at uniform temperature*

Therefore, at any specified value of $x$,

$$\left(\frac{\partial T}{\partial t}\right)_{y=0} = 0.332\,\frac{\mathrm{Re}_x^{1/2}\ \mathrm{Pr}^{1/3}}{x}\,(T_\infty - T_w) \tag{4.95}$$

The local rate of convection heat transfer per unit area becomes

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$$q_c'' = -k\left(\frac{\partial T}{\partial y}\right)_{y=0}$$

$$= -k\,0.332\,\frac{\text{Re}_x^{1/2}\,\text{Pr}^{1/3}}{x}(T_\infty - T_w) \quad (4.96)$$

The total rate of heat transfer from a plate of width $b$ and length $L$ is

$$Q = \int_{x=0}^{L} q_c''\,\text{d}x = 0.664\,k\,\text{Re}_L^{1/2}\,\text{Pr}^{1/3}\,b\,(T_w - T_\infty) \quad (4.97)$$

The local heat transfer coefficient obtained from Eq. (4.96),

$$h_{cx} = \frac{q_c''}{T_w - T_\infty} = 0.332\,\frac{k}{x}\,\text{Re}_x^{1/2}\,\text{Pr}^{1/3} \quad (4.98)$$

The local Nusselt number is

$$\mathbf{Nu_x = \frac{h_{cx}\,x}{k} = 0.332\,Re_x^{1/2}\,Pr^{1/3}} \quad (4.99)$$

The average Nusselt number over the entire plate of length $L$ is

$$\text{Nu}_L = \int_o^L \text{Nu}_x\,\text{d}x = 2\,(\text{Nu}_x)_{x=L}$$

$$= \mathbf{0.664\,Re_L^{1/2}\,Pr^{1/3}} \quad (4.100)$$

and the average heat transfer coefficient

$$\bar{h}_c = 2\,(h_{cx})_{x=L} = 0.664\,\frac{k}{L}\,\text{Re}_L^{1/2}\,\text{Pr}^{1/3} \quad (4.101)$$

The physical properties in Eqs. (4.95)–(4.101) vary with temperature. Experimental data are found to agree satisfactorily with the results predicted analytically using the above equations if the properties are evaluated at a mean temperature $T^*$ given by $T^* = (T_w + T_\infty)/2$, often called the *film temperature*.

## 4.7 APPROXIMATE INTEGRAL BOUNDARY LAYER ANALYSIS

We had earlier developed the exact mathematical solution of the differential equations describing the laminar flow of a fluid over a flat surface in deriving the boundary layer thickness and the heat transfer coefficient. To circumvent the problems involved in solving the partial differential equations of the boundary layer. Theodore von Karman suggested the approximate integral method in which he considered a control volume that extends from the wall to beyond the boundary layer. Let us consider a control volume (CV) bounded by the two planes $AB$ and $CD$ normal to the surface, a distance d$x$ apart, and a parallel plane in the free stream at a distance $l$ from the surface (Fig. 4.23). Let us consider unit width of the plate in $z$-direction.

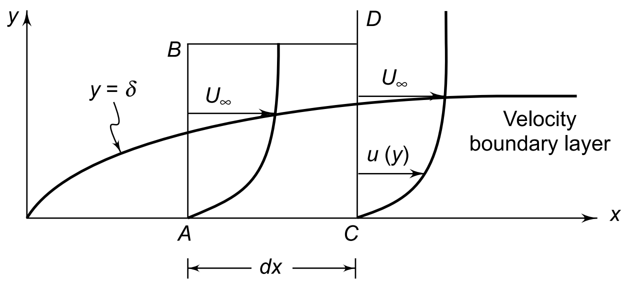


**Fig. 4.23** *Control volume for integral momentum conservation*

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Momentum flow across face *AB* into the CV in Fig. 4.23 is

$$= \int_0^1 \rho u^2 \, dy$$

Similarly, momentum flow across face *CD* is

$$= \int_0^1 \rho u^2 dy + \frac{d}{dx}\left(\int_0^1 \rho u^2 \, dy\right) dx$$

Fluid entering across *BD* at the rate

$$= \frac{d}{dx}\left(\int_0^1 \rho u \, dy\right) dx$$

This quantity is the difference between the rate of flow leaving across *CD* and that entering across *AB*. Since the fluid entering across *BD* has a velocity component in the *x*-direction equal to $u_\infty$, the flow of *x*-momentum across the upper face into the CV is

$$u_\infty \frac{d}{dx}\left(\int_0^1 \rho u \, dy\right) dx$$

Net *x*-momentum transfer

$$I = \text{outflow} - \text{inflow}$$

$$= \int_0^1 \rho u^2 \, dy + \frac{d}{dx}\left(\int_0^1 \rho u^2 \, dy\right) dx - \int_0^1 \rho u^2 \, dy$$

$$- u_\infty \left(\frac{d}{dx} \int_0^1 \rho u \, dy\right) dx$$

$$= -\frac{d}{dx}\left[\int_0^1 \rho u \, (u_\infty - u) dy\right] dx$$

For $y \geq \delta$, $u = u_\infty$ and the integrand *I* will be zero. We have to consider the integrand only within the limits from $y = 0$ to $y = \delta$.

There will be no shear across face *BD* outside the boundary layer where d*u*/d*y* is zero. A shear force $\tau_w$ acts at the fluid–solid interface, and there will be pressure forces acting on faces *AB* and *CD*.

Net forces acting on the CV are

$$p\delta - \left(p + \frac{dp}{dx} dx\right)\delta - \tau_w \, dx = -\delta \frac{dp}{dx} dx - \tau_w \, dx$$

By Newton's second law of motion

$$-\delta \frac{dp}{dx} dx - \tau_w \, dx = -\frac{d}{dx}\left[\int_o^\delta \rho u \, (u_\infty - u) \, dy\right] dx$$

For flow over a flat plate the pressure gradient in the *x*-direction, d*p*/d*x* can be neglected. Therefore,

$$\frac{d}{dx}\int_o^\delta \rho u (u_\infty - u) dy = \tau_w \tag{4.102}$$

The above equation is often called *von Kàrmàn's momentum integral equation*.

(I) Assuming a four-term polynomial for the velocity distribution [11]

$$u(y) = a + by + cy^2 + dy^3 \tag{4.103}$$

where the constants are evaluated from the boundary conditions:

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at $\quad y = 0,\ u = 0$ and so $a = 0$

$$u = v = 0 \text{ and } \frac{\partial^2 u}{\partial y^2} = 0$$

$$y = \delta,\ u = u_\infty \text{ and } \frac{\partial u}{\partial y} = 0$$

From these conditions we find

$$a = 0, b = \frac{3}{2}\frac{u_\infty}{\delta},\ c = 0,\ d = -\frac{u_\infty}{2\delta^3}$$

Substituting in Eq. (4.103)

$$u = \frac{3}{2}\frac{u_\infty}{\delta} y - \frac{u_\infty}{2}\frac{y^3}{\delta^3}$$

$$\frac{u}{u_\infty} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3 \tag{4.104}$$

Substituting Eq. (4.104) for the velocity distribution in the integral momentum equation, Eq. (4.102),

$$\frac{d}{dx}\int_o^\delta (\rho u\, u_\infty - \rho u^2)\, dy = \tau_w$$

$$\text{L.H.S.} = \frac{d}{dx}\int_o^\delta \left[\rho u_\infty^2\left(\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\frac{y^3}{\delta^3}\right) - u_\infty^2\,\rho\left(\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\frac{y^3}{\delta^3}\right)^2\right] dy$$

$$= \frac{d}{dx}\int_o^\delta \left[\rho u_\infty^2\left(\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\frac{y^3}{\delta^3}\right) - \rho u_\infty^2\left(\frac{9}{4}\frac{y^2}{\delta^2} - \frac{3}{2}\frac{y^4}{\delta^4} + \frac{1}{4}\frac{y^6}{\delta^6}\right)\right] dy$$

$$= \frac{d}{dx}\left[\rho u_\infty^2\left\{\left(\frac{3}{2\delta}\frac{\delta^2}{2} - \frac{1}{2\delta^3}\frac{\delta^4}{4}\right) - \frac{9}{4\delta^2}\frac{\delta^3}{3} + \frac{3}{2\delta^4}\frac{\delta^5}{5} - \frac{1}{4\delta^6}\frac{\delta^7}{7}\right\}\right] dy$$

$$= \frac{d}{dx}\left(\rho u_\infty^2 \frac{39}{280}\delta\right)$$

Again, $$\tau_w = \left(\frac{du}{dy}\right)_{y=0} = \mu \frac{3}{2}\frac{u_\infty}{\delta} \tag{4.105}$$

Therefore, $$\frac{d}{dx}\left(\rho u_\infty^2 \frac{39}{280}\delta\right) = \mu\frac{3}{2}\frac{u_\infty}{\delta}$$

$$\frac{39}{280}\rho u_\infty^2 \frac{d\delta}{dx} = \frac{3}{2}\mu\frac{u_\infty}{\delta} \tag{4.105a}$$

$$\delta = \left(\frac{280}{13}\frac{\nu x}{u_\infty}\right)^{1/2} = 4.64\left(\frac{\nu x}{u_\infty}\right)^{1/2}$$

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When $x = 0$, $\delta = 0$ and $\delta \propto (x)^{1/2}$

Also,
$$\frac{\boldsymbol{\delta}}{\mathbf{x}} = \frac{\mathbf{4.64}}{(\mathbf{Re_x})^{\mathbf{1/2}}} \tag{4.106}$$

where local Reynolds number $\mathrm{Re}_x = \dfrac{u_\infty x}{\nu}$.

Equation (4.106) gives a value of $\delta$, only 8% below that of the exact analysis, given by Eq. (4.89) where

$$\frac{\delta}{x} = \frac{5}{(\mathrm{Re}_x)^{1/2}}$$

The results of approximate analysis are thus satisfactory in practice.

Substituting for $\delta$ from Eq. (4.106) in Eq. (4.105),

$$\tau_w = \mu \frac{3}{2} \frac{u_\infty (\mathrm{Re}_x)^{1/2}}{4.64x}$$

Dividing both sides by $\dfrac{1}{2} \rho u_\infty^2$

$$\frac{\tau}{1/2\rho u_\infty^2} = C_{f_x} = \frac{3u_\infty}{9.28} \frac{\mu (\mathrm{Re_x})^{1/2} \times 2}{x\rho u_\infty^2}$$

$$\mathbf{C_{f_x} = 0.647 \frac{v}{u_\infty x} (Re_x)1/2 = \frac{0.647}{(Re_x)^{1/2}}} \tag{4.107}$$

The exact analysis gave us (Eq. 4.91)

$$\mathbf{C_{f_x} = \frac{0.664}{(Re_x)^{1/2}}}$$

(II) If the velocity profile in laminar boundary layer over a flat plate is assumed to be a second order polynomial $u = a + by + cy^2$, substituting the boundary conditions (i) $y = 0$, $u = 0$, (ii) $y = \delta$, $u = u_\infty$, $\dfrac{du}{dy} = 0$, velocity distribution becomes

$$\frac{\mathbf{u}}{\mathbf{u}_\infty} = \mathbf{2}\left(\frac{\mathbf{y}}{\boldsymbol{\delta}}\right) - \left(\frac{\mathbf{y}}{\boldsymbol{\delta}}\right)^{\mathbf{2}} \tag{4.108}$$

and the boundary layer thickness is

$$\frac{\delta}{x} = \frac{5.48}{\sqrt{\mathrm{Re}_x}} \tag{4.109}$$

and the mean skin friction coefficient is

$$\bar{c}_f = \frac{1.46}{\sqrt{\mathrm{Re}_L}} \tag{4.110}$$

(III) Similarly, if the velocity profile is assumed to be

$$\frac{u}{u_\infty} = \sin \frac{\pi y}{2\delta} \tag{4.111}$$

the corresponding parameters can be derived to be

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$$\frac{\delta}{x} = \frac{4.795}{\sqrt{\mathrm{Re}_x}} \tag{4.112}$$

and
$$\bar{c}_f = \frac{1.31}{\sqrt{\mathrm{Re}_L}} \tag{4.113}$$

The results for the boundary layer thickness and average skin friction coefficient yielded by different velocity profiles are shown in Table 4.3.

**Table 4.3** *Boundary Layer Parameters for Different Velocity Profiles*

| *S. No.* | *Velocity Profile* | *Boundary conditions* | | *d* | $\bar{c}_f$ |
|---|---|---|---|---|---|
| | | *At y = 0* | *At y = d* | | |
| 1. | $\frac{u}{u_\infty} = \frac{y}{\delta}$ | $u = 0$ | $u = u_\infty$ | $\frac{3.46\,x}{\sqrt{\mathrm{Re}_x}}$ | $\frac{1.155}{\sqrt{\mathrm{Re}_L}}$ |
| 2. | $\frac{u}{u_\infty} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$ | $u = 0$ | $u = u_\infty$<br>$\frac{\partial u}{\partial y} = 0$ | $\frac{5.48\,x}{\sqrt{\mathrm{Re}_x}}$ | $\frac{1.46}{\sqrt{\mathrm{Re}_L}}$ |
| 3. | $\frac{u}{u_\infty} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$ | $u = 0$<br>$\frac{\partial^2 u}{\partial y^2} = 0$ | $u = u_\infty$<br>$\frac{\partial u}{\partial y} = 0$ | $\frac{4.64\,x}{\sqrt{\mathrm{Re}_x}}$ | $\frac{1.292}{\sqrt{\mathrm{Re}_L}}$ |
| 4. | $\frac{u}{u_\infty} = \sin\left(\frac{\pi}{2} \cdot \frac{y}{\delta}\right)$ | $u = 0$ | $u = u_\infty$ | $\frac{4.795\,x}{\sqrt{\mathrm{Re}_x}}$ | $\frac{1.31}{\sqrt{\mathrm{Re}_L}}$ |
| 5. | Blasius exact solution | | | $\frac{5\,x}{\sqrt{\mathrm{Re}_x}}$ | $\frac{1.328}{\sqrt{\mathrm{Re}_L}}$ |

The integral energy equation can be derived in a similar fashion. A CV extending beyond the limits of both the temperature and the velocity boundary layers is to be used in the derivation (Fig. 4.24).

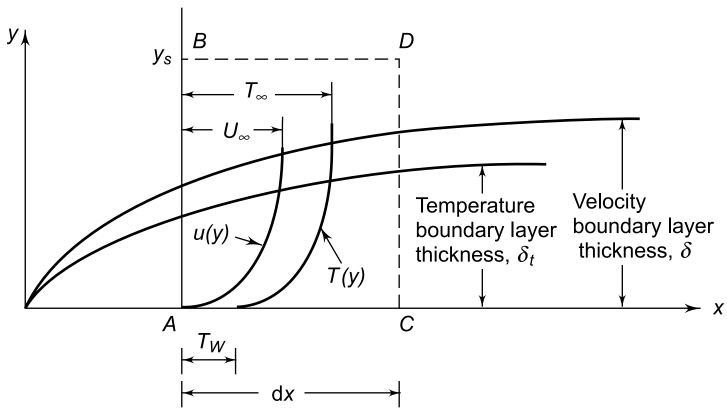


**Fig. 4.24** *Control volume for integral energy conservation analysis*

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Energy entering the CV across face $AB$

$$= \int_o^1 c_p \rho u T \, dy$$

Energy leaving the CV across face $CD$

$$= \int_o^1 c_p \rho u T \, dy + \frac{d}{dx}\left(\int_o^1 c_p \rho u T \, dy\right) dx$$

Energy carried into the CV across the upper face

$$= c_p T_\infty \frac{d}{dx}\left(\int_o^1 \rho u \, dy\right) dx$$

Heat conducted across the wall at the interface

$$= -k \, dx \left(\frac{\partial T}{\partial y}\right)_{y=0}$$

Making an energy balance

$$\int_o^1 c_p \rho u T \, dy - k \, dx \left(\frac{\partial T}{\partial y}\right)_{y=0} - \int_o^1 c_p \rho u T \, dy$$

$$- \frac{d}{dx}\left(\int_o^1 c_p \rho T \, dy\right) dx + c_p T_\infty \frac{d}{dx}\left(\int_o^1 \rho u \, dy\right) dx = 0$$

At $l \geq \delta_t$, $T = T_\infty$, and integration need be taken up to $y = \delta_t$.

$$\frac{d}{dx}\int_o^{\delta_t} u(T_\infty - T) \, dy = \frac{k}{\rho c_p}\left(\frac{\partial T}{\partial y}\right)_{y=0}$$

or
$$\frac{d}{dx}\int_o^{\delta_t} u(T_\infty - T) dy = \alpha \left(\frac{\partial T}{\partial y}\right)_{y=0} \tag{4.114}$$

This is known as the *integral energy equation* of the laminar boundary layer at low speed. Let us assume the temperature distribution in the thermal boundary layer as

$$T(y) = e + fy + gy^2 + hy^3 \tag{4.115}$$

The boundary conditions are:

at $\quad y = 0,\ T = T_w,\ \dfrac{\partial^2 T}{\partial y^2} = 0 \qquad$ (since $u = v = 0$)

at $\quad y = \delta_t,\ T = T_\infty$ and $\dfrac{dT}{dy} = 0$

From these conditions, we get

$$e = T_w,\ f = \frac{3}{2}\frac{T_\infty - T_w}{\delta_t},\ g = 0$$

$$h = \frac{T_\infty - T_w}{2\delta_t^3}$$

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$$T = T_w + \frac{3}{2}\frac{T_\infty - T_w}{\delta_t}y + \frac{T_\infty - T_w}{2\delta_t^3}y^3$$

or
$$\frac{\mathbf{T - T_w}}{\mathbf{T_\infty - T_w}} = \frac{\mathbf{3}}{\mathbf{2}}\frac{\mathbf{y}}{\boldsymbol{\delta}_\mathbf{t}} - \frac{\mathbf{1}}{\mathbf{2}}\left(\frac{\mathbf{y}}{\boldsymbol{\delta}_\mathbf{t}}\right)^{\mathbf{3}} \tag{4.116}$$

This is the temperature distribution in the thermal boundary layer. Substituting in energy equation, Eq. (4.114),

$$\int_o^{\delta_t}(T_\infty - T)u\,dy = \int_o^{\delta_t}[(T_\infty - T) - (T - T_w)]\,u\,dy$$

$$= (T_\infty - T_w)u_\infty\int_o^{\delta_t}\left[1 - \frac{3}{2}\frac{y}{\delta_t} + \frac{1}{2}\left(\frac{y}{\delta_t}\right)^3\right]\left[\frac{3y}{2\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right]dy$$

$$= (T_\infty - T_w)u_\infty\int_o^{\delta_t}\left[\left(\frac{3}{2\delta}\right)y - \left(\frac{9}{4\delta\,\delta_t}\right)y^2 + \left(\frac{3}{4\delta\,\delta_t^3}\right)y^4\right.$$

$$\left. - \left(\frac{1}{2\delta^3}\right)y^3 + \left(\frac{3}{4\delta_t\,\delta^3}\right)y^4 - \left(\frac{1}{4\delta_t^3\,\delta^3}\right)y^3\right]dy$$

$$= (T_\infty - T_w)\,u_\infty\left[\frac{3}{2\delta}\frac{\delta_t^2}{2} - \frac{3}{4}\frac{\delta_t^2}{\delta} + \frac{3}{4}\frac{\delta_t^2}{\delta} - \frac{1}{8}\frac{\delta_t^4}{\delta^3} + \frac{3}{20}\frac{\delta_t^4}{\delta^3} - \frac{1}{28}\frac{\delta_t^4}{\delta^3}\right]$$

If we let $\zeta = \delta_t/\delta$ the above expression can be written as

$$(T_\infty - T_w)u_\infty\,\delta\left(\frac{3}{20}\zeta^2 - \frac{3}{280}\zeta^4\right)$$

For fluids having $\Pr \geq 1$, $\zeta \leq 1$, the second term in the parentheses can be neglected compared to the first. Substituting in the approximate integral equation, Eq. (4.114),

$$\frac{d}{dx}[(T_\infty - T_w)\,u_\infty\,\delta\,\frac{2}{20}\zeta^2 = \alpha\left(\frac{\partial T}{\partial y}\right)_{y=0}$$

$$= \frac{3}{2}\alpha(T_\infty - T_w)\frac{1}{\delta_t}$$

$$= \frac{3}{2}\frac{\alpha\,(T_\infty - T_w)}{\zeta\delta}$$

$$\frac{1}{10}u_\infty\,\zeta^3\delta\,\frac{\partial\delta}{\partial x} = \alpha$$

From Eq. (4.105a),

$$\delta\frac{\partial\delta}{\partial x} = \frac{140}{13}\frac{\mu}{\rho}\frac{1}{u_\infty};\frac{140}{13}\frac{\mu}{\rho u_\infty} = \frac{10\alpha}{u_\infty}\frac{1}{\zeta^3}$$

$$\zeta^3 = \frac{13}{14}\frac{1}{\Pr}\delta_t = 0.976\,\delta\,\Pr^{-1/3} \tag{4.117}$$

If we compare with Eq. (4.94), $\delta/\delta_t = \Pr^{1/3}$ we find that except for the numerical constant (0.976 instead of 1.0), the above result is in agreement with the exact calculation of Pohlhausen.

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The rate of heat transfer per unit area is

$$q_c'' = -k\left(\frac{\partial T}{\partial y}\right)_{y=0} = -\frac{3}{2}\frac{k}{\delta_t}(T_\infty - T_w)$$

Substituting Eqs. (4.106) and (4.117) for $\delta$ and $\delta_t$,

$$q_c'' = \frac{3}{2}\,k\,\frac{(T_\infty - T_w)}{0.976\,\delta\,\mathrm{Pr}^{-1/3}}$$

$$= \frac{3}{2}\,\frac{k\,(T_\infty - T_w)\mathrm{Pr}^{1/3}\,\mathrm{Re}_x^{1/2}}{0.976 \times 4.64x}$$

$$= 0.331\,\frac{k}{x}\,\mathrm{Re}_x^{1/2}\,\mathrm{Pr}^{1/3}\,(T_w - T_\infty) \tag{4.118}$$

The local heat transfer coefficient

$$h_c = \frac{q_c''}{T_w - T_\infty} = 0.331\frac{k}{x}\mathrm{Re}_x^{1/2}\,\mathrm{Pr}^{1/3} \tag{4.119}$$

The local Nusselt number

$$\mathbf{Nu_x = \frac{h_c x}{k} = 0.331\,Re_x^{1/2}\,Pr^{1/3}} \tag{4.120}$$

This result is in excellent agreement with Eq. (4.99), the result of an exact analysis, which gives $\mathrm{Nu}_x = 0.332\,\mathrm{Re}_x^{1/2}\,\mathrm{Pr}^{1/3}$.

## 4.8 TURBULENT FLOW OVER A FLAT PLATE: ANALOGY BETWEEN MOMENTUM AND HEAT TRANSFER

The flow in the boundary layer is more often turbulent, rather than laminar. In laminar flow where the streamlines slide over one another, momentum and heat transfer take place by molecular diffusion. In turbulent flow, the transport mechanism is aided by innumerable eddies. Irregular velocity fluctuations are superimposed upon the motion of the main stream, and these fluctuations are primarily responsible for the transfer of heat as well as momentum. The rates of momentum and heat transfer in turbulent flow and the associated friction and heat transfer coefficients are many times larger than in laminar flow because of better mixing in which groups of particles collide with one another at random, establish cross-flow on a macroscopic scale and effectively mix the fluid.

Instantaneous streamlines in turbulent flows are highly torn and uneven, and it is difficult to trace the path of individual fluid elements. But if the flow at a point is averaged over a period of time, large compared to the period of a single fluctuation, the time-mean properties and the velocity of the fluid are constant, if the average flow remains steady. It is, therefore, possible to describe each fluid property and the velocity in turbulent flow in terms of a mean value which does not vary with time and a fluctuating component which is a function of time.

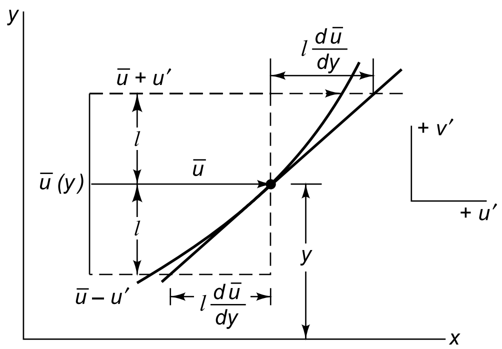


**Fig. 4.25** *Mixing length for momentum transfer in turbulent flow*

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Let us consider a two-dimensional flow in which the mean velocity is parallel to the $x$-direction (Fig. 4.25). The instantaneous velocity components $u$ and $v$ can then be expressed in the form

$$u = \bar{u} + u', \; v = v'$$
$$p = \bar{p} + p', \; T = \bar{T} + T' \text{ and so on} \tag{4.121}$$

where the bar over a symbol denotes the time-mean value and the prime denotes the instantaneous deviation from the time-mean value. According to the model used to describe the flow,

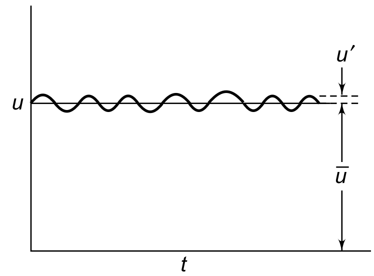


**Fig. 4.26** *Time variation of instantaneous velocity in turbulent flow*

$$\bar{u} = \frac{1}{t^*}\int_0^{t^*} u \, dt \tag{4.122}$$

where $t^*$ is a time interval that is large compared to the period of the fluctuations. The time variation of $u$ and $u'$ is qualitatively shown in Fig. 4.26. From Eq. (4.122) and Fig. 4.26 it is apparent that the time average of $u'$ is zero (i.e., $\bar{u}' = 0$). Similarly, $\bar{v}' = 0$ and $\rho\bar{v}' = 0$.

The fluctuating velocity components continuously transport mass and so momentum across a plane normal to the $y$-direction.

Instantaneous rate of transfer of $x$-momentum in $y$-direction per unit area

$$-(\rho v')(\bar{u} + u')$$

where the minus sign has a special significance which will be discussed later.

Time-average of the instantaneous rate of $x$-momentum transfer per unit area

$$\tau_t = -\frac{1}{t^*}\int_0^{t^*} (\rho v')(\bar{u} + u')\, dt \tag{4.123}$$

This is also called "*apparent turbulent shear stress*" or "*Reynolds stress*", $\tau_t$.

Breaking the term in Eq. (4.123) into parts

$$\tau_1 = -\frac{1}{t^*}\int_0^{t^*} (\rho v')\bar{u}\, dt - \frac{1}{t^*}\int_0^{t^*} (\rho v)' u'\, dt$$
$$= -\overline{(\rho v)'}\,\bar{u} - \rho\overline{u' v'}$$

Since $\bar{u}$ is constant and the time-average of $(\rho v)'$ is zero, the first term of the above equation will be zero. Therefore,

$$\boldsymbol{\tau_t = -\rho\, \overline{u'v'}} \tag{4.124}$$

where $\overline{u'v'}$ is the time average of the product of the fluctuating components $u'$ and $v'$, which is not zero, but a negative quantity. If $v'$ is positive i.e., the fluid particles with a certain velocity $\bar{u}$ travel upward to a $y$-plane where the velocity $\bar{u}$ is more (Fig. 4.25), these will tend to slow down the particles in that plane giving rise to a negative velocity component $u'$. And if $v'$ is negative, it will tend to accelerate the flow, giving rise to a positive $u'$. So a positive $v'$ is associated with a negative $u'$ and vice versa, so that the time-average of the product $u'$ $v'$ is not zero, but a negative quantity. The turbulent shear stress as defined by Eq. (4.124) is thus positive and has the same sign as the corresponding laminar shearing stress

$$\tau_1 = \mu\frac{d\bar{u}}{dy} = \rho v\frac{du}{dy}$$

It may be noted that the laminar shearing stress $\tau_1$ is a true stress, whereas the apparent turbulent shearing stress or Reynolds stress $\tau_t$ is a concept introduced to account for the effects of momentum transfer due to turbulent fluctuations.

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We can express the total shear stress in turbulent flow as

$$\tau = \frac{\textbf{viscous force}}{\textbf{area}} + \textbf{turbulent momentum flux}$$

To relate the turbulent momentum flux with the time-average velocity gradient $d\bar{u}/dy$, Prandtl postulated that the fluctuations of macroscopic fluid particles in turbulent flow are, on the average, similar to the motion of molecules in a gas i.e., they travel a distance $l$ perpendicular to $\bar{u}$ before coming to rest in another $y$-plane (Fig. 4.25). This distance $l$ is known as *Prandtl's mixing length* and qualitatively corresponds to the mean free path of a gas molecule. Prandtl further argued that the fluid particles retain their identity and physical properties during the cross motion and that the turbulent fluctuations arise because of the difference of the time-mean properties between $y$ planes distance $l$ apart. If a fluid particle travels from a layer at $y$ plane to a layer at $(y + 1)$ plane,

$$u' \simeq 1\frac{d\bar{u}}{dy} \tag{4.125}$$

The turbulent shearing stress is then

$$\tau_t = \rho\overline{u'v'} = -\rho\,\overline{v'1}\,\frac{d\bar{u}}{dy} = \rho\varepsilon_M\frac{d\bar{u}}{dy} \tag{4.126}$$

where $\varepsilon_M$ $(= -\overline{v'1})$ is called the *eddy viscosity* or the turbulent exchange coefficient for momentum. The eddy viscosity $\varepsilon_M$ is analogous to the kinematic viscosity $\nu$. But $\nu$ is a physical property, whereas $\varepsilon_M$ is not, and it depends on the dynamics of flow.

Total shearing stress,

$$\tau = \tau_1 + \tau_t = \rho\nu\frac{d\bar{u}}{dy} + \rho\varepsilon_M\frac{d\bar{u}}{dy}$$

$$= \rho\,(\nu + \varepsilon_M)\,\frac{d\bar{u}}{dy} \tag{4.127}$$

For turbulent flow,

$$\varepsilon_M >> \nu \text{ and } \tau = \rho\,\varepsilon_M\frac{d\bar{u}}{dy}$$

For laminar flow,

$$\varepsilon_M = 0 \text{ and } \tau = \rho\nu\frac{d\bar{u}}{dy}$$

For buffer layer (transition zone),

$$\tau = \rho(\nu + \varepsilon_M)\frac{d\bar{u}}{dy}$$

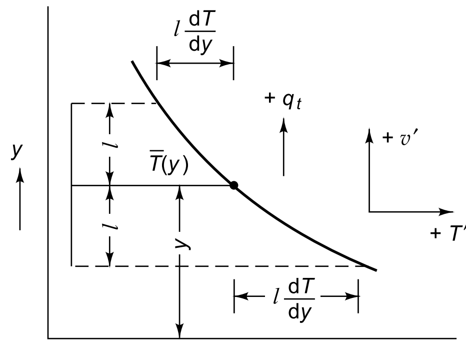


**Fig. 4.27** *Mixing length for energy transfer in turbulent flow*

The transfer of energy as heat in turbulent flow can be visualised in a similar fashion. Let us consider a two-dimensional time-mean temperature distribution (Fig. 4.27). The fluctuating velocity components transport fluid particles and the energy stored in them across a plane normal to the $y$-direction.

Instantaneous rate of energy transfer per unit area at any point in the $y$-direction.

$$= (\rho v)'\,c_p\,T$$

where $\quad T = \bar{T} + T'$

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The time average of the instantaneous rate of heat transfer per unit area is equal to turbulent rate of heat transfer $Q_t$.

$$\therefore \qquad \frac{Q_t}{A} = \frac{1}{t^*}\int_0^{t^*} (\rho v)' e_p(\overline{T} + T')\,\mathrm{d}t = \rho c_p\,\overline{v'T'} \tag{4.128}$$

Using Prandtl's concept of mixing length, we can relate the temperature fluctuation to the time-mean temperature gradient by the equation

$$T' \simeq 1\frac{\mathrm{d}\overline{T}}{\mathrm{d}y} \tag{4.129}$$

When a fluid particle migrates from the $y$ plane to the $(y \pm 1)$ plane, the resulting temperature fluctuation is caused by the difference between the time-mean temperatures of the two planes. Assuming that the transport mechanisms of temperature (or energy) and velocity (or momentum) are similar, the mixing lengths in Eqs (4.125) and (4.129) are equal. The product $\overline{v'\,T'}$ in Eq. (4.128) however, is positive, because a positive $v'$ is accompanied by a positive $T'$, and vice versa.

Combining Eqs (4.128) and (4.129), the turbulent rate of heat transfer per unit area

$$\frac{Q_t}{A} = -\rho c_p\,\overline{v'1}\,\frac{\mathrm{d}\overline{T}}{\mathrm{d}y} \tag{4.130}$$

where the minus sign is a consequence of the second law of thermodynamics, with heat dissipation from the system. To express the turbulent heat flux in a form analogous to the Fourier conduction equation, we define $\varepsilon_H$, a quantity called *eddy diffusivity of heat* or the turbulent exchange coefficient for temperature, by the equation $\varepsilon_H = \overline{v}'1$. Therefore,

$$\mathbf{\frac{Q_t}{A} = \rho c_p\,\varepsilon_H\,\frac{d\overline{T}}{dy}} \tag{4.131}$$

The total rate of heat transfer per unit area

$$\frac{Q_t}{A} = \text{(molecular conduction)/area + (Turbulent transfer through eddies)/area}$$

$$= -\alpha\rho c_p\frac{\mathrm{d}\overline{T}}{\mathrm{d}y} - \rho c_p\,\varepsilon_H\,\frac{\mathrm{d}T}{\mathrm{d}y}$$

or
$$\frac{Q_t}{A} = -\rho\,c_p\,(\alpha + \varepsilon_H)\,\frac{\mathrm{d}\overline{T}}{\mathrm{d}y} \tag{4.132}$$

where $\alpha = k/\rho c_p$. **The contribution to the heat transfer by molecular conduction is proportional to $\alpha$, and turbulent contribution is proportional to $\varepsilon_H$.**

For all fluids except liquid metals, $\varepsilon_H >> a$ in turbulent flow. For laminar flow,

$$\varepsilon_H = 0 \text{ and } Q/A = -\rho c_p\,\alpha\,\frac{\mathrm{d}\overline{T}}{\mathrm{d}y} = -k\,\frac{\mathrm{d}\overline{T}}{\mathrm{d}y}$$

In transition zone, $Q/A = -\rho c_p\,(\alpha + \varepsilon_H)\,\frac{\mathrm{d}\overline{T}}{\mathrm{d}y}$. Prandtl number was defined as the ratio of two transport properties, $\nu/\alpha$, those of momentum and energy. Similarly, the ratio of the turbulent eddy viscosity to the eddy diffusivity, $\varepsilon_M/\varepsilon_H$, is called the *turbulent Prandtl number*, $\mathrm{Pr}_t$. Therefore,

$$\mathbf{Pr_t = \frac{\varepsilon_M}{\varepsilon_H}} \tag{4.133}$$

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According to Prandtl's mixing length theory, since $\varepsilon_M = \varepsilon_H = \overline{v'l}$, $Pr_t$ is unity.

For $Pr_t = 1$, the turbulent heat flux can be related to the turbulent shear stress. By combining Eqs (4.126) and (4.131),

$$\frac{Q_t}{A\tau_t} = \frac{-\rho c_p\, \varepsilon_H\, \mathrm{d}\bar{T} / \mathrm{d}y}{\rho \varepsilon_M\, \mathrm{d}\bar{u} / \mathrm{d}y}$$

or

$$\frac{\mathbf{Q_t}}{\mathbf{A}} = -\bar{\tau}_t\, c_p \frac{\mathrm{d}\bar{T}}{\mathrm{d}\bar{u}} \tag{4.134}$$

This relationship was first derived in 1874 by the British physicist Osborne Reynolds and is called **Reynolds analogy**. It is a good approximation whenever the flow is turbulent and can be applied in turbulent boundary layers as well as to turbulent flow in pipes and ducts. This analogy, however, does not hold good in the viscous sublayer where the flow is laminar.

## 4.9 REYNOLDS' ANALOGY FOR TURBULENT FLOW OVER A FLAT PLATE

To derive a relation between heat transfer and friction coefficients in flow over a flat plate for a fluid having Pr = 1, we recall that the laminar shear stress

$$\tau = \mu \frac{\mathrm{d}u}{\mathrm{d}y}$$

and the rate of heat transfer per unit area across any plane normal to the $y$-direction is

$$q'' = -k \frac{\mathrm{d}T}{\mathrm{d}y}$$

Combining these two equations, we obtain

$$q'' = -\tau \frac{k}{\mu} \frac{\mathrm{d}T}{\mathrm{d}u} \tag{4.135}$$

In Eqs. (4.134) and (4.135), we observe that if $k/\mu = c_p$ or, $c_p\, \mu/k = 1$ i.e., Pr = 1, the same equation of heat flow holds good in the laminar and turbulent layers.

To determine the rate of heat transfer from a flat plate to a fluid with Pr = 1 flowing over it in turbulent flow, we replace $k/\mu$ by $c_p$ and separate the variables in Eq. (4.135). Assuming that $q''$ and $\tau$ are constant, we get

$$\frac{q''_w}{\tau_w c_p}\, \mathrm{d}u = -\mathrm{d}T \tag{4.136}$$

where the subscript $w$ indicates that $q''$ and $\tau$ are taken at the wall of the plate. Integrating Eq. (4.136) between the limits $u = 0$ when $T = T_w$ and $u = u_\infty$ when $T = T_\infty$ gives

$$\frac{q''_w}{\tau_w c_p}\, u_\infty = T_w - T_\infty \tag{4.137}$$

Now, by definition, the local heat transfer coefficient $h_{cx}$ and local friction coefficient $C_{f_x}$ are

$$h_{cx} = \frac{q''_w}{T_w - T_\infty} \text{ and } \tau_{wx} = C_{f_x} \frac{\rho u_\infty^2}{2}$$

$$\frac{h_{cx}\, 2u_\infty}{C_{f_x}\, \rho u_\infty^2\, c_p} = 1$$

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or, $$\frac{h_{cx}}{\rho c_p u_\infty} = \frac{C_{f_x}}{2} = St_x \tag{4.138}$$

where $St_x$ = the local Stanton number = $Nu_x/Re_x\,Pr = h_{cx}/\rho c_p u_\infty$.

Equation (4.138) is satisfactory for gases in which Pr = 1. Colburn [12] has shown that this equation can also be used for fluids having 0.6 < Pr < 50 if it is modified in accordance with

$$St_x\,Pr^{2/3} = \frac{C_{f_x}}{2} \tag{4.139}$$

where the subscript $x$ denotes the distance from the leading edge of the plate. This expression is referred to as the *Reynolds–Colburn analogy*, and **$St_x\,Pr^{2/3}$ is called Colburn's j-factor**.

To apply the analogy between heat transfer and momentum transfer, in practice it is necessary to know the friction coefficient $C_{f_x}$. For turbulent flow over a flat plate the empirical equation for the local friction coefficient

$$C_{f_x} = 0.0576\left(\frac{u_\infty x}{\nu}\right)^{-1/5} = \frac{0.0576}{(Re_x)^{0.2}} \tag{4.140}$$

is in good agreement with experimental results for Reynolds number varying between $5 \times 10^5$ and $5 \times 10^7$ [10].

If the turbulent boundary layer is assumed to start at the leading edge, the average friction coefficient over a plane surface of length $L$ is

$$\overline{C}_f = \frac{1}{L}\int_0^L C_{f_x}\,dx = 0.072\left(\frac{u_\infty L}{\nu}\right)^{-1/5} = \frac{0.072}{(Re_L)^{0.2}} \tag{4.141}$$

Again, from Eq. (4.139), for turbulent flow

$$St_x\,Pr^{2/3} = \frac{Nu_x}{Re_x\,Pr}Pr^{2/3} = \frac{C_{f_x}}{2} = \frac{0.0576}{2 \times (Re_x)^{0.2}}$$

$$= 0.0288\,Re_x^{0.2}$$

$$Nu_x = 0.0288\,Re_x^{0.8}\,Pr^{1/3} \tag{4.142}$$

It is the local Nusselt number at any value of $x$ larger than $x_c$, since between $x = 0$ to $x = x_c$ the flow is laminar.

The average heat transfer coefficient in turbulent flow over a plane surface of length $L$ is

$$h_m = \bar{h}_c = \frac{1}{L}\int_0^L h_{cx}\,dx$$

$$= \frac{1}{L}\int_0^L \frac{k}{x}0.0288\,Pr^{1/3}\left(\frac{u_\infty}{\nu}\right)^{0.8} x^{0.8}\,dx$$

On integrating and nondimensionalising

$$Nu_L = 0.036\,Re_L^{0.8}\,Pr^{1/3} \tag{4.143}$$

This is valid only when $L >> x_c$.

To consider the mixed boundary layer in a flat plate with laminar flow from $x = 0$ to $x = x_c$ and turbulent flow from $x = x_c$ to $x = L$ (Fig. 4.28).

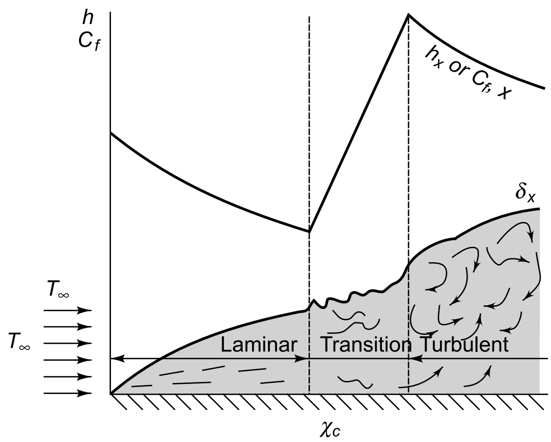


**Fig. 4.28** *Variation of local friction and heat transfer coefficients, $C_{f_x}$ and $h_x$, for flow over a flat plate*

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$$h_m = \frac{1}{L - x_c} \int_{x_c}^{L} 0.0288 \left(\frac{u_\infty}{\nu}\right)^{0.8} \frac{k}{x} x^{0.8}\, \mathrm{Pr}^{1/3}\, \mathrm{d}x$$

$$\frac{h_c\,(L - x_c)}{k} = 0.036\,(\mathrm{Re}_L^{0.8} - \mathrm{Re}_{x_c}^{0.8})\,\mathrm{Pr}^{1/3}$$

$$= 0.036\,(\mathrm{Re}_L^{0.8} - 36{,}239)\,\mathrm{Pr}^{1/3} \tag{4.144}$$

If Eq. (4.120) is used between $x = 0$ and $x = x_c$ and Eq. (4.142) between $x = x_c$ and $x = L$ for the integration of $h_{cx}$ it yields, with $\mathrm{Re}_{xc} = 500{,}000$.

$$\bar{h} = \frac{1}{L}\left(\int_0^{x_c} h_{\text{lam.}}\, \mathrm{d}x + \int_{x_c}^{L} h_{\text{turb.}}\, \mathrm{d}x\right)$$

Putting $$\mathrm{Nu}_{\text{lam.}} = 0.332 \left(\frac{u_\infty x}{\nu}\right)^{0.5} \mathrm{Pr}^{1/3} = \frac{h_{\text{lam.}} x}{k}$$

and $$\mathrm{Nu}_{\text{turb.}} = 0.0288 \left(\frac{u_\infty x}{\nu}\right)^{0.8} \mathrm{Pr}^{1/3} = \frac{h_{\text{turb.}} x}{k}$$

$$\bar{h} = \frac{k}{L}\left[0.332\left(\frac{u_\infty}{\nu}\right)^{0.5} \int_0^{x_c} x^{-1/2}\, \mathrm{d}x + 0.0288\left(\frac{u_\infty}{\nu}\right)^{0.8} \int_{x_c}^{L} x^{-0.2}\, \mathrm{d}x\right] \mathrm{Pr}^{1/3}$$

$$\overline{\mathrm{Nu}}_L = [0.664\, \mathrm{Re}_{xc}^{1/2} + 0.036\,(\mathrm{Re}_L^{0.8} - \mathrm{Re}_{x_c}^{0.8})]\, \mathrm{Pr}^{1/3}$$

or, $$\mathbf{Nu_L = (0.036\, Re_L^{0.8} - 835)\, Pr^{1/3}} \tag{4.145}$$

Adding the laminar friction drag between $x = 0$ and $x = x_c$ to the turbulent drag between $x = x_c$ and $x = L$ gives per unit width,

$$\bar{C}_f = \frac{(0.072\, \mathrm{Re}_L^{-1/5}\, L - 0.072\, \mathrm{Re}_{x_c}^{-1/5}\, x_c + 1.33\, \mathrm{Re}_{x_c}^{-1/2}\, x_c)}{L}$$

For a critical Reynolds number, $\mathrm{Re}_{xc} = 500{,}000$, this reduces to

$$\bar{C}_f = 0.072\left(\mathrm{Re}_L^{-1/5} - \frac{0.0464\, x_c}{L}\right)$$

$$= \frac{0.072}{\mathrm{Re}_L^{0.2}} - \frac{0.0464 \times 0.072 \times 500{,}000}{\mathrm{Re}_L}$$

$$= \mathbf{\frac{0.072}{Re_L^{0.2}} - \frac{1670}{Re_L}} \tag{4.146}$$

## 4.10 CONSTANT HEAT FLUX BOUNDARY CONDITION

The analysis given in Section 4.7 has considered the laminar heat transfer from an isothermal surface. In many practical problems the surface heat flux is essentially constant, and the objective is to find the distribution of the plate-surface temperature for given fluid-flow conditions. For the constant heat flux case it can be shown that the local Nusselt number is given by [13]

$$\mathrm{Nu}_x = 0.453\, \mathrm{Re}_x^{1/2}\, \mathrm{Pr}^{1/3} \tag{4.147}$$

which may be expressed in terms of wall heat flux and temperature difference as

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$$\mathrm{Nu}_x = \frac{q_w\, x}{k(T_w - T_\infty)} \tag{4.148}$$

The average temperature difference along the plate, for the constant heat flux condition, may be obtained from

$$\overline{T_w - T_\infty} = \frac{1}{L}\int_0^L (T_w - T_\infty)\,\mathrm{d}x = \frac{1}{L}\int_0^L \frac{q_w x}{k\mathrm{Nu}_x}\,\mathrm{d}x$$

$$= \frac{q_w\, L/k}{0.6795\ \mathrm{Re}_L^{1/2}\ \mathrm{Pr}^{1/3}} \tag{4.149}$$

$$\mathbf{q_w = \frac{3}{2} h_{x=L}(T_w - T_\infty)}$$

where $q_w$ is the wall heat flux in W/m$^2$.

***Other Relations:*** Equation (4.99) is applicable to fluids having Prandtl numbers between 0.6 and 50. It would not apply to fluids with very low Prandtl numbers like liquid metals or to high Prandtl number fluids like heavy oils or silicones. For a very wide range of Prandtl numbers, Churchill and Ozoe [30] have correlated a large amount of data to give the following relation for laminar flow on an isothermal plate:

$$\mathrm{Nu}_x = \frac{0.3387\ \mathrm{Re}_x^{1/2}\ \mathrm{Pr}^{1/3}}{\left[1 + \left(\dfrac{0.0468}{\mathrm{Pr}}\right)^{2/3}\right]^{1/4}} \quad \text{for } \mathrm{Re}_x\,\mathrm{Pr} > 100 \tag{4.150}$$

For the constant heat flux case, 0.3387 is changed to 0.4637 and 0.0468 is changed to 0.0207. The properties are evaluated at the film temperature.

## 4.11 BOUNDARY LAYER THICKNESS IN TURBULENT FLOW

In the turbulent boundary layer the shape of the velocity profile is much more curved than in the laminar layer [Fig. 4.29(a)]. The measured velocity profile agrees satisfactorily with the equation proposed by Prandtl:

$$u = u_\infty \left(\frac{y}{\delta}\right)^{1/7} \tag{4.151}$$

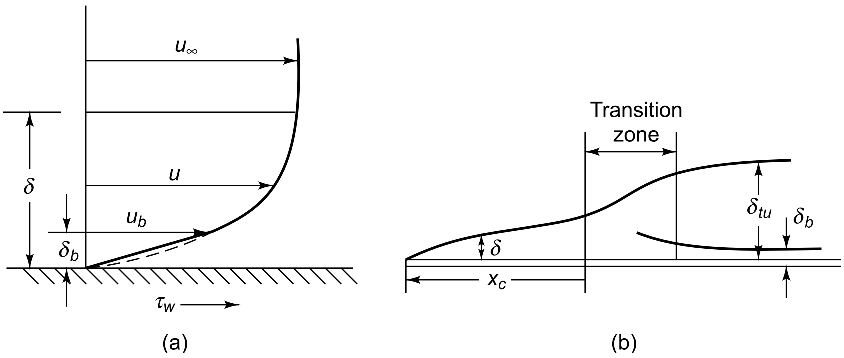


**Fig. 4.29** *Laminar sublayer ($\delta_b$) in a turbulent boundary layer ($\delta_{tu}$)*

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This equation does not hold good near the wall. The velocity gradient

$$\frac{du}{dy} = \frac{1}{7}\frac{u_\infty}{\delta^{1/7} y^{6/7}}$$

and at the wall where $y = 0$, $du/dy = \infty$. An infinite value of shear stress at the wall is impossible. In fact the turbulence always dies down near the wall, where a laminar sublayer exists and where the velocity increases linearly with the distance $y$ [Fig. 4.29(b)]. Outside this sublayer Eq. (4.151) holds true. For not too large Reynolds numbers and smooth surfaces, Blasius equation holds

$$\tau_w = 0.0228\ \rho u_\infty^2 \left(\frac{\nu}{u_\infty \delta}\right)^{1/4} \tag{4.152}$$

The momentum integral equation, Eq. (4.102),

$$\frac{d}{dx}\int_0^\delta \rho u (u_\infty - u) dy = \tau_w$$

Substituting

$$u = \frac{u_\infty}{\delta^{1/7}} y^{1/7}$$

Integrand,

$$I = \rho \int_0^\delta \left[u_\infty - u_\infty \left(\frac{y}{\delta}\right)^{1/7}\right] u_\infty \left(\frac{y}{\delta}\right)^{1/7} dy$$

$$= \rho u_\infty^2 \int_0^\delta \left[1 - \left(\frac{y}{\delta}\right)^{1/7}\right]\left(\frac{y}{\delta}\right)^{1/7} dy$$

$$= \rho u_\infty^2 \left[\int_0^\delta \left(\frac{y}{\delta}\right)^{1/7} dy - \int_0^\delta \left(\frac{y}{\delta}\right)^{2/7} dy\right]$$

$$= \rho u_\infty^2 \left(\frac{1}{\delta^{1/7}}\frac{\delta^{8/7}}{8/7} - \frac{1}{\delta^{2/7}}\frac{\delta^{9/7}}{9/7}\right)$$

$$= \rho u_\infty^2\, \delta \left(\frac{7}{8} - \frac{7}{9}\right) = \frac{7\rho u_\infty^2 \delta}{72}$$

Substituting,

$$\frac{d}{dx}\left(\frac{7\rho u_\infty^2 \delta}{72}\right) = \tau_w = 0.0228\ \rho u_\infty^2 \left(\frac{\nu}{u_\infty \delta}\right)^{1/4}$$

$$\frac{d\delta}{dx} = \frac{72 \times 0.0228}{7}\left(\frac{\nu}{u_\infty \delta}\right)^{1/4}$$

$$\int_0^\delta \delta^{1/4} d\delta = \int_0^x 0.235 \left(\frac{\nu}{u_\infty}\right)^{1/4} dx$$

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$$\frac{4}{5}\delta^{5/4} = 0.294\left(\frac{\nu}{u_\infty}\right)^{1/4} x$$

$$\delta = \left(0.294 \times \frac{5}{4}\right)^{4/5}\left(\frac{u}{u_\infty}\right)^{1/5}(x)^{4/5} = 0.376\left(\frac{\nu}{u_\infty x}\right)^{1/5} x$$

$$\frac{\boldsymbol{\delta}}{\mathbf{x}} = \frac{\mathbf{0.376}}{(\mathbf{Re_x})^{1/5}} \tag{4.153}$$

The displacement thickness

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{u_\infty}\right) dy = \int_0^\delta \left[1 - \left(\frac{y}{\delta}\right)^{1/7}\right] dy$$

$$= \int_0^\delta dy - \frac{1}{\delta^{1/7}} \int_0^\delta y^{1/7} dy$$

$$= \delta - \frac{\mathbf{1}}{\delta^{1/7}} \frac{\mathbf{7}}{\mathbf{8}} \delta^{8/7} = \frac{\delta}{\mathbf{8}} \tag{4.154}$$

If the laminar and the turbulent boundary layers are calculated for the critical distance $x_c$, it can be seen that the turbulent layer is thicker. In reality an instantaneous increase in boundary layer thickness is not possible. The transition from the laminar to the turbulent boundary layer takes place in a transition zone as is indicated in Fig. 4.30.

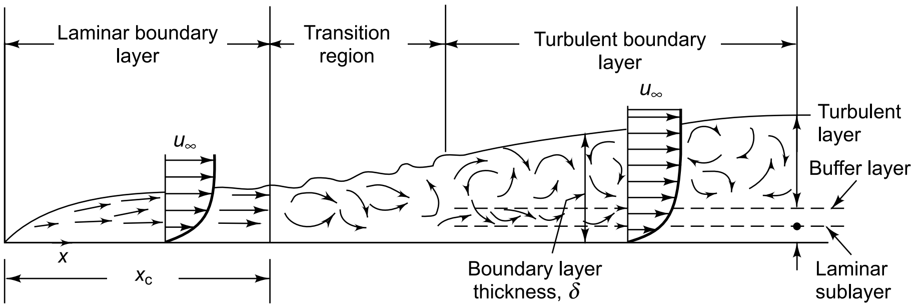


**Fig. 4.30** *Transition from laminar to turbulent boundary layer*

For heat transfer calculations the thickness $\delta_b$ of the laminar sublayer is needed. For this we must first calculate the velocity $u_b$ at the border between the turbulent layer and the laminar sublayer. The linear velocity increase in the sublayer is derived from the shear stress on the wall.

$$\tau_w = \mu \frac{du}{dy} = \mu \frac{u}{y}$$

Now, $$\tau_w = 0.0228\ \rho u_\infty^2 \left(\frac{\nu}{u_\infty \delta}\right)^{1/4} = \mu \frac{u}{y}$$

$\therefore$ $$u = 0.0228\ \rho u_\infty^2 \left(\frac{\nu}{u_\infty \delta}\right)^{1/4} \frac{y}{\mu}$$

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When $y = \delta_b$, $u = u_b$

$$\therefore \qquad u_b = 0.0228 \frac{\rho u_\infty^2}{\mu}\left(\frac{\nu}{u_\infty \delta}\right)^{1/4} \delta_b$$

$$= 0.0228 \frac{\rho u_\infty^2}{\mu}\left(\frac{\nu\delta^3}{u_\infty}\right)^{1/4} \frac{\delta_b}{\delta}$$

$$\frac{\delta_b}{\delta} = \frac{1}{0.0228}\frac{\nu}{u_\infty^2}\left(\frac{u_\infty}{\nu\delta^3}\right)^{1/4} u_b \quad u_b$$

$$= \frac{1}{0.0228}\frac{u_b}{u_\infty}\left(\frac{\nu^3}{u_\infty^3\,\delta^3}\right)^{1/4} \tag{4.155}$$

Now,
$$\frac{u}{u_\infty} = \left(\frac{y}{\delta}\right)^{1/7}$$

when $y = \delta_b$, $u = u_b$

$$\frac{u_b}{u_\infty} = \left(\frac{\delta_b}{\delta}\right)^{1/7}$$

Substituting in Eq. (4.154),

$$\frac{\delta_b}{\delta} = \frac{1}{0.0228}\frac{u_b}{u_\infty}\left(\frac{\nu}{u_\infty\delta}\right)^{3/4} = \left(\frac{u_b}{u_\infty}\right)^7$$

$$\frac{u_b}{u_\infty} = \left(\frac{1}{0.0228}\right)^{1/6}\left(\frac{\nu}{u_\infty\delta}\right)^{1/8}$$

or
$$\frac{u_b}{u_\infty} = \frac{1.869}{(\mathrm{Re}_\delta)^{1/8}} \tag{4.156}$$

Since,
$$\frac{\delta}{x} = \frac{0.376}{(\mathrm{Re}_x)^{1/5}}$$

$$\frac{u_b}{u_\infty} = 1.869\left(\frac{u_\infty\delta}{\nu}\right)^{-1/8} = 1.869\left(\frac{u_\infty}{\nu}\frac{0.376\,x}{(u_\infty x/\nu)^{1/5}}\right)^{-1/8}$$

$$= 2.11\left(\frac{u^{4/5}\,x^{4/5}}{\nu^{4/5}}\right)^{-1/8} = 2.11\left(\frac{u_\infty x}{\nu}\right)^{-1/10}$$

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$$\frac{u_b}{u_\infty} = \frac{2.11}{(\text{Re}_x)^{0.1}} \tag{4.157}$$

$$\frac{\delta_b}{\delta} = \left(\frac{u_b}{u_\infty}\right)^7 = \frac{(2.11)^7}{(\text{Re}_x^{0.1})^7} = \frac{191}{\text{Re}_x^{0.7}}$$

$$\frac{\delta_b}{\delta} = \frac{191}{(\text{Re}_x)^{0.7}} \tag{4.158}$$

$$\tau_w = \frac{u_b}{\delta_b} = 0.0228\,\rho u_\infty^2 \left(\frac{\nu}{u_\infty \delta}\right)^{1/4}$$

$$= 0.0228\,\rho u_\infty \left[\frac{\nu}{u_\infty}\,\frac{(u_\infty x/\nu)^{1/5}}{0.376\,x}\right]^{1/4}$$

$$= \boldsymbol{\frac{0.0296\,\rho u_\infty^2}{(\text{Re}_x)^{0.2}}} \tag{4.159}$$

At any given value of $x$, a turbulent boundary layer increases at a faster rate than a laminar boundary layer ($\delta_{\text{tu}} \propto x^{4/5}$ and $\delta_{\text{lam}} \propto x^{1/2}$). Despite its greater thickness, the turbulent boundary layer offers less resistance to heat flow than a laminar layer because the turbulent eddies produce continuous mixing between the warmer and cooler fluids on a macroscopic scale. These eddies diminish in intensity in the buffer layer and hardly penetrate the laminar sublayer.

Prandtl divided the flow field into a laminar and a turbulent layer, but neglected the buffer layer in his analysis and obtained

$$\text{St}_x = \frac{\text{Nu}_x}{\text{Re}_x\,\text{Pr}} = \frac{C_{f_x}/2}{1 + 2.1\,\text{Re}_x^{-0.1}\,(\text{Pr} - 1)} \tag{4.160}$$

For Pr = 1, $\text{St}_x = C_{f_x}/2$, which is Reynolds analogy. The second term in the denominator is a measure of the thermal resistance in the laminar sublayer. This part of the total thermal resistance increases as the Prandtl number becomes larger.

Prandtl's analysis was later refined by von Kàrmàn who divided the flow field into three zones: a laminar sublayer adjacent to the surface in which the eddy diffusivity $\varepsilon_H$ is zero, and heat flows only by conduction. Next to it is a buffer layer in which both conduction and convection contribute to the heat transfer mechanism i.e., $\alpha$ and $\varepsilon_H$ are of the same order of magnitude. Finally, a turbulent region in which conduction is negligible compared to convection, and the Reynolds analogy applies. He used experimental data for the velocity distribution and the shear stress to evaluate $\varepsilon_M$ and assumed $\varepsilon_M = \varepsilon_H$. Also assuming the physical properties of the fluid to be independent of temperature, he determined the thermal resistances in each of the three zones. The von Kàrmàn analogy gives the following equation:

$$\text{Nu}_x = \frac{0.0296\,\text{Re}_x^{0.8}\,\text{Pr}}{1 + 0.86\,\text{Re}_x^{-0.1}\,(\text{Pr} - 1) + \ln\left[1 + \frac{5}{6}(\text{Pr} - 1)\right]} \tag{4.161}$$

for $5 \times 10^5 < \text{Re}_x < 10^7$ [14]

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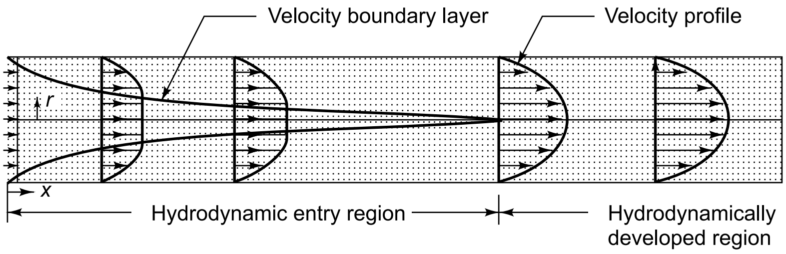
## 4.12 FORCED CONVECTION INSIDE TUBES AND DUCTS

Let us consider a fluid entering a circular tube at a uniform velocity. The fluid particles in the layer in contact with the surface of the tube will come to a complete stop. The layer will cause the fluid particles in the adjacent layers to slow down gradually as a result of friction. To make up for this velocity reduction, the velocity of the fluid at the mid-section of the tube will have to increase to keep the mass flow rate through the tube constant. As a result, *velocity boundary layer* develops along the tube. The thickness of this boundary layer increases in the flow direction until the boundary layer reaches the tube centre and thus fills the entire tube [Fig. 4.31(a)]. The region from the tube inlet to the point at which the boundary layers merge at the centreline is called the *hydrodynamic entry region*, and the length of this region is called the *hydrodynamic entry length* $(L_e)_h$. The region beyond this entry region where the velocity profile is *fully developed* and remains unchanged is called the hydrodynamically developed region. The velocity in this region is *parabolic* for laminar flow, and somewhat *flatter* in turbulent flow [Fig. 4.31(b)].



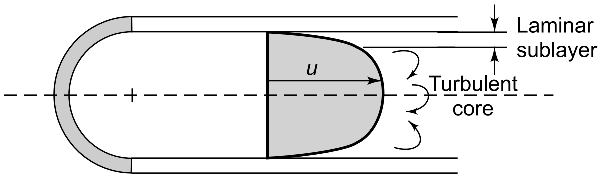




**Fig. 4.31** *(a) Development of velocity boundary layer for laminar flow in a tube and (b) velocity profile in turbulent flow*

Now, let us consider that a fluid at a uniform temperature enters a circular tube with its wall at a different temperature. The fluid particles in the layer in contact with the surface of the tube will assume the tube surface or wall temperature $T_w$. This will initiate convection heat transfer in the tube, and the development of the *thermal boundary layer* along the tube (Fig. 4.32). The thickness of this boundary layer also increases in the flow direction until the boundary layer reaches the tube centre and thus fills the entire tube. The region of flow over which the thermal boundary layer develops and reaches the tube centre is called the *thermal entry region*, and the length of this region is called the *thermal entry length* $(L_e)_t$. The region beyond the thermal entry region in which the temperature profile remains unchanged is called the *thermally developed region*. The dimensionless temperature profile $(T - T_w)/(T_c - T_w)$ does not also change upstream of $(L_e)_t$. The region in which the flow is both hydrodynamically and thermally developed is called the *fully developed region*. The shape of the fully developed temperature profile $T(r, x)$ differs according to whether a uniform surface temperature $(T_w)$

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or a uniform heat flux is maintained. For both surface conditions, however, the amount by which fluid temperatures exceed the entrance temperature increases with increasing $x$.

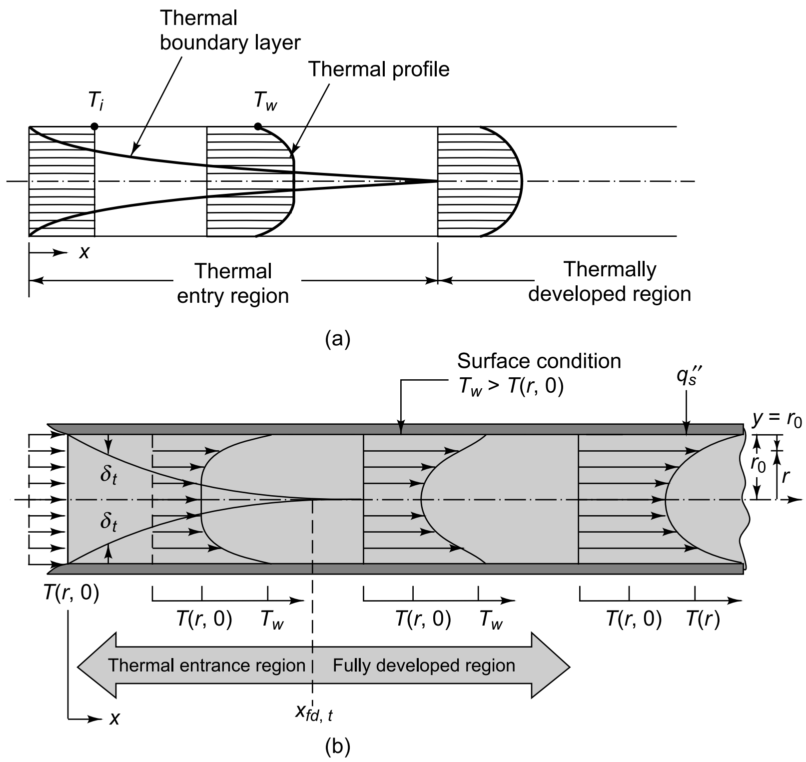


**Fig. 4.32** *Development of thermal boundary layer in a tube (a) when $T > T_w$ and (b) when $T_w > T$*

The hydrodynamic and thermal entry lengths in laminar flow are given approximately as

$$(L_e)_h = 0.05\ \mathrm{Re}_d \mathrm{D} \tag{4.162}$$

$$\mathbf{(L_e)_t = 0.05\ Re_d\ Pr\ D}$$

In turbulent flow the hydrodynamic and thermal entry lengths are independent of $\mathrm{Re}_d$ and Pr, and are generally taken to be

$$\mathbf{(L_e)_h = (L_e)_t = 10D} \tag{4.163}$$

The final shapes of the velocity and temperature profiles depend on whether the fully developed flow is laminar or turbulent. Figures 4.33 and 4.34 illustrate qualitatively the growth of boundary layers as well as the variations in the local heat transfer coefficient near the entrance of a tube for laminar and turbulent flow conditions, respectively. The heat transfer coefficient is largest near the entrance and decreases along the duct until both the velocity and temperature profiles for the fully developed flow have been established. If $\mathrm{Re_d}$ for the fully developed flow is below 2100, the entrance effects may be appreciable for a length as much as 100D from the entrance.

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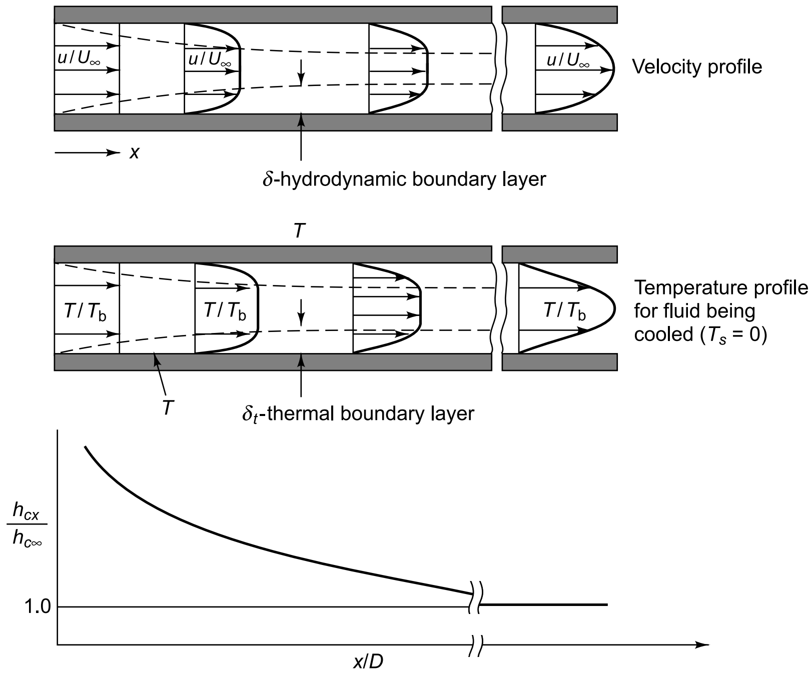


**Fig. 4.33** *Velocity and temperature profiles and variation of local heat transfer coefficient near the inlet for a fluid being cooled in laminar flow through an isothermal tube*

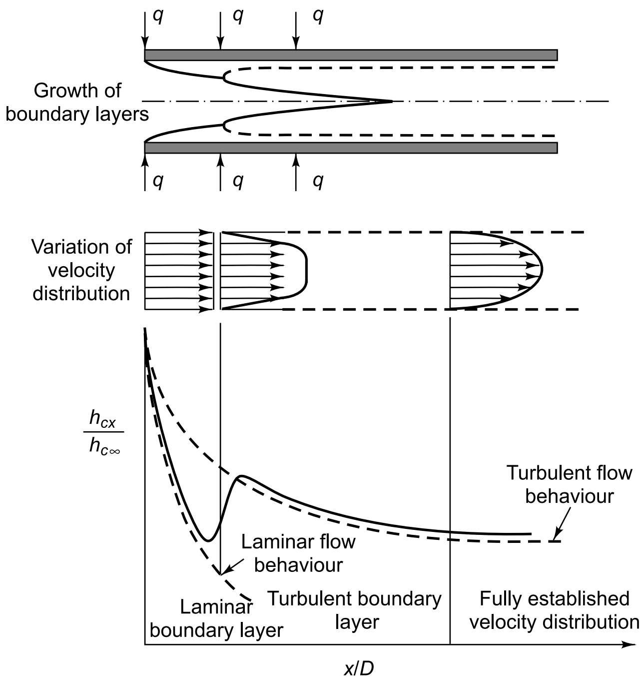


**Fig. 4.34** *Velocity distribution and variation of local heat transfer coefficient near the entrance of a uniformly heated tube for a fluid in turbulent flow*

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For a given fluid the Nusselt number depends primarily on the flow conditions which can be characterised by the Reynolds number, $Re_d$. For flow on long conduits the characteristic length in Reynolds number as well as in Nusselt number is the hydraulic diameter $D_H$ and the velocity to be used is the mean over the flow cross-sectional area, $u_m$, or

$$Re_d = \frac{u_m D_H \rho}{\mu} = \frac{u_m D_H}{\nu} \text{ and } Nu_d = \frac{\bar{h}_c D_H}{k} \tag{4.164}$$

where $$\mathbf{D_H = \frac{4A}{P}}$$

$A$ being the flow cross-sectional area and $P$ the wetted perimeter [Fig. 4.35(a)]. For a circular tube or a pipe, $A = \frac{\pi}{4} D^2$, $P = \pi D$, so $D_H = D$, the inside tube diameter. For an annulus formed between two concentric tubes [Fig. 4.35(b)], $A = \frac{\pi}{4} (D_2^2 - D_1^2)$ and $P = \pi(D_1 + D_2)$ so that

$$\mathbf{D_H = D_2 - D_1}$$

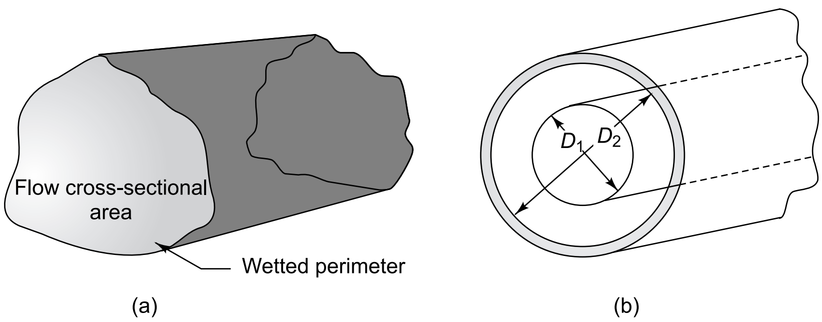


**Fig. 4.35** *Hydraulic diameter for (a) irregular cross-section and (b) annulus*

For a rectangular conduit of sides $a$ and $b$,

$$\mathbf{D_H = \frac{4ab}{2(a+b)} = \frac{2ab}{a+b}}$$

In engineering practice the Nusselt number for flow in conduits is usually evaluated from empirical equations based on experimental results. From a dimensional analysis, as shown in Section 4.5, the experimental results obtained in forced convection heat transfer experiments in long conduits can be correlated by an equation of the form

$$Nu_d = \phi(Re_d)\, \psi(Pr) \tag{4.165}$$

where the symbols $\phi$ and $\psi$ denote functions of the Reynolds number and Prandtl number, respectively.

In long ducts, where the entrance effects are not important, the flow is laminar when $Re_d \le 2100$. In the range $2100 \le Re_d \le 10{,}000$, a transition from laminar to turbulent flow takes place. The flow in this region is transitional. At $Re_d > 10{,}000$, the flow becomes fully turbulent.

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In laminar flow through a duct, just as in laminar flow over a plate, there is no mixing of warmer and colder fluid particles by eddy motion, and the heat transfer takes place solely by conduction. Since all fluids except liquid metals have small thermal conductivities, the heat transfer coefficients in laminar flow are relatively small. In transitional flow a certain amount of mixing occurs through eddies that carry warmer fluid into cooler regions and vice versa. Since the mixing motion, though on a small scale, accelerates the transfer heat considerably, a marked increase in heat transfer coefficient occurs above $Re_d = 2100$, as can be seen in Fig. 4.36 where experimentally measured values of the average Nusselt number for atmospheric air flowing through a long heated tube are plotted as a function of the Reynolds number. Since the Prandtl number for air does not vary appreciably, Eq. (4.165) reduces to $Nu = \phi(Re)$ and the curve in Fig. 4.35 shows the dependence of $Nu_d$ on the flow conditions. In the laminar regime we find that $Nu_d$ remains small increasing from about 2.3 at $Re_d = 200$ to 5.0 at $Re_d = 2100$. At $Re_d > 2100$, $Nu_d$ begins to increase rapidly until $Re_d \cong 8000$. As $Re_d$ is further increased, $Nu_d$ continues to increase, but at a slower rate.

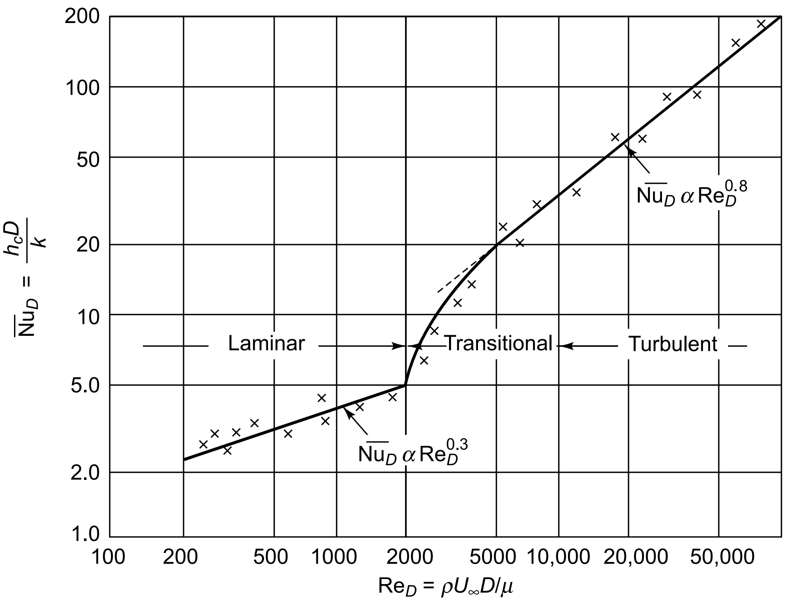


**Fig. 4.36** *Nusselt number varying with Reynolds number for air flowing in a long heated tube at uniform wall temperature*

A qualitative description of this behaviour can be given by observing the fluid flow field shown schematically in Fig. 4.37. At $Re_d > 8000$, the flow inside the conduit is fully turbulent except for a very thin layer of fluid adjacent to the wall. In this layer turbulent eddies are damped out due to viscous forces near the wall, and therefore heat flows through this layer mainly by conduction. The edge of this sublayer is shown by a dashed line (Fig. 4.37). The flow beyond it is turbulent, and the circular arrows represent the eddies that sweep the edge of the layer, probably penetrate it, and carry along with them fluid at the temperature prevailing there. The eddies mix the warmer and cooler fluids so effectively that heat is transferred very rapidly between the edge of the viscous sublayer and the turbulent bulk of the fluid. It thus appears that except for liquid metals of high $k$, the thermal resistance of the sublayer controls the rate of heat transfer, and most of the temperature drop occurs here. The turbulent portion of the flow field offers little resistance to the flow of heat. The only effective method of increasing the heat transfer coefficient is therefore to decrease the thermal resistance of the sublayer. This can be accomplished by increasing the turbulence in the main stream so that the eddies can penetrate deeper into the layer. An increase in turbulence, however, is accompanied by large energy losses that increase the frictional pressure drop in the

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conduit. An increase in the flow velocity increases heat transfer coefficients and thus decreases the size and hence the initial cost of the equipment for a certain heat transfer. At the same time, however, the pumping cost increases ($V\Delta p$). The optimum design, therefore, requires a compromise between the initial and operating costs.

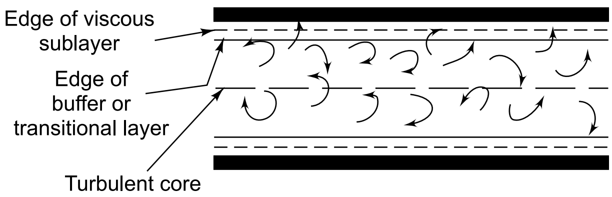


**Fig. 4.37** *Flow mechanism for a fluid in turbulent flow through a tube*

Equation (4.165) shows that the Nusselt number is a function of Prandtl number. The Prandtl number ($\text{Pr} = \nu/\alpha = c_p\,\mu/k$] is a function of the fluid properties alone. It is a ratio of kinematic viscosity $\nu$, which is often referred to as molecular diffusivity of momentum because it controls the rate of momentum transfer between the molecules, and thermal diffusivity $\alpha$ ($=k/\rho\, c_p$), which is often called the molecular diffusivity of heat, because it is a measure of the ratio of the heat transfer and energy storage capacities of the molecules. The Prandtl number relates the temperature distribution to the velocity distribution. For flow in a pipe, just as over a flat plate, the velocity and temperature distributions are similar for fluids having Pr = 1. When Pr < 1, the temperature gradient near the wall is less steep than the velocity gradient.

For fluids having Pr > 1, the temperature gradient is steeper than the velocity gradient. The effect of Prandtl number on the temperature gradient in turbulent flow at a given Reynolds number in tubes is shown in Fig. 4.38, where temperature profiles at different Prandtl numbers are shown at $\text{Re}_d = 10{,}000$. It shows

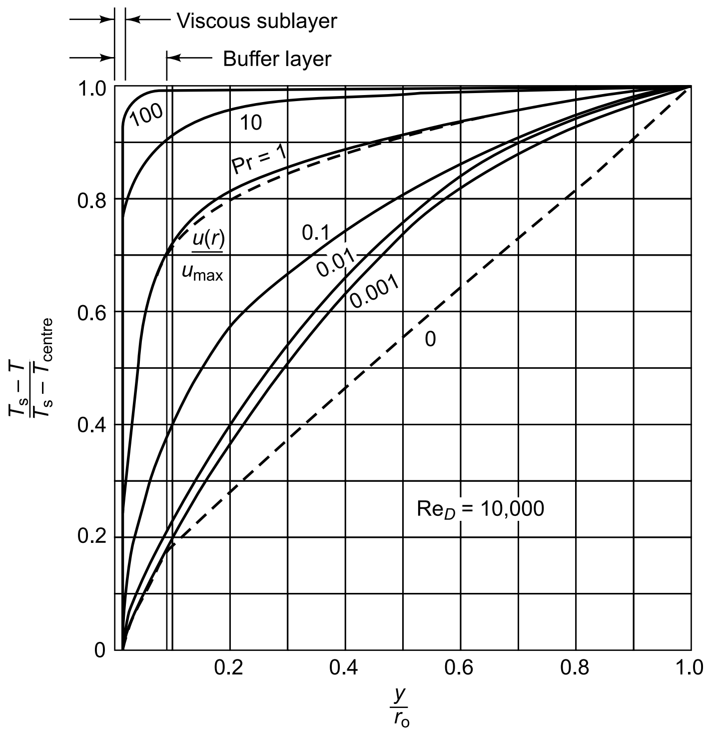


**Fig. 4.38** *Effect of Prandtl number on temperature profile for turbulent flow in a long tube where y = distance from the tube wall*

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that at the given $Re_d$, the temperature gradient at the wall is steeper in a fluid having a large Prandtl number than in a fluid having a small Prandtl number. Therefore, at a given $Re_d$, fluids having larger Prandtl numbers have larger Nusselt numbers. However, the higher Nusselt number here should not be taken to mean a higher heat transfer coefficient.

Most oils have high viscosity and small thermal conductivity and so these have large Prandtl number, some upto 5000 or more. Liquid metals have a high thermal conductivity and a small specific heat and so Prandtl number ranges from 0.005 to 0.01. Gases have Prandtl number varying from 0.6 to 1.0.

## 4.13 ANALYSIS OF LAMINAR FORCED CONVECTION IN A LONG TUBE

Let us consider laminar flow through a tube under fully developed conditions with constant heat flux at the wall. In the fluid element (Fig. 4.39), the pressure is uniform over the cross-section, and the pressure forces are balanced by the viscous shear forces acting over the surface:

$$[p - (p + \mathrm{d}p)]\,\pi r^2 = \tau 2\,\pi r\,\mathrm{d}x = -\left(\mu \frac{\mathrm{d}u}{\mathrm{d}r}\right) 2\pi r\,\mathrm{d}x$$

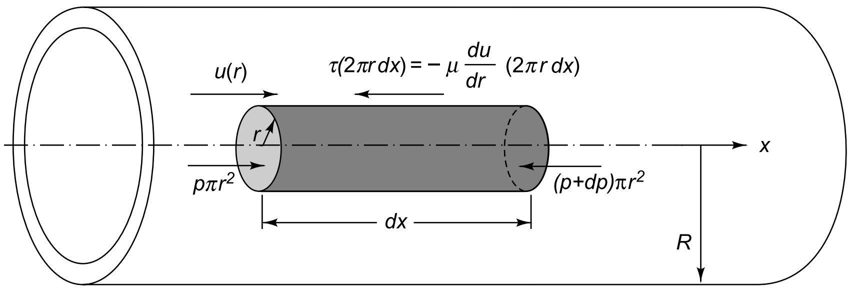


**Fig. 4.39** *Force balance on a cylindrical fluid element inside a tube of radius R*

$$\therefore \qquad \mathrm{d}u = \frac{1}{2\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right) r\,\mathrm{d}r$$

where $\mathrm{d}p/\mathrm{d}x$ is the axial pressure gradient. The radial distribution of the axial velocity is then

$$u(r) = \frac{1}{4\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right) r^2 + C$$

where $C$ is a constant of integration. When $r = R$, $u = 0$

$$\therefore \qquad C = -\frac{1}{4\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right) R^2$$

Substituting

$$u(r) = \frac{r^2 - R^2}{4\mu}\frac{\mathrm{d}p}{\mathrm{d}x} \tag{4.166}$$

The maximum velocity occurs at the centre where $r = 0$.

$$\therefore \qquad \mathbf{u_{max}} = -\frac{\mathbf{R^2}}{\mathbf{4\mu}}\frac{\mathbf{dp}}{\mathbf{dx}} = \mathbf{u_0} \tag{4.167}$$

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There, the velocity distribution in dimensionless form becomes

$$\frac{u}{u_{max}} = 1 - \left(\frac{r}{R}\right)^2 = \frac{u}{u_0} \tag{4.168}$$

Thus the velocity distribution in fully developed laminar flow is parabolic.
The mean velocity of fluid, $u_m$, is

$$u_m = \frac{\int_0^R u r \mathrm{d}r}{\int_0^R r \mathrm{d}r} = \frac{\int_0^R \frac{1}{4\mu}(r^2 - R^2)\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right) r \mathrm{d}r}{R^2/2}$$

$$= -\frac{1}{2\mu R^2}\frac{\mathrm{d}p}{\mathrm{d}x}\left(R^2 \cdot \frac{R^2}{2} - \frac{R^4}{4}\right) = -\frac{R^2}{8\mu}\frac{\mathrm{d}p}{\mathrm{d}x} \tag{4.169}$$

$$u_0 = \text{fluid velocity at the centreline} = -\frac{R^2}{4\mu}\frac{\mathrm{d}p}{\mathrm{d}x}$$

∴ From Eq. (4.167), $\mathbf{u_0 = 2u_m}$ (4.170)

To obtain the pressure drop of the fluid in the tube of length $L$, a force balance (Fig. 4.39) gives

$$\Delta p \pi R^2 = 2\pi R \tau_w L$$

$$\mathbf{\Delta p = \frac{2\tau_w L}{R}}$$

The pressure drop can also be related in the form

$$\mathbf{\Delta p = \frac{fL}{D}\frac{\rho U_m^2}{2}} \tag{4.171}$$

where $f$ is the *Darcy friction factor*.

$$\therefore \quad \frac{2\tau_w L}{R} = \frac{fL}{D}\frac{\rho u_m^2}{2}$$

or
$$\tau_w = \frac{f}{8}\rho u_m^2 = C_f \frac{u_m^2}{2}$$

where $C_f$ is the Fanning friction coefficient

$$\therefore \quad \mathbf{C_f = \frac{f}{4}} \tag{4.172}$$

From Eq. (4.169), putting $\mathrm{d}p/\mathrm{d}x = \Delta p/L$

$$u_m = -\frac{R^2}{8\mu}\frac{\Delta p}{L}$$

$$-\Delta p = 8\mu L\, u_m \frac{4}{D^2} = \frac{32\mu L u_m}{D^2}$$

or
$$\mathbf{\Delta p = p_1 - p_2 = \frac{32\mu L u_m}{D^2}} \tag{4.173}$$

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This is known as **Hagen–Poiseuille equation** for laminar flow. If a fluid flows through a capillary tube of length $L$ and diameter $D$, and the mass flow rate ($u_m = \dot{m}/\rho A$) and the pressure drop $\Delta p$ are measured, the viscosity of the fluid $\mu$ can be estimated from the above equation.

Using Eq. (4.171),

$$\Delta p = \frac{32\mu L u_m}{D^2} = \frac{fL}{D}\frac{\rho u_m^2}{2}$$

$$\therefore \qquad \mathbf{f} = \frac{64\mu}{\rho u_m D} = \frac{64}{Re_d} \tag{4.174}$$

If the volumetric flow rate of the fluid is $\dot{V}$ ($\dot{m}v$) and $\Delta p$ is the pressure drop, then the pumping power

$$P_p = \Delta p \frac{\dot{V}}{\eta_p} \tag{4.175}$$

where $\eta_p$ is the pump efficiency.

### 4.13.1 Heat Transfer Coefficient for Laminar Flow in a Tube

There are two boundary conditions in which heat transfer coefficient can be determined:

(a) Constant Heat Flux

(b) Constant Wall Temperature

Other analytical methods for arbitrarily varying temperature or arbitrarily varying heat flux are quite complex.

#### (a) Constant Heat Flux

Let us consider the control volume (Fig. 4.40) for laminar flow through a tube where heat is transferred by conduction into and out of the element in a radial direction.

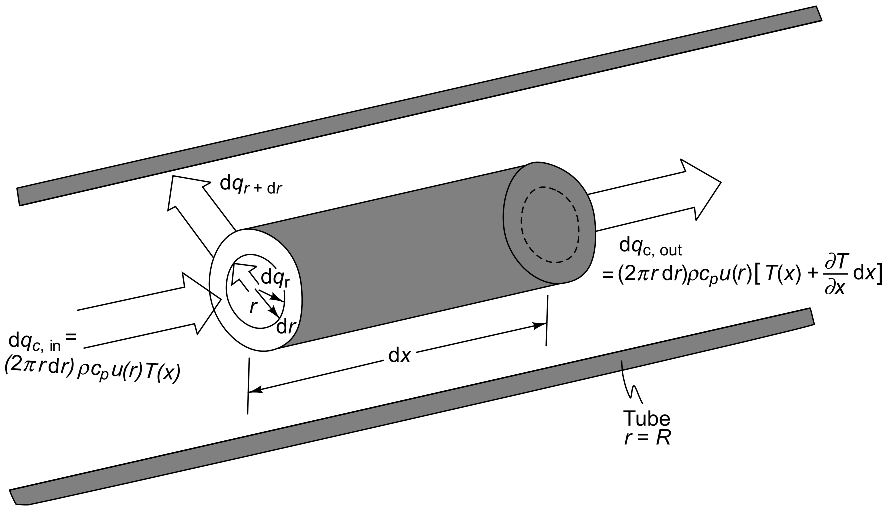


**Fig. 4.40** *Control volume for energy analysis in flow through a pipe*

Rate of heat conduction into the element

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$$dQ_r = -k2\pi r\,dx\,\frac{dT}{dr}$$

Rate of heat conduction out of the element

$$dQ_{r+dr} = -k2\pi(r+dr)\,dx\left(\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial r^2}dr\right)$$

Heat carried away by the fluid

$$dQ_c = \rho u 2\pi r\,dr\,c_p\,\frac{\partial T}{\partial x}dx$$

By energy balance,

$$dQ_r - dQ_{r+dr} = dQ_c$$

$\therefore$

$$-k2\pi r\,dx\,\frac{\partial T}{\partial r} + k2\pi r\,dx\,\frac{\partial^2 T}{\partial r^2} + k2\pi r\,dx\,\frac{\partial T}{\partial x}\,dr$$

$$+k2\pi dr\,dx\,\frac{\partial^2 T}{\partial x^2} + k2\pi dr\,dx\,\frac{\partial^2 T}{\partial r^2}\,dr$$

$$= \rho u 2\pi r\,dr\,c_p\,\frac{\partial T}{\partial x}\,dx$$

Neglecting the last term of the L.H.S. of the equation and simplifying,

$$k\left(\frac{\partial T}{\partial r}+r\frac{\partial^2 T}{\partial r^2}\right) = \rho u c_p r\,\frac{\partial T}{\partial x}$$

or

$$\frac{1}{ur}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial x} \tag{4.176}$$

Since the heat flux over the surface is uniform, the fluid temperature increases linearly with distance $x$, and so,

$$\frac{\partial T}{\partial x} = \text{constant} \tag{4.177}$$

Equation (4.176) then becomes an ordinary differential equation with $r$ as the only variable.

At $r = 0,\ \frac{\partial T}{\partial r} = 0$ (at the centre)

At $r = R,\ k\left(\frac{\partial T}{\partial r}\right)_{r=R} = q''_w = \text{constant}$

From Eq. (4.176),

$$\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = u_o\left(1-\frac{r^2}{R^2}\right)\frac{r}{\alpha}\frac{\partial T}{\partial x}$$

$$r\frac{\partial T}{\partial r} = \frac{u_0}{\alpha}\frac{\partial T}{\partial x}\left(\frac{r^2}{2}-\frac{r^4}{4R^2}\right)+C_1$$

$$\frac{\partial T}{\partial r} = \frac{u_0}{\alpha}\frac{\partial T}{\partial x}\left(\frac{r}{2}-\frac{r^3}{4R^2}\right)+\frac{C_1}{r}$$

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$$\therefore \qquad T(r, z) = \frac{u_0}{\alpha}\frac{\partial T}{\partial x}\left(\frac{r^2}{4} - \frac{r^4}{16R^2}\right) + C_1 \ln r + C_2$$

At $\qquad r = 0, \quad \dfrac{\partial T}{\partial r} = 0, \quad \therefore C_1 = 0$

$r = 0$, $T = T_0$, $\quad C_2 = T_0$, the centreline temperature

$$T - T_0 = \frac{u_0}{\alpha}\frac{\partial T}{\partial x}\frac{R^2}{4}\left[\left(\frac{r}{R}\right)^2 - \frac{1}{4}\left(\frac{r}{R}\right)^4\right] \tag{4.178}$$

This is the temperature distribution of the fluid in the radial direction. The average bulk temperature of the fluid ($T_b$) is the temperature if the fluid were well mixed adiabatically so that there is no radial variation of temperature at any cross-section. It is also called the **mixing cup temperature**. Therefore,

$$T_b = \frac{\int_0^R \rho u 2\pi r \, \mathrm{d}r c_p T}{\int_0^R \rho u 2\pi r \mathrm{d}r c_p} = \frac{\int_0^R T u r \mathrm{d}r}{\int_0^R u r \mathrm{d}r} \tag{4.179}$$

At $\qquad r = R$, $T = T_w$. Therefore, Eq. (4.178) becomes

$$T_w = T_0 + \frac{u_0}{\alpha}\frac{\partial T}{\partial x}\frac{R^2}{4}\frac{3}{4}$$

$$= T_0 + \frac{3}{16}\frac{u_0 R^2}{\alpha}\frac{\partial T}{\partial x} \tag{4.180}$$

Substituting Eqs (4.178) and (4.168) for $T$ and $u$ respectively in Eq. (4.179), we obtain

$$T_\mathrm{b} = \frac{\int_0^R T u r \, \mathrm{d}r}{\int_0^R u r \, \mathrm{d}r}$$

$$= \frac{\int_0^R u_0\left(1 - \frac{r^2}{R^2}\right)\left[T_0 + \frac{u_0}{\alpha}\frac{\partial T}{\partial x}\cdot\frac{R^2}{4}\left(\frac{r^2}{R^2} - \frac{1}{4}\frac{r^2}{R^4}\right)\right] r \, \mathrm{d}r}{\int_0^R u_0\left(1 - \frac{r^2}{R^2}\right) r \, \mathrm{d}r}$$

$$= \frac{2u_m \int_0^R \left(1 - \frac{r^2}{R^2}\right)\left[T_0 + \frac{u_0}{\alpha}\frac{\partial T}{\partial x}\cdot\left(\frac{r^2}{4} - \frac{r^4}{16R^2}\right)\right] r \mathrm{d}r}{2u_m\left(\frac{R^2}{2} - \frac{1}{R^2}\frac{R^4}{4}\right)}$$

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$$= \frac{4}{R^2}\int_0^R \left[T_0\left(r - \frac{r^3}{R^2}\right) + \frac{u_0}{\alpha}\frac{\partial T}{\partial x}\cdot\left(\frac{r^3}{4} - \frac{r^5}{16R^2} - \frac{r^5}{4R^2} + \frac{r^7}{16R^4}\right)\right]\mathrm{d}r$$

$$= \frac{4}{R^2}\left[T_0\left(\frac{R^2}{2} - \frac{1}{R^2}\frac{R^4}{4}\right) + \frac{u_0}{\alpha}\frac{\partial T}{\partial x}\cdot\left(\frac{R^4}{16} - \frac{R^6}{96R^2} - \frac{R^6}{24R^2} + \frac{R^8}{128R^4}\right)\right]$$

$$= \frac{4}{R^2}\left(T_0\frac{R^2}{4} + \frac{u_0}{\alpha}\frac{\partial T}{\partial x}R^4\frac{7}{384}\right)$$

$$= T_0 + \frac{7}{96}\frac{u_0 R^2}{\alpha}\frac{\partial T}{\partial x} \tag{4.181}$$

Rate of heat transfer,

$$Q = \bar{h}_c A\,(T_w - T_b) = kA\left(\frac{\partial T}{\partial r}\right)_{r=R}$$

$$\bar{h}_c = \frac{k\left(\dfrac{\partial T}{\partial r}\right)_{r=R}}{T_w - T_u} \tag{4.182}$$

From Eq. (4.178)

$$\left(\frac{\partial T}{\partial r}\right)_{r=R} = \frac{u_0}{\alpha}\frac{\partial T}{\partial x}\frac{R^2}{4}\left(\frac{1}{R^2}2R - \frac{1}{4R^4}4R^3\right)$$

$$= \frac{u_0}{\alpha}\frac{\partial T}{\partial x}\frac{R}{4} \tag{4.183}$$

Substituting Eqs (4.183), (4.181) and (4.180) in Eq. (4.182).

$$\bar{h}_c = \frac{k\dfrac{u_0}{\alpha}\dfrac{\partial T}{\partial x}\dfrac{R}{4}}{T_w - T_b} = \frac{k\dfrac{u_0}{\alpha}\dfrac{\partial T}{\partial x}\dfrac{R}{4}}{T_0 + \dfrac{3}{16}\dfrac{u_0R^2}{\alpha}\dfrac{\partial T}{\partial x} - T_0 - \dfrac{7}{96}\dfrac{u_0R^2}{\alpha}\dfrac{\partial T}{\partial x}}$$

$$= \frac{k}{(11/24)\,R} = \frac{24}{11}\frac{k\times 2}{D} = \frac{48}{11}\frac{k}{D}$$

$$\mathbf{Nu_d} = \frac{\mathbf{hD}}{\mathbf{k}} = \frac{\mathbf{48}}{\mathbf{11}} = \mathbf{4.364} \tag{4.184}$$

(for $q''_w$ = constant)

This shows that the Nusselt number for fully developed laminar flow in a tube is constant and is independent of Reynolds number and Prandtl number.

For constant heat flux boundary condition,

$$\frac{Q}{A} = q_c = \text{constant}$$

$$q_c\,\pi D\,\mathrm{d}x = \dot{m}c_p\mathrm{d}T_b = \frac{\pi}{4}D^2u_mc_p\mathrm{d}T_b$$

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$$\therefore \qquad \frac{dT_b}{dx} = \frac{q_c \pi D}{\dot{m}c_p} = \frac{4q_c}{D} \cdot \frac{1}{\rho c_p u_m} = \text{constant}$$

Bulk fluid temperature $T_b$ varies linearly with $x$ [Fig. 4.40(a)]. Also,

$$q_c = h_c (T_w - T_b) = \frac{k}{D} \text{Nu}_d (T_w - T_b)$$

$$T_w - T_b = \frac{q_c D}{4.364 k} = \text{constant}$$

In the fully developed laminar forced convection heat transfer, $(T_w - T_b)$ also remains constant with $x$, as shown in Fig. 4.41(a).

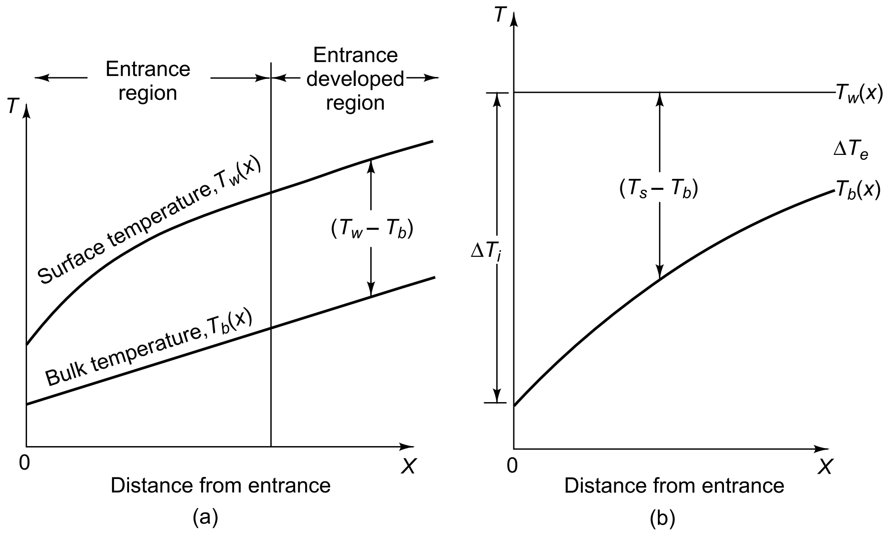


**Fig. 4.41** *Variation of average bulk temperature of a fluid in a pipe for (a) constant heat flux and (b) constant wall temperature*

### (b) Uniform Wall Temperature

When the tube wall temperature is constant, the analysis is more difficult because the temperature difference between the wall and the bulk fluid, $T_w - T_b$, varies along the tube i.e., $\partial T_b/\partial x = f(x)$, as shown in Fig. 4.41(b). Equation (4.176) can be solved subject to the boundary condition that at $r = R$, $T(x, R)$ = constant, which needs an iterative procedure. Here too, the Nusselt number is a constant (see [11]) as follows:

$$\therefore \qquad \mathbf{Nu_d = 3.66\ (T_w = constant)} \qquad (4.185)$$

The energy balance for a differential length d$x$ gives

$$dQ_c = \dot{m}c_p dT_b = q''_w P dx$$

where $P$ is the perimeter of the duct and $q''_w$ is the wall heat flux.

$$\frac{dT_b}{dx} \frac{q''_w P}{\dot{m}c_p} = \frac{P}{\dot{m}c_p} = h_c (T_w - T_b)$$

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Since $dT_b/dx = d(T_b - T_w)/dx$ for a constant wall temperature,

$$\frac{d(T_b - T_w)}{dx} = \frac{P}{\dot{m}c_p} = h_c(T_w - T_b)$$

Putting $\Delta T = T_w - T_b$,

$$-\frac{d(\Delta T)}{dx} = \frac{P}{\dot{m}c_p} h_c \Delta T$$

$$\int_{T_i}^{T_e} \frac{d(\Delta T)}{\Delta T} = -\frac{P}{\dot{m}c_p}\int_0^L h_c dx$$

$$\ln\frac{\Delta T_e}{\Delta T_i} = -\frac{PL}{\dot{m}c_p}\bar{h}_c \tag{4.186}$$

where $$\bar{h}_c = \frac{1}{L}\int_0^L h_c dx$$

Rearranging Eq. (4.186),

$$\mathbf{\frac{\Delta T_e}{\Delta T_i} = \exp\left(-\frac{\bar{h}_c PL}{\dot{m}c_p}\right)} \tag{4.187}$$

The rate of heat transfer by convection to or from a fluid through a duct with $T_w$ = constant can be written as

$$Q_c = \dot{m}c_p[(T_w - T_{b,i}) - (T_w - T_{b,e})] = \dot{m}c_p(\Delta T_i - \Delta T_e)$$

Substituting $\dot{m}c_p$ from Eq. (4.186),

$$Q_c = -\frac{\bar{h}_c Pl}{\ln\dfrac{\Delta T_e}{\Delta T_i}}(\Delta T_i - \Delta T_e) = \bar{h}_c A_w \frac{\Delta T_e - \Delta T_i}{\ln\dfrac{\Delta T_e}{\Delta T_i}}$$

$$= \mathbf{\bar{h}_c A_w \Delta T_{lm}} \tag{4.188}$$

where $\Delta T_{lm}$ is the log mean temperature difference (LMTD).

Hausen [18] recommended the following relation for the average convection coefficient in laminar flow through ducts with uniform surface temperature:

$$\mathrm{Nu}_d = 3.66 + \frac{0.0668\,\mathrm{Re}_{D_H}\cdot \mathrm{Pr}\cdot D/L}{1 + 0.045\,(\mathrm{Re}_{D_H}\cdot \mathrm{Pr}\cdot D/L)^{0.66}}\left(\frac{\mu_b}{\mu_w}\right)^{0.14} \tag{4.189}$$

A relatively simple empirical equation suggested by Sieder and Tate [19] has been widely used to correlate experimental results for liquids in tubes and can be written in the form

$$\mathrm{Nu}_{D_H} = 1.86\left(\frac{\mathrm{Re}_{D_H}\cdot \mathrm{Pr}\cdot D_H}{L}\right)^{0.33}\left(\frac{\mu_b}{\mu_w}\right)^{0.14} \tag{4.190}$$

where all the properties in Eqs. (4.189) and (4.190) are based on the bulk temperature and the empirical correction factor $(\mu_b/\mu_w)^{0.14}$ is introduced to account for the temperature variation on the physical properties.

An additional complication in the determination of a heat transfer coefficient in laminar flow arises when the buoyancy forces are of the same order of magnitude as the external forces due to the forced convection.

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Such a condition may arise in oil coolers when low flow velocities are used. Also, in the cooling of the rotor blades of gas turbines, the natural convection effects may be so large that their effect on the velocity pattern cannot be neglected even in high-velocity flow. When the buoyancy forces are in the same direction as the external forces, they increase the rate of heat transfer. When the external and buoyancy forces act in opposite directions, the heat transfer is reduced. Eckert *et al*. [20, 21] studied heat transfer in mixed flow, and their results are shown in Fig. 4.42(a) and (b). In practice, natural convection effects are hardly ever significant in turbulent flow.

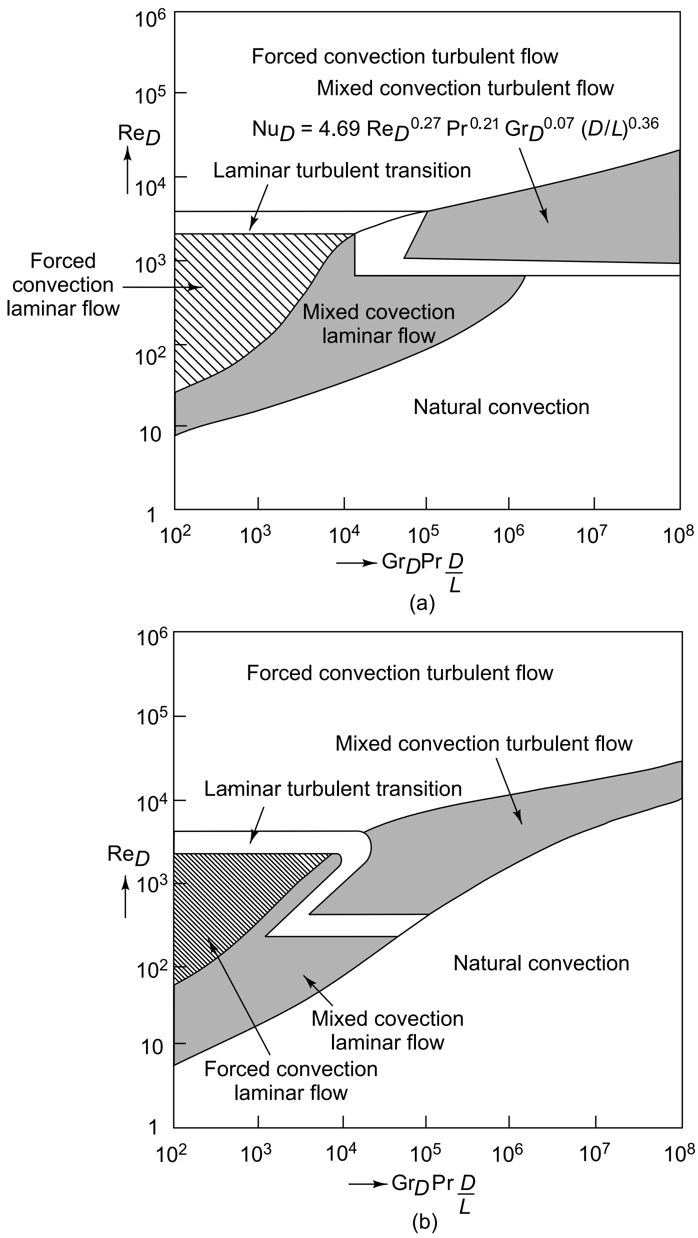


**Fig. 4.42** *Forced, natural and mixed convection regimes for (a) horizontal pipe flow and (b) vertical pipe flow*

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The influence of natural convection on the heat transfer to fluids in horizontal isothermal tubes was investigated by Depew and August [22] who correlated the data by the equation

$$\mathrm{Nu}_d = 1.75\left(\frac{\mu_b}{\mu_w}\right)^{0.14}[\mathrm{Gz} + 0.12\,(\mathrm{Gz}\,\mathrm{Gr}_d^{1/3}\,\mathrm{Pr}^{0.36})^{0.88}]^{1/3} \tag{1.191}$$

where Gz is the Graetz number, defined by

$$\mathrm{Gz} = \frac{\pi}{4}\mathrm{Re}_d\,\mathrm{Pr}\,\frac{D}{L}$$

and $\mathrm{Gr}_d = \dfrac{g\beta\theta D^3}{\nu^2}$ is the Grasof number.

It is valid in the range $25 < \mathrm{Gz} < 700$, $5 < \mathrm{Pr} < 400$ and $250 < \mathrm{Gr}_d < 10^5$. Physical properties except for $\mu_w$ are to be evaluated at the average bulk temperature.

## 4.14 ANALYSIS OF COUETTE FLOW FOR LAMINAR FORCED CONVECTION

The space between the two infinite parallel plates separated by a distance $L$ is filled with a liquid having viscosity $\mu$, density $\rho$ and thermal conductivity $k$ (Fig. 4.43). The upper plate at $y = L$ moves with a constant velocity $u_1$ and sets the fluid particles moving in the direction parallel to the plates, while the lower plate remains stationary. The lower and upper plates are kept at temperatures $T_0$ and $T_1$, respectively. The heat transfer problem characterised with this simple model is important for a journal and its bearing in which one surface is stationary while the other is rotating and the clearance between them is filled with a lubricating oil of high viscosity. When the clearance is small in comparison with the radius of the bearing, the geometry can be considered as two parallel plates. The oil being viscous, the temperature rise in the fluid due to friction (i.e., viscous dissipation) may become considerable even at moderate flow velocities. Therefore, the temperature rise in the fluid and the amount of heat transfer through the walls are of interest in engineering applications. In solving this heat transfer problem, we determine the velocity distribution in the flow and then the temperature distribution.

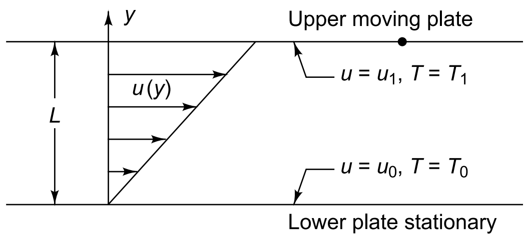


**Fig. 4.43** *Heat transfer in Couette flow*

The flow is fully developed since all the fluid particles move in the direction parallel to the plates. The velocity component normal to the wall is zero (i.e., $v = 0$). Then the continuity equation reduces to

$$\frac{\partial u}{\partial x} = 0$$

which implies that the axial velocity component $u$ does not depend on $x$. Therefore, $u = u(y)$ only. The $y$-momentum equation subject to conditions $v = 0$ and body force = 0

$$\frac{\partial p}{\partial y} = 0$$

which implies that the pressure $p$ does not depend on $y$. Therefore, $p = p(x)$ only.

Finally, the $x$-momentum equation, subject to conditions $v = 0$, $u = u(y)$, $p = p(x)$ and body force = 0, reduces to

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$$\frac{d^2u(y)}{dy^2} = \frac{1}{\mu}\frac{dp(x)}{dx}$$

Since the fluid motion is set by simple shear flow due to viscosity and no pressure gradient is involved in the direction of motion, $dp/dx = 0$

$$\frac{d^2u}{dy^2} = 0$$

When $y = 0$, $u = 0$ and at $y = L$, $u = u_1$. Therefore, the velocity distribution in the Couette flow

$$u = \frac{y}{L}u_1 \tag{4.192}$$

The energy equation is

$$\rho c_p u\frac{\partial T}{\partial x} = k\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right] + \mu\phi$$

where the energy dissipation function $\phi = (du/dy)^2$.

Since $T$ varies only in $y$-direction, the energy equation reduces to

$$\frac{d^2T(y)}{dy^2} = -\frac{\mu}{k}\left(\frac{du}{dy}\right)^2 \tag{4.193}$$

By introducing the velocity profile given by Eq. (4.192),

$$\frac{d^2T}{dy^2} = -\frac{\mu}{k}\frac{u_1^2}{L^2}$$

At $\quad y = 0$, $T = T_0$ and $y = L$, $T = T_1$.

$$T(y) - T_0 = \frac{y}{L}\left[(T_1 - T_0) + \frac{\mu u_1^2}{2k}\left(1 - \frac{y}{L}\right)\right] \tag{4.194}$$

(i) When $T_0 \neq T_1$, both sides are divided by $(T_1 - T_0)$,

$$\frac{T(y) - T_0}{T_1 - T_0} = \frac{y}{L}\left[1 + \frac{u_1^2}{2k(T_1 - T_0)}\left(1 - \frac{y}{L}\right)\right]$$

$$= \frac{\mathbf{y}}{\mathbf{L}}\left[\mathbf{1} + \frac{\mathbf{1}}{\mathbf{2}}\mathbf{PrE}\left(\mathbf{1} - \frac{\mathbf{y}}{\mathbf{L}}\right)\right] \tag{4.195}$$

where $\mathrm{Pr} = \dfrac{c_p\mu}{k}$ and E = Eckert number $= \dfrac{u_1^2}{c_p(T_1 - T_0)}$

Figure 4.44 shows a plot of $[T(y) - T_0]/(T_1 - T_0)$ as a function of $y/L$ for several values of Pr. E. The case Pr. E = 0 indicates no viscous dissipation in the medium and the temperature distribution is a straight line, which means pure conduction across the fluid layer.

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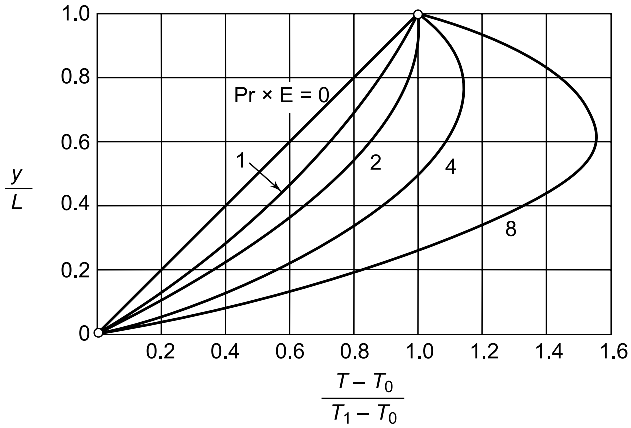


**Fig. 4.44** *Temperature distribution in Couette flow ($T_1 > T_0$)*

The heat flux at the upper wall

$$(q)_{y=L} = -k \left.\frac{\mathrm{d}T(y)}{\mathrm{d}y}\right|_{y=L}$$

$$= \frac{T_1 - T_0}{L}\left(1 - \frac{1}{2}\mathrm{Pr}\cdot\mathrm{E}\right) \quad (4.196)$$

For Pr. E > 2, the r.h.s. of the above equation is positive i.e., the heat flows from the liquid into the walls even though the upper wall is at a higher temperature than the lower wall.

For Pr. E < 2, the r.h.s. of the equation is negative, and heat flows from the upper wall to the fluid.

For Pr. E = 2, the r.h.s. = 0, and there is no heat flow at the upper wall.

When $T_0 = T_1$, both plates are at the same temperature and

$$T(y) - T_0 = \frac{\mu u_1^2}{2k}\frac{y}{L}\left(1 - \frac{y}{L}\right) \quad (4.197)$$

The maximum temperature occurs at the midpoint between the plates. By setting $y = L/2$

$$T_{\max} = T_0 - \frac{\mu u_1^2}{8k} \quad (4.198)$$

By combining Eqs (4.197) and (4.198)

$$\frac{T(y) - T_0}{T_{\max} - T_0} = 4\frac{y}{L}\left(1 - \frac{y}{L}\right) \quad (4.199)$$

## 4.15 VELOCITY DISTRIBUTION IN TURBULENT FLOW THROUGH A PIPE

Velocity distribution in turbulent flow has been investigated extensively because of its importance in practice, but no fundamental theory is yet available to determine this velocity distribution rigorously by purely theoretical approaches. Therefore, empirical and semi-empirical relations are used to correlate the velocity field in turbulent flow.

Nikuradse [26] was an early investigator who presented a careful measurement of velocity distribution in turbulent flow through a smooth pipe. Later more experiments were conducted to develop empirical relations to fit the velocity distribution. We would discuss the flow field into three distinct layers: (a) a very thin layer at the wall where viscous shear stress dominates, called the *viscous sublayer*, (b) a *buffer*

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*layer* where viscous and turbulent shear stresses are equally important and (c) a *turbulent layer*, in which turbulent shear stress dominates.

In the study of velocity distribution for turbulent flow, the following two dimensionless quantities are introduced:

$$u^+ = \frac{u}{(\tau_w/\rho)^{1/2}} = \text{dimensionless velocity}$$

$$y^+ = \frac{y}{\nu}\left(\frac{\tau_w}{\rho}\right)^{1/2} = \text{dimensionless distance}$$

where $\tau_w$ is the shear stress at the wall and $u$ is the velocity component parallel to the wall surface.

Experiments have shown that the viscous sublayer is maintained in the region $y^+ < 5$. The shear stress is $\tau_w = \mu \frac{\partial u}{\partial y}$ and integrating it with $u = 0$ at $y = 0$, we get $u^+ = y^+$ for viscous sublayer, $0 < y^+ < 5$.

The buffer layer is considered to extend from $y^+ = 5$ to $y^+ = 30$ and a logarithmic velocity distribution law $u^+ = A \ln y^+ + B$ is assumed, where constants $A$ and $B$ are determined from the condition that $u^+$ be equal to that of the laminar sublayer and of the turbulent layer at $y^+ = 5$ and $y^+ = 30$, respectively. The resulting velocity distribution becomes

$$u^+ = 5.0 \ln y^+ - 3.05 \text{ for buffer layer} \quad 5 \le y^+ \le 30$$

The region $y^+ > 30$ is considered to be the turbulent layer. By utilising the mixing-length concept and assuming that the mixing length varies linearly with the distance from the wall, $1 = \lambda y$, it can be shown that the velocity distribution in turbulent layer has a logarithmic profile in the form

$$u^+ = \frac{1}{\lambda} \ln y^+ + C$$

where $\lambda$ is called the universal constant. For smooth pipes, $C = 5.5$, and the velocity distribution in the turbulent layer becomes

$$u^+ = 2.5 \ln y^+ + 5.5 \text{ for turbulent layer } y^+ > 30$$

Figure 4.45 shows a correlation of the velocity distribution law as obtained above for the three layers along with Nikuradse's measured experimental data. It is often termed as "*universal velocity profile*".

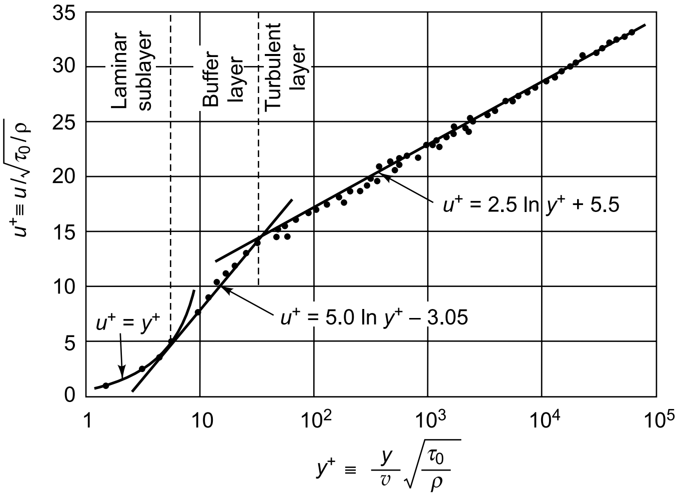


**Fig. 4.45** *Logarithms velocity distribution law and Nikuradse's experimental data for turbulent flow inside smooth pipe*

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In Section 4.12, the turbulent flow regime along with the other two regimes has also been discussed, and Fig. 4.36 shows Nusselt number varying with Reynolds number for air in different regimes of fluid flow in a pipe.

## 4.16 ANALOGY BETWEEN HEAT AND MOMENTUM TRANSFER IN TURBULENT FLOW

The analysis of heat transfer for turbulent flow is much more involved than that for laminar flow. In Section 4.5, based on the analysis of laminar flow along a flat plate, we developed a relation between the heat transfer coefficient and the local drag coefficient $C_{f_x}$ as

$$\mathrm{St}_x \, \mathrm{Pr}^{2/3} = C_{f_x}/2$$

where $$\mathrm{St}_x = \frac{\mathrm{Nu}_x}{\mathrm{Re}_x \, \mathrm{Pr}} = \frac{h_x}{\rho c_p u_\infty} \quad \text{and} \quad C_{f_x} = \frac{\tau_w}{\frac{1}{2}\rho u_\infty^2}$$

For turbulent flow inside a circular pipe a similar expression can be found out. Making a force balance,

$$\tau_w \pi D L = \Delta p \frac{\pi}{4} D^2 = \frac{fL}{D}\frac{\rho u_m^2}{2}\frac{\pi}{4} D^2$$

$$\boldsymbol{\tau_w = \frac{f}{8} \rho u_m^2} \tag{4.200}$$

where $f$ is the Darcy friction factor. Heat transfer rate at the wall

$$q_w = \frac{Q}{A} = -k\frac{dT}{dy}$$

and the shear stress

$$\tau_w = \mu \frac{du}{dy}$$

Dividing the two equations

$$\frac{Q}{A\tau_w} = -\frac{k}{\mu}\frac{dT}{du}$$

For fluids having Pr = 1, i.e., $\frac{c_p \mu}{k} = 1$

$$\frac{Q}{A\tau_w} = -c_p \frac{dT}{du}$$

$$\frac{Q}{A}\int_0^{u_m} du = -c_p \tau_w \int_{T_w}^{T_b} dT$$

$$\frac{Q u_m}{A} = c_p \tau_w (T_w - T_b)$$

$$\frac{Q}{A(T_w - T_b)} = \bar{h}_c \frac{c_p \tau_w}{u_m} \tag{4.201}$$

Substituting $\tau_w$ from Eq. (4.200),

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$$\bar{h}_c = \frac{c_p}{u_m} f \frac{\rho u_m^2}{8}$$

$$\mathbf{St_d} = \frac{\mathbf{Nu_d}}{\mathbf{Re_d Pr}} = \frac{\mathbf{h_c}}{\rho \mathbf{c_p u_m}} = \frac{\mathbf{f}}{\mathbf{8}} \tag{4.202}$$

This is known as *Reynolds analogy* for momentum and heat transfer. Reynolds first assumed that the entire flow field consists of a single zone of highly turbulent region. He neglected the presence of the viscous sublayer and the buffer layer. In such a turbulent core, the molecular diffusivity of heat $\alpha$ and that of momentum $\nu$ are negligible compared with turbulent diffusivities, i.e.

$$\nu << \varepsilon_M \quad \text{and} \quad \alpha << \varepsilon_H$$

Then, the equations

$$\tau = \rho\,(\nu + \varepsilon_M)\,\frac{\mathrm{d}u}{\mathrm{d}y}$$

and
$$\frac{Q}{A\rho c_p} = -\,(\alpha + \varepsilon_H)\,\frac{\mathrm{d}T}{\mathrm{d}y} \tag{4.203}$$

where $y$ is the distance measured from the tube wall, become

$$\frac{\tau}{\rho} = \varepsilon_M \frac{\mathrm{d}u}{\mathrm{d}y} \quad \text{and} \quad \frac{q_w}{\rho c_p} = -\,\varepsilon_H \frac{\mathrm{d}T}{\mathrm{d}y}$$

Assuming the turbulent diffusivities to be equal, $\varepsilon_M = \varepsilon_H$, i.e. $\mathrm{Pr}_t$, turbulent Prandtl number = 1, we obtain by division,

$$\frac{q_w}{c_p \tau} = \frac{\mathrm{d}T}{\mathrm{d}u}$$

On integration, we obtain the same Eq. (4.201). The Eq. (4.202), called Reynolds analogy, is valid for both laminar and turbulent flow in a pipe for Pr = 1 or $\mathrm{Pr}_t = 1$.

Prandtl assumed that the flow field consisted of two layers, a viscous sublayer where the molecular diffusivities are dominant, i.e.,

$$\varepsilon_M << \nu \quad \text{and} \quad \varepsilon_H << \alpha$$

and a turbulent core where the turbulent diffusivities are dominant, i.e.,

$$\nu << \varepsilon_M \quad \text{and} \quad \alpha << \varepsilon_H \text{ and } \varepsilon_M = \varepsilon_H$$

These assumptions are utilized to simplify Eq. (4.203) for each layer, the equations are integrated, and the definitions of the friction factor and the heat transfer coefficient are introduced. The following result is obtained.

$$\mathbf{St_d} = \frac{\mathbf{h_c}}{\rho \mathbf{c_p u_m}} = \frac{\mathbf{f}}{\mathbf{8}} \frac{\mathbf{1}}{\mathbf{1 + 5\,[f/8]^{1/2}\,(Pr - 1)}} \tag{4.204}$$

This relationship is known as the *Prandtl analogy* for momentum and heat transfer for fully developed turbulent flow in a pipe. We note that for Pr = 1 the Prandtl analogy reduces to the Reynolds analogy.

Von Kàrmàn extended Prandtl's analogy by separating the flow field into three distinct layers: a viscous sublayer, a buffer layer and a turbulent core. He made assumptions about the relative magnitudes of the molecular and turbulent diffusivities of heat and momentum in the three layers and obtained the following result:

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$$\mathbf{St_d = \frac{\bar{h}_c}{\rho c_p u_m} = \frac{f}{8} \frac{1}{1 + 5\,(f/8)^{1/2}\ (Pr - 1) + \ln[(5\,Pr + 1)/6]}} \tag{4.205}$$

which is known as *von Kàrmàn analogy* for momentum and heat transfer for fully developed turbulent flow in a pipe.

According to experimental data for fluids flowing in smooth pipes in the range of Reynolds numbers from 10,000 to 1,000,000, the friction factor is given by the empirical relation [16].

$$\mathbf{f = 0.184\ Re_d^{-0.2}} \tag{4.206}$$

For $\quad Re_d << 20{,}000, \quad f = 0.316\ (Re_d)^{-1/4}$ (4.206a)

Using this relation, Reynolds analogy becomes

$$St_d = \frac{Nu_d}{Re_d\ Pr} = \frac{f}{8} = \frac{0.184\ Re_d^{-0.2}}{8}$$

$$\mathbf{= 0.023\ Re_d^{-0.2}} \tag{4.207}$$

Since Pr = 1,

$$Nu_d = 0.023\ Re_d^{0.8} \tag{4.208}$$

or,
$$\mathbf{h_c = 0.023 u_m^{0.8}\ D^{-0.2}\ k \left(\frac{\mu}{\rho}\right)^{-0.8}} \tag{4.209}$$

In fully developed turbulent flow, $\bar{h}_c \propto u_m^{0.8}$ and $\bar{h}_c \propto \dfrac{1}{D^{0.2}}$. For a given flow rate, an increase in the tube diameter $D$ reduces the velocity and hence $\bar{h}$. The use of small tubes and high velocities result in high $\bar{h}_c$. But large velocities require more pumping power. In the design of heat exchanger equipment, it is necessary to strike a balance between the gain in heat transfer rates and the increase in pumping requirements.

Figure 4.46 (Moody's chart) shows the effect of surface roughness on the friction factor. We observe that the friction factor increases appreciably with the relative roughness, defined as the ratio of the average asperity height $\varepsilon$ to the diameter $D$. According to Eq. (4.199), it appears that roughening the surface would increase the friction factor and hence the heat transfer coefficient. Measurements by Dipprey and Sabersky [17] in tubes artificially roughened with sand grains are shown in Fig. 4.47, where Stanton number is plotted against the Reynolds number for various values of the roughness ratio, $\varepsilon/D$. At small $Re_d$, $St_d$ has the same value for rough and smooth surfaces. For each value of $\varepsilon/D$, $St_d$ reaches a maximum, and with a further increase in $Re_d$, it begins to decrease.

## 4.17 EMPIRICAL CORRELATIONS

The phenomena of turbulent forced convection are so complex that empirical correlations are used in practice in engineering design.

(a) The Dittus–Boelter equation, given below, extends the Reynolds analogy to fluids with Prandtl numbers between 0.7 and 160 by multiplying the right hand side of Eq. (4.208) by a correction factor of the form $Pr^n$:

$$\mathbf{Nu_d = \frac{\bar{h}_c D}{k} = 0.023\ Re_d^{0.8}\ Pr^n} \tag{4.210}$$

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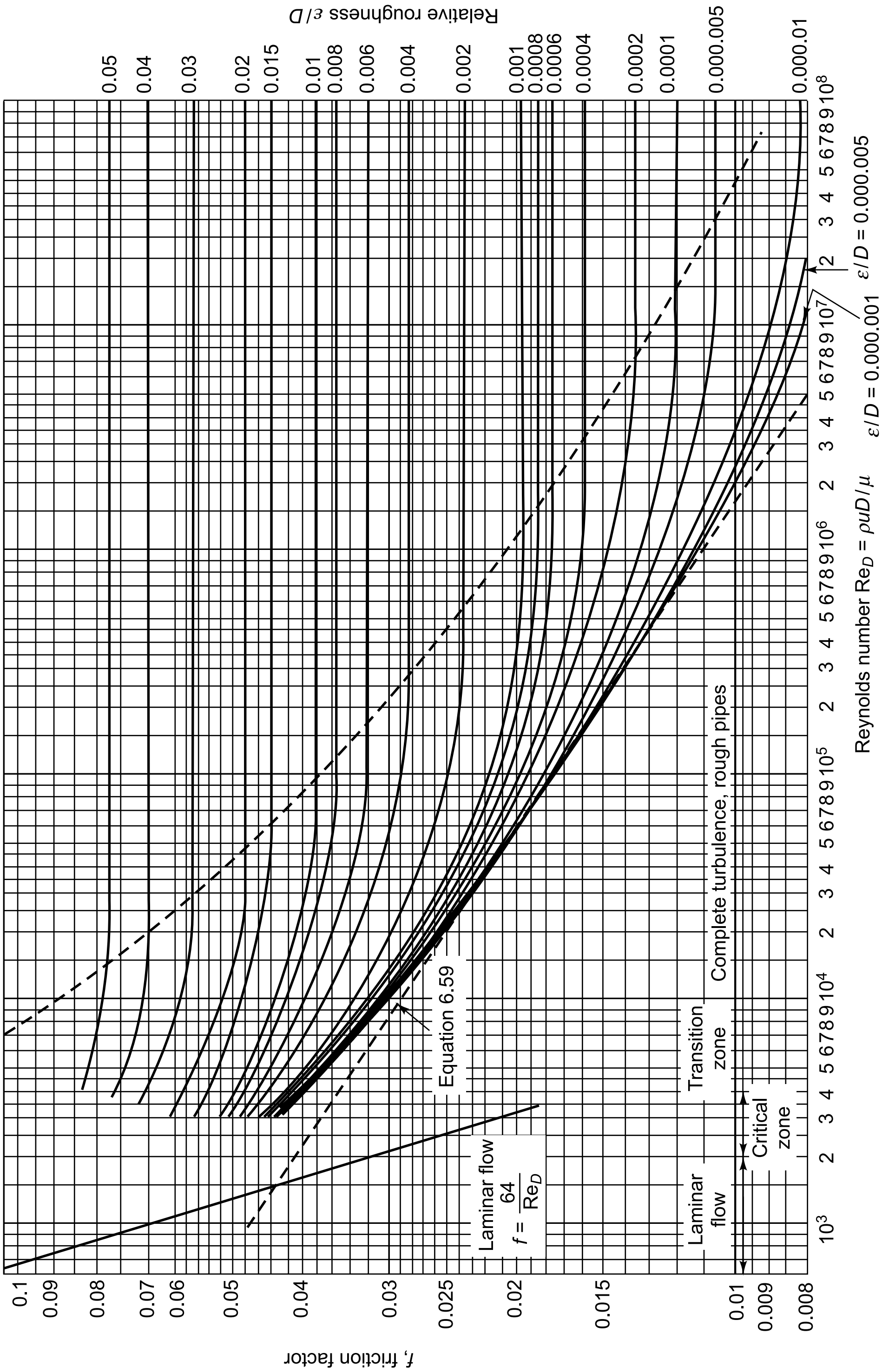


**Fig. 4.46** *Moody's diagram: friction factor varying with Reynolds number for laminar and turbulent flows in tubes with various surface roughnesses*

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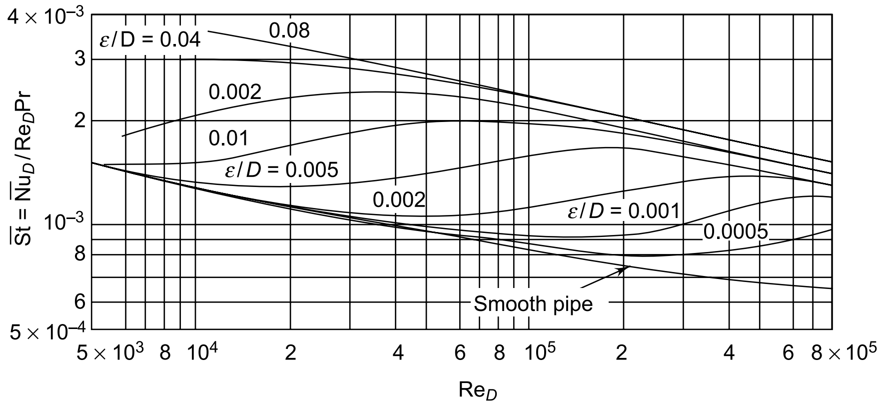


**Fig. 4.47** *Heat transfer in artificially roughened tubes, $St_d$ vs $Re_d$ for various values of $\varepsilon/D$.*

where $n$ = 0.4 for heating ($T_w > T_b$) and $n$ = 0.3 for cooling ($T_w < T_b$). It is valid within ± 20% for uniform wall temperature as well as uniform heat flux conditions within the following ranges of parameters:

$6000 < \text{Re}_d < 10^7$

$\mathbf{0.5 < Pr < 120}$

$\mathbf{L/D > 60}$

It should be used only for situations with moderate temperature differences ($T_w - T_b$), since variations in physical properties due to temperature gradient at a given cross-section are not taken into account by the correlation.

(b) For situations in which significant property variations exist due to a large temperature difference ($T_w - T_b$), a correlation developed by Sieder and Tate [19] is recommended:

$$\mathbf{Nu_d = 0.027\, Re_d^{0.8}\, Pr^{1/3} \left(\frac{\mu_b}{\mu_w}\right)^{0.14}} \tag{4.211}$$

where all properties except $\mu_w$ are evaluated at the bulk temperature. The viscosity $\mu_w$ is evaluated at the wall temperature. The correlation is valid both for uniform wall temperature and uniform heat flux in the following range of conditions:

$6000 < \text{Re}_d < 10^7$

$0.7 < \text{Pr} < 10{,}000$

$L/D > 60$

To account for the variation in physical properties due to the temperature gradient in the flow direction, the wall and bulk temperatures should be the mean values between the inlet and outlet of the duct. For noncircular ducts, hydraulic diameter $D_H$ should be used.

(c) A similar correlation, but restricted to gases was suggested by Kays and London [16] for long ducts

$$\overline{\text{Nu}}_d = C\text{Re}_d^{0.8}\,\text{Pr}^{0.3}\left(\frac{T_b}{T_w}\right)^n \tag{4.212}$$

where all properties are based on the bulk temperature $T_b$. The constant $C$ and exponent $n$ are

$$C = \begin{cases} 0.020 \text{ for uniform wall temperature, } T_w \\ 0.021 \text{ for uniform heat flux, } q''_w \end{cases}$$

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$$n = \begin{cases} 0.575 \text{ for heating} \\ 0.150 \text{ for cooling} \end{cases}$$

(d) Petukhov [31] developed a more accurate expression for fully developed turbulent flow in smooth tubes:

$$\overline{\mathrm{Nu}}_d = \frac{(f/8)\,\mathrm{Re}_d\,\mathrm{Pr}}{1.07 + 12.7(f/8)^{1/2}(\mathrm{Pr}^{2/3} - 1)}\left(\frac{\mu_b}{\mu_w}\right)^n \tag{4.213}$$

where $n = 0.11$ for $T_w > T_b$, $n = 0.25$ for $T_w < T_b$ and $n = 0$ for constant heat flux or for gases. All properties are evaluated at $T_f = (T_w + T_b)/2$ except for $\mu_b$ and $\mu_w$. The friction factor may be obtained from Moody's chart or from

$$f = (1.82 \log_{10}, \mathrm{Re}_d - 1.64)^{-2} \tag{4.213a}$$

(e) Hausen [18] presents the following empirical relation for fully developed laminar flow in tubes at constant wall temperature (as stated earlier)

$$\overline{\mathrm{Nu}}_d = 3.66 + \frac{0.0668\,(\mathrm{d}/L)\,\mathrm{Re}_d\,\mathrm{Pr}}{1 + 0.04\,[(\mathrm{d}/L)\,\mathrm{Re}_d\,\mathrm{Pr}]^{2/3}} \tag{4.214}$$

The Nusselt number approaches a constant value of 3.66 when the tube is sufficiently long.

### 4.17.1 Liquid Metals

Liquid metals, such as sodium, mercury, lead and lead-bismuth alloys have certain advantages like low melting point, high density, low vapour pressure at high temperature and high thermal conductivity (10 to 100 W/m K). These metals can be used over wide ranges of temperatures, have a large heat capacity per unit volume and have large heat transfer coefficients. They are particularly suitable for use in nuclear power plants, where large amounts of heat are released and must be removed in a small volume. Liquid metals have some safety problem in handling and pumping due to induced radioactivity. The development of electromagnetic pumps, however, has removed some of these problems.

The effect of turbulent eddies in liquid metals is of less importance compared to conduction. The temperature profile is established much more rapidly than the velocity profile. Lubarsky and Kaufman [23] proposed the following empirical relation for fully developed turbulent flow of liquid metals under uniform heat flux condition:

$$\overline{\mathbf{Nu}}_{\mathbf{d}} = \mathbf{0.625\,(Re_d\,Pr)^{0.4}} \tag{4.215}$$

for $100 < \mathrm{Re}_d\,\mathrm{Pr} < 10{,}000$, $L/D > 10$, and properties evaluated at the bulk temperature.

According to Skupinski et al. the Nusselt number for liquid metals (Na–K mixture) flowing in smooth tubes [10] can be obtained from

$$\overline{\mathrm{Nu}}_d = 4.82 + 0.0185\,(\mathrm{Re}_d\mathrm{Pr})^{0.827} \tag{4.216}$$

if the heat flux is uniform in the range of $\mathrm{Re}_d\,\mathrm{Pr} > 100$ and $L/D > 30$, with all properties evaluated at the bulk temperature.

For constant wall temperature the data are correlated, according to Seban and Shimazaki [24], by the equation

$$\overline{\mathrm{Nu}}_d = 5.0 + 0.025\,(\mathrm{Re}_d\,\mathrm{Pr})^{0.8} \tag{4.217}$$

in the range $\mathrm{RePr} > 100$, $L/D > 30$.

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### 4.17.2 Coiled Tubes

Coiled tubes are used in heat exchange equipment to obtain a large heat transfer area per unit volume and to enhance the heat transfer coefficient on the inside surface (Fig. 4.48). As a result of centrifugal forces, a secondary flow pattern consisting of two vortices perpendicular to the axial flow direction is set up and heat transfer occurs not only by diffusion in the radial direction but also by convection. The contribution of this secondary convective transfer dominates the overall process and enhances the rate of heat transfer per unit length of tube compared to a straight tube of equal length.

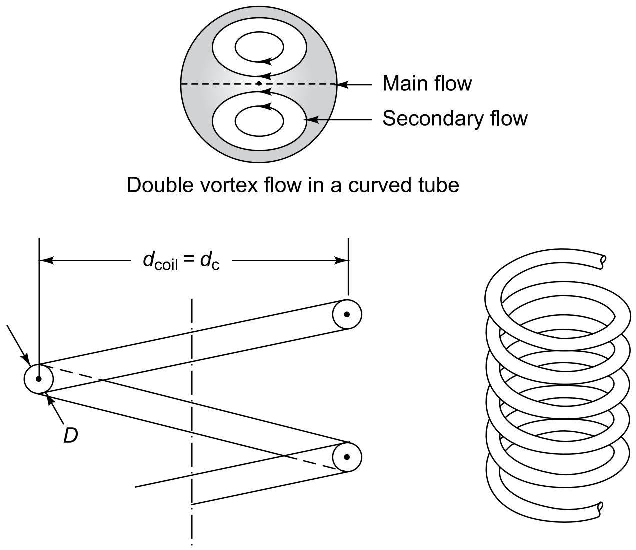


**Fig. 4.48** *Schematic diagram illustrating flow and nomenclature for heat transfer in helically coiled tubes*

The flow and heat transfer in coiled tubes are governed by *Dean number,* $\text{Dn} = \text{Re}_d [D/d_c]^{1/2}$, where $D$ is the tube diameter and $d_c$ is the coil diameter. Three regions are distinguished [25]:

1. The region of small Dean numbers, Dn < 20, where inertial forces due to secondary flow are negligible

$$\overline{\text{Nu}} = 1.7\,(\text{Dn}^2\,\text{Pr})^{1/6} \tag{4.218}$$

if $\text{Dn}^2\,\text{Pr} > 10{,}000$.

2. The region of intermediate Dean numbers, 20 < Dn < 100, in which the inertial forces due to secondary flow balance the viscous forces

$$\overline{\text{Nu}} = 0.9\,(\text{Re}^2\,\text{Pr})^{1/6} \tag{4.218a}$$

3. The region of large Dean numbers, Dn > 100, where the viscous forces are significant only in the boundary near the wall

$$\overline{\text{Nu}} = 0.7\,\text{Re}_d^{0.43}\,\text{Pr}^{1/6}\,(D/d_c)^{0.07} \tag{4.218b}$$

These equations are valid both for a uniform heat flux and a uniform wall temperature. In laminar flow, the friction factor in a coiled tube is

$$f = \frac{64}{\text{Re}_d} \cdot \frac{21.5\,\text{Dn}}{(1.56 + \log_{10}\text{Dn})^{5.73}} \tag{4.219}$$

Transition to turbulent flow occurs at

$$(\text{Re}_D)_{\text{critical}} = 2\left(\frac{D}{d_c}\right)^{0.32} \times 10^4 \tag{4.220}$$

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## 4.18 FLOW ACROSS A SINGLE CIRCULAR CYLINDER: FORCED CONVECTION OVER EXTERIOR SURFACES

Flow across a single circular cylinder is frequently encountered in practice, but the determination of the drag and heat transfer coefficients is a very complicated matter because of the complexity of the flow patterns around the cylinder. The boundary layer of a fluid flowing over the surface of a cylinder separates when the pressure rise along the surface becomes too large. Beyond the point of separation the fluid near the surface flows in a direction *opposite* to the main stream (Fig. 4.49). This reversal in the flow produces turbulent eddies. These eddies form on both sides of the cylinder and extend downstream to produce a turbulent wake at the rear of the cylinder. Fluid particles striking the cylinder at the stagnation points are brought to rest and the pressure there, $p_o$, rises equal to the velocity head $\rho u_\infty^2/2g$ over and above the free stream pressure $p_\infty$. The flow then divides, building up the boundary layer along the surface. The fluid accelerates as it flows past the cylinder surface with the crowding of streamlines (Fig. 4.50). This flow pattern for an inviscid fluid in irrotational flow (the potential flow) allows the velocity to reach a maximum at both sides of the cylinder and fall again to zero at the stagnation point in the rear. The corresponding pressure distribution for different Reynolds numbers is shown in Fig. 4.51. The flow pattern around the cylinder undergoes a series of

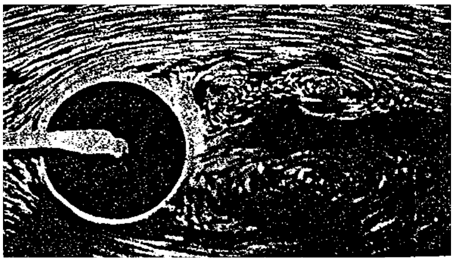

**Fig. 4.49** *Flow pattern in cross flow over a single horizontal cylinder*

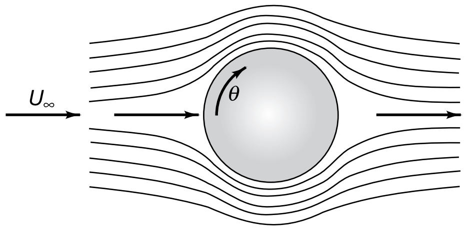


**Fig. 4.50** *Streamlines for potential flow over a circular cylinder*

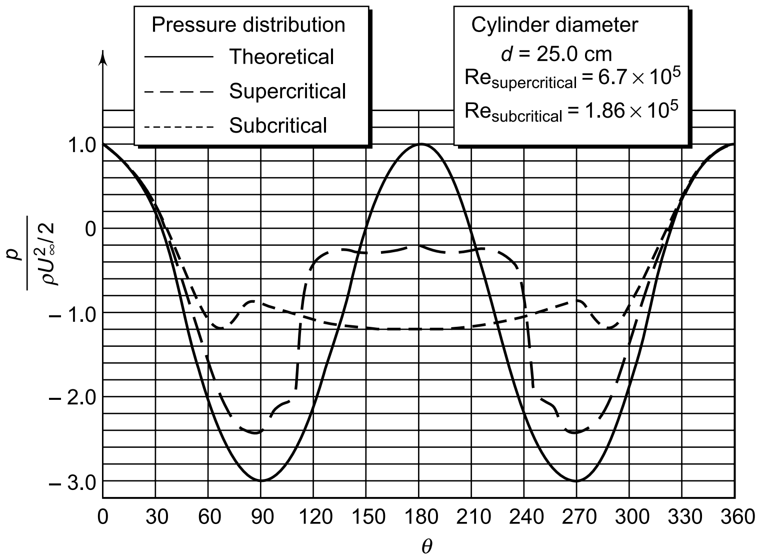


**Fig. 4.51** *Pressure distribution around a circular cylinder in cross flow at various Reynolds numbers*

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changes as the Reynolds number is increased (Fig. 4.52) and the heat transfer essentially depends on this flow pattern.

The total drag coefficient $C_D$ is defined by

$$C_D = \frac{F}{A_f\ \rho u_\infty^2/2}$$

where $F$ = the sum of pressure and frictional forces,

$A_f$ = frontal projected area = $\pi DL$ (cylinder) or $\frac{\pi}{4}D^2$ (sphere)

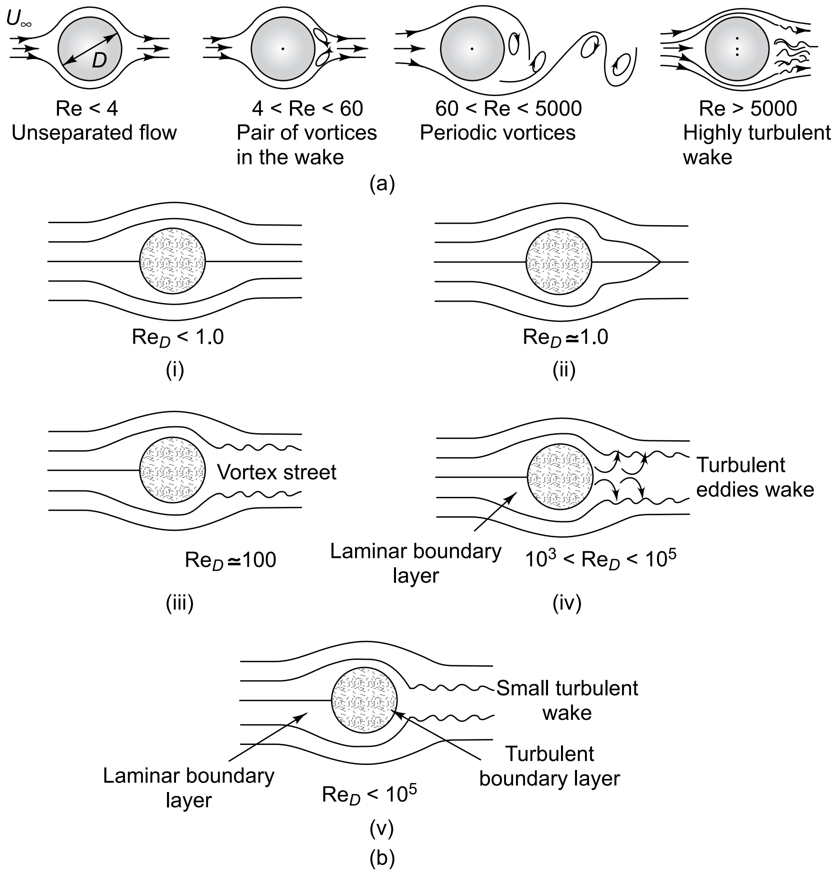


**Fig. 4.52** *Flow patterns for cross flow over a circular cylinder at various Reynolds numbers*

$D$ = outside diameter of cylinder or sphere

$L$ = cylinder length

Figure 4.53 shows the drag coefficient varying with Reynolds number for long cylinders and spheres. With reference to Figs 4.52(b) and 4.53 at $\text{Re}_d \leq 1.0$, the flow adheres to the surface and the streamlines follow those predicted from potential flow theory. The inertia forces are negligibly small, and the drag is caused only by viscous forces, since there is no flow separation. Heat is transferred by conduction alone.

At $\text{Re}_d \simeq 10$, the inertia forces become appreciable and two weak eddies stand in the rear of the cylinder. The pressure drag accounts now for about half of the total drag.

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At $Re_d \simeq 100$, vortices separate alternatively from the two sides of the cylinder and stretch a considerable distance downstream. These vortices are referred to as *von Kármán vortex streets* after Theodore von Kármán who studied the shedding of vortices from bluff bodies. The pressure drag now dominates.

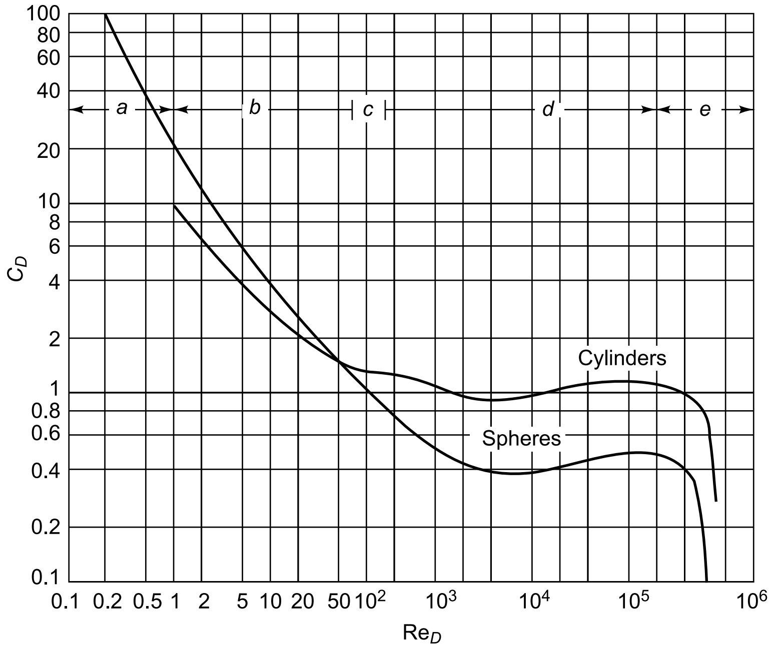


**Fig. 4.53** *Drag coefficient varying with Reynolds number for cylinders and spheres in unconfined cross flow*

For $10^3 \le Re_d \le 10^5$, the skin friction drag becomes negligible compared to the pressure drag caused by turbulent eddies in the wake. The drag coefficient remains approximately constant because the boundary layer remains laminar from the leading edge to the point of separation with $\theta$ between 80° and 85° measured from the direction of the flow.

For $Re_d > 10^5$, the flow in the boundary layer becomes turbulent while it is still attained, and the separation point moves toward the rear.

Over the forward portion of the cylinder ($0 < \theta < 80°$), the empirical equation for $h_c(\theta)$, the local value of the heat transfer coefficient,

$$\overline{Nu}(\theta) = \frac{h_c(\theta)D}{k} = 1.14\left(\frac{\rho u_\infty D}{\mu}\right)^{0.5} (Pr)^{0.4} \left[1-\left(\frac{\theta}{90}\right)^3\right] \tag{4.221}$$

Giedt [27] measured the local pressures and local heat transfer coefficients over the entire circumference of a long 10.2 cm outer diameter cylinder in an air stream over a $Re_d$ varying from 70,000 to 220,000. Giedt's results are shown in Fig. 4.54, and similar data for lower Reynolds numbers are shown in Fig. 4.55. At $Re_d$ < 100, 000, separation of the laminar boundary layer occurs at $\theta$ of about 80°. The local heat transfer is largest at the stagnation point and decreases with distance along the surface as the boundary layer thickness increases. The heat transfer reaches a minimum on the sides of the cylinder near the separation point. Beyond the separation point, the local heat transfer coefficient increases because of eddies.

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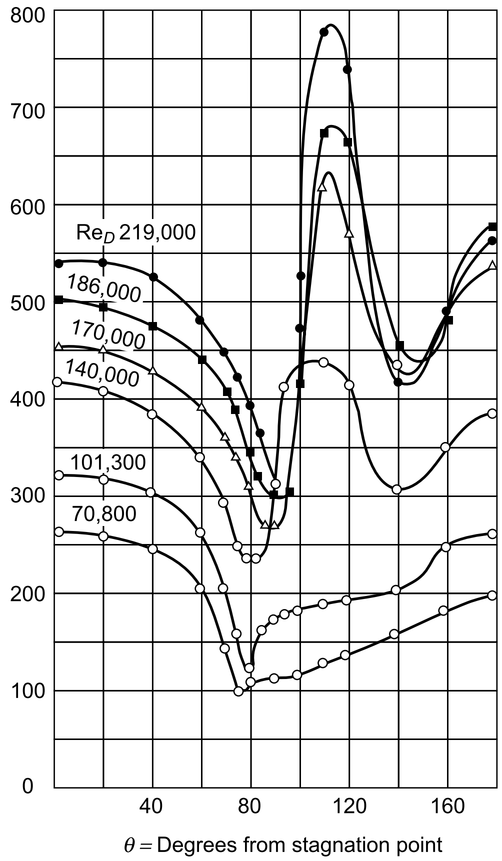


**Fig. 4.54** *Circumferential variation of heat transfer coefficient at high Reynolds numbers for a circular cylinder in cross flow*

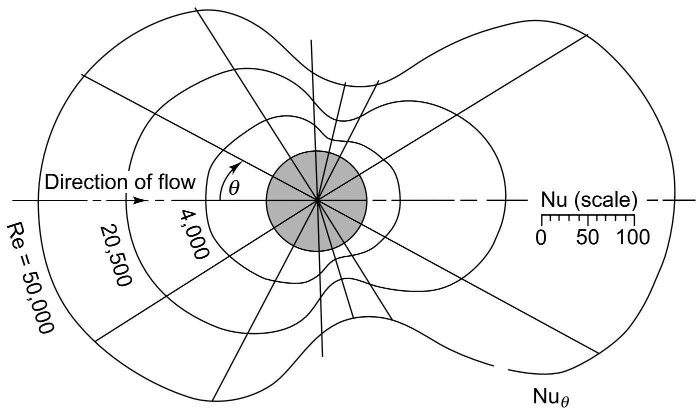


**Fig. 4.55** *Circumferential variation of local Nusselt number at low Reynolds numbers for a circular cylinder in cross flow*

A correlation for a cylinder at uniform temperature $T_w$ in cross-flow of liquids and gases has been proposed by Zukauskas [28]:

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$$\overline{\text{Nu}}_d = \frac{h_c D}{k} = C\left(\frac{u_\infty D}{\nu}\right)^m \text{Pr}^n \left(\frac{\text{Pr}}{\text{Pr}_w}\right)^{0.25} \tag{4.222}$$

where all fluid properties are evaluated at the free-stream fluid temperature $T_\infty$ except for $\text{Pr}_w$ which is evaluated at the wall temperature. The constants are given below.

| $Re_d$ | $C$ | $m$ |
|---|---|---|
| 1 – 40 | 0.75 | 0.4 |
| $40 - 1 \times 10^3$ | 0.51 | 0.5 |
| $1 \times 10^3 - 2 \times 10^5$ | 0.26 | 0.6 |
| $2 \times 10^5 - 1 \times 10^6$ | 0.076 | 0.7 |

For Pr < 10, $n = 0.37$, and for Pr > 10, $n = 0.36$.

Whitaker [see Ozisik] correlated the average heat transfer coefficient $h_m$ for the flow of gases or liquids across a single cylinder by

$$\text{Nu}_m = \frac{h_m D}{k} = (0.4\,\text{Re}^{0.5} + 0.06\,\text{Re}^{2/3})\,\text{Pr}^{0.4}(\mu_\infty/\mu_w)^{0.25} \tag{4.223}$$

which agrees with the experimental data within 25% in the range of variables: $40 < \text{Re} < 10^5$, $0.67 < \text{Pr} < 300$, $0.25 < \frac{\mu_\infty}{\mu_w} < 5.2$, where the physical properties are evaluated at $T_\infty$, except $\mu_w$ evaluated at $T_w$. The functional dependence $\text{Re}^{0.5}$ characterizes the contribution from the undetached laminar boundary region, and $\text{Re}^{2/3}$ characterizes the contribution from the wake region around the cylinder.

A more elaborate but more general correlation is given by Churchill and Bernstein [see Ozisik] for the average heat transfer coefficient $h_m$ for flow across a single cylinder as

$$\text{Nu}_m = 0.3 + \frac{0.62\,\text{Re}^{1/2}\,\text{Pr}^{1/3}}{\left[1 + (0.4/\text{Pr})^{2/3}\right]^{1/4}}\left[1 + \left(\frac{\text{Re}}{282000}\right)^{1/2}\right] \tag{4.224}$$

for 20,000 < Re < 400,000.

### 4.18.1 Flow Across a Sphere

A knowledge of heat transfer characteristics to or from spherical bodies is important for predicting the thermal performance of systems where clouds of particles are heated or cooled in a stream of fluid, as in packed or fluidised beds. In the limit of very small Reynoldo numbers (creeping flow), the drag coefficient is inversely proportional to Reynolds number and the specific relation is termed Stoke's law: $C_D = \frac{24}{\text{Re}_d}$, for $\text{Re}_d < 0.5$. The total drag coefficient of a sphere as a function of Reynolds number is shown in Fig. 4.53 and corresponding data for heat transfer are shown in Fig. 4.56. For $25 \le \text{Re}_b \le 100{,}000$, the equation recommended by McAdams [29] for estimating the average heat transfer coefficient for spheres heated or cooled by a gas is

$$\overline{\text{Nu}}_d = \frac{h_c D}{k} = 0.37\left(\frac{\rho D u_\infty}{\mu}\right)^{0.6} = 0.37\,\text{Re}_d^{0.6} \tag{4.225}$$

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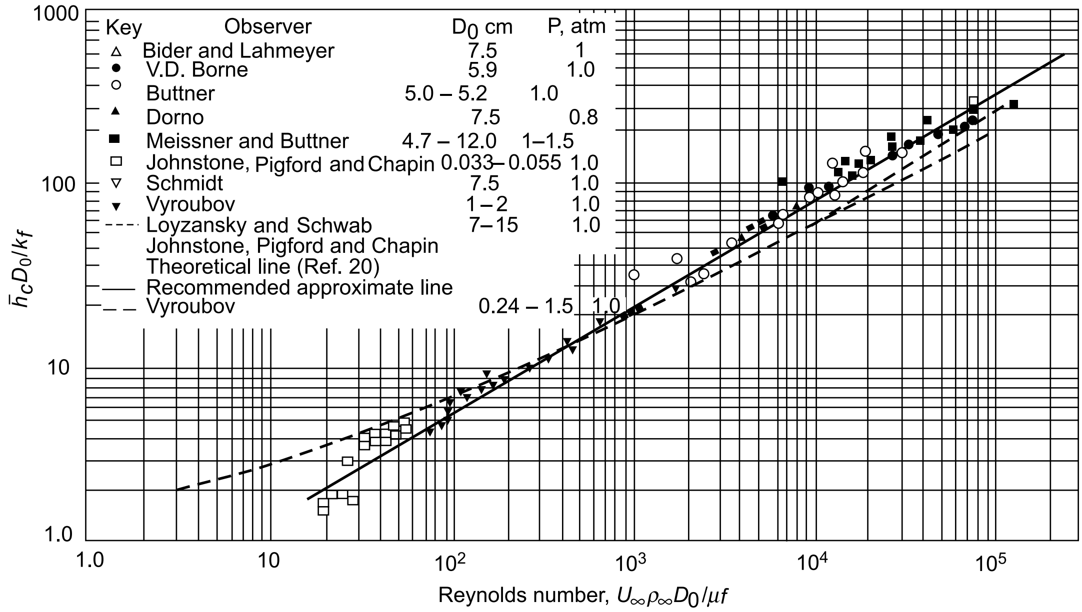


**Fig. 4.56** *Correlation of experimental heat transfer coefficients for flow over a sphere (from McAdams [29])*

For $1 < \mathrm{Re}_d < 25$, the equation

$$\bar{h}_c = c_p u_\infty \rho \left( \frac{2.2}{\mathrm{Re}_d} + \frac{0.48}{\mathrm{Re}_d^{0.5}} \right) \tag{4.226}$$

can be used for heat transfer in a gas. For heat transfer in liquids as well as gases, the equation

$$\overline{\mathrm{Nu}}_d = \frac{\bar{h}_c D}{k} = 2 + (0.4\,\mathrm{Re}_d^{0.5} + 0.06\,\mathrm{Re}_d^{0.67})\,\mathrm{Pr}^{0.4} \left( \frac{\mu_\infty}{\mu_w} \right)^{0.25} \tag{4.227}$$

correlates available data in the Reynolds number ranges between 3.5 and $7.6 \times 10^4$ and Prandtl numbers between 0.7 and 380.

A special case of convective heat (and mass) transfer from spheres relates to transport from freely falling liquid drops and correlation often used is

$$\mathrm{Nu}_d = 2 + 0.6\,\mathrm{Re}_d^{1/2}\,\mathrm{Pr}^{1/3} \tag{4.228}$$

In the limit $\mathrm{Re}_d \to 0$, both the above equations reduce to $\mathrm{Nu}_d = 2$. It is valid for heat transfer by conduction from a sphere to a stationary medium around the surface.

### 4.18.2 Tube Bundles in Cross Flow

Heat transfer and pressure drop characteristics of tube bundles have numerous applications in the design of heat exchangers and industrial heat transfer equipment like air preheaters, economisers and superheaters in steam generators, and shell-and-tube heat exchangers. Heat transfer and pressure drop data for a large number of these heat exchangers have been compiled by Kays and London [16].

The heat transfer in flow over tube bundles depends largely on the flow pattern and the degree of turbulence, which in turn are functions of the velocity of the fluid and the size and arrangement of the tubes. The tube bundles can be in *in-line and staggered arrangements* (Figs 4.57 and 4.58). The bundle

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geometry is characterised by the *transverse pitch* $S_T$ and the *longitudinal pitch* $S_L$ between the tube centres. The *diagonal pitch* $S_D$ is sometimes used for the staggered arrangement. To define the Reynolds number for flow through the tube bank, the flow velocity is based on the *minimum flow area*

$$\mathrm{Re}_d = \frac{DG_{\max}}{\mu} \tag{4.229}$$

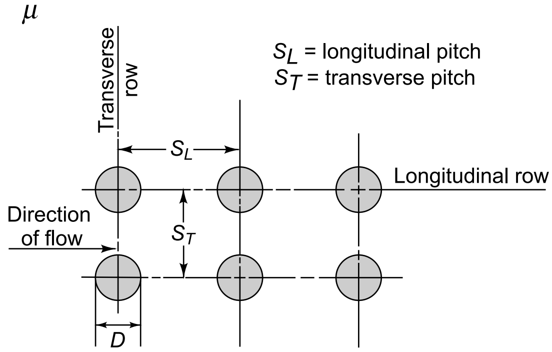


**Fig. 4.57** *In-line tube arrangement in cross flow*

where $G_{\max} = \rho u_{\max}$ = maximum mass flow velocity, $u_{\max}$ being the maximum velocity based on the minimum flow area available for flow. If $u_\infty$ is the fluid velocity at entry to the tube bank, then the in-line arrangement,

$$u_{\max} = u_\infty \frac{S_T}{S_T - D} = u_\infty \frac{S_T/D}{S_T/D - 1} \tag{4.230}$$

where $S_T$ is the transverse pitch. For staggered arrangement,

$$u_{\max} = u_\infty \frac{S_T}{2\,(S_D - D)} = \frac{1}{2} u_\infty \frac{S_T/D}{S_D/D - 1} \tag{4.231}$$

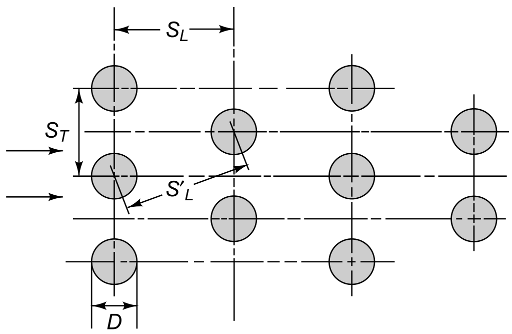


**Fig. 4.58** *Staggered tube arrangement in cross flow*

The Reynolds number is

$$\mathrm{Re}_d = \frac{u_{\max} D}{\nu}$$

Zukauskas [28] developed correlation for $0.7 < \mathrm{Pr} < 500$ in the form

$$\overline{\mathrm{Nu}}_d = C\,\mathrm{Re}_d^m\,\mathrm{Pr}^{0.36} \left(\frac{\mathrm{Pr}}{\mathrm{Pr}_w}\right)^{0.25} \tag{4.232}$$

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where the fluid properties are evaluated at the bulk fluid temperature except $Pr_w$ which is evaluated at the tube wall temperature.

For laminar flow in the range $10 < Re_d < 100$

$$\overline{Nu}_d = 0.8\ Re_d^{0.4}\ Pr^{0.36}\left(\frac{Pr}{Pr_w}\right)^{0.25} \tag{4.233}$$

for in-line tubes and

$$\overline{Nu}_d = 0.9\ Re_d^{0.4}\ Pr^{0.36}\left(\frac{Pr}{Pr_w}\right)^{0.25} \tag{4.234}$$

for staggered tubes.

In the transition regime $10^3 < Re_d < 2 \times 10^5$,

$$\overline{Nu}_d = 0.27\ Re_d^{0.63}\ Pr^{0.36}\left(\frac{Pr}{Pr_w}\right)^{0.25} \tag{4.235}$$

for in-line tubes and

$$\overline{Nu}_d = 0.35\left(\frac{S_T}{S_L}\right)^{0.2} Re_d^{0.60}\ Pr^{0.36}\left(\frac{Pr}{Pr_w}\right)^{0.25} \tag{4.236}$$

for staggered tubes with $S_T/S_L < 2$ and

$$\overline{Nu}_d = 0.40\ (Re_d)^{0.60}\ Pr^{0.36}\left(\frac{Pr}{Pr_w}\right)^{0.25} \tag{4.237}$$

for staggered tubes with $S_T/S_L \geq 2$.

In turbulent regime, $Re_d > 2 \times 10^5$,

$$\overline{Nu}_d = 0.021\ Re_d^{0.84}\ Pr^{0.36}\left(\frac{Pr}{Pr_w}\right)^{0.25} \tag{4.238}$$

for in-line tubes and

$$\overline{Nu}_d = 0.022\ Re_d^{0.84}\ Pr^{0.36}\left(\frac{Pr}{Pr_w}\right)^{0.25} \tag{4.239}$$

for staggered tubes with Pr > 1, and if Pr = 0.7

$$\overline{Nu}_d = 0.019\ Re_d^{0.84} \tag{4.240}$$

Achenbach [30] extended the tube-bundle data upto $Re_d = 7 \times 10^6$ for staggered arrangement with $S_T/D = 2$ and lateral pitch $S_L/D = 1.4$. His correlation is

$$\overline{Nu}_d = 0.0131\ Re_d^{0.883}\ Pr^{0.36} \tag{4.241}$$

which is valid in the range $4.5 \times 10^5 < Re_d < 7 \times 10^6$.

## 4.19 HEAT TRANSFER ENHANCEMENT

Considerations to save energy and material in recent years have spurred research at producing more efficient heat exchange equipment by augmentation of heat transfer. The potentials of heat transfer augmentation in engineering applications are numerous. For example, the heat exchanger for a projected ocean thermal energy conversion (OTEC) plant or a desalination plant requires a heat transfer surface area on the order of 10,000 m$^2$/MW(*e*). Clearly, an increase in the efficiency or the heat exchanger through augmentation may

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result in considerable savings in the material requirement. A large amount of literature is available on the subject [33]. We discuss briefly principal augmentation techniques for single-phase forced flow in ducts.

We recall the analogies between momentum and heat transfer. Increasing the friction factor increases the heat transfer coefficient. The Moody chart (Fig. 4.46, p. 316), for example, shows that in turbulent flow increasing the relative roughness of the surface increases the friction factor. This chart is based on the random sand-grain type of surface roughness. Other types of surface roughness have been produced and their friction factors and heat transfer characteristics have been tested for possible use in heat transfer augmentation as discussed below.

#### (a) Roughened surfaces

Surface roughness can be produced by the machining of the surface (like knurling) as well as by casting, forming, and welding processes. A large number of geometric configurations are possible, each having its own heat transfer and pressure drop characteristics.

#### (b) Extended surfaces

The use of fins on the outer surface of tubes to enhance heat transfer is well known. Internally finned tubes have also been used to enhance heat transfer to fluids flowing inside tubes.

#### (c) Coiled tubes

Coil tubes can serve as a heat transfer enhancement device because the secondary flow produced by the curvature causes an increase in the heat transfer coefficient.

#### Optimization Method

The increase in heat transfer with augmentation is accompanied by an increase in friction factor. In some situations, the heat transfer coefficients are increased at most about 4 times while the friction factors are increased as much as 50 times [33]. An increased friction factor implies an increased power for pumping the fluid. So the results of angmentation in enhancing heat transfer should be weighed against the increased power requirement for pumping the fluid. For a given heat transfer technique if the heat transfer and the friction factor data are available as a function of Reynolds number, it may be possible to optimize the system to reduce the heat transfer surface, to obtain increased heat transfer capacity, or to reduce the power required for pumping the fluid. The manufacturing procedure, manufacturing cost, materials, and factors like increase in operating cost associated with the augmentation technique used should be considered before a final dicision is made.

## Solved Exampes

**Example 4.1** An application involving exact solution of momentum and energy equations is met within a situation called Couette flow with heat transfer. Couette flow provides a simple model for flow between two parallel plates. The lower plate is stationary and the upper plate at a distance $L$ is moving with a velocity $U$. The lower and upper plates are maintained at uniform temperatures $T_0$ and $T_L$ respectively. This model can be applied to the case of a shaft rotating in its stationary bearing with a heavy lubricating oil in the clearance. If the clearance is small, the situation can be considered as that of flow between two parallel plates. Determine the velocity and temperature distribution between the two plates. Find also the surface heat fluxes to the plates.

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**Solution** With reference to Fig. Ex. 4.1, the continuity equation for steady-state incompressible flow

$$\frac{\partial u}{\partial x} + \underbrace{\cancel{\frac{\partial v}{\partial y}}}_{=0} + \underbrace{\cancel{\frac{\partial w}{\partial z}}}_{=0} = 0$$

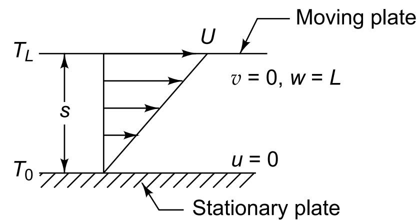


**Fig.** *Ex. 4.1*

$$\frac{\partial u}{\partial x} = 0 \text{ or } u \neq f(x)$$

The velocity does not vary with $x$.

The $x$-momentum equation.

$$\rho\left[\underbrace{\cancel{\frac{\partial u}{\partial t}}}_{=0} + \underbrace{u\cancel{\frac{\partial u}{\partial x}}}_{=0} + \underbrace{v\cancel{\frac{\partial v}{\partial y}}}_{=0} + \underbrace{w\cancel{\frac{\partial w}{\partial z}}}_{=0} = \underbrace{F_x}_{=0}\ \underbrace{\cancel{\frac{\partial p}{\partial x}}}_{=0} + \mu\nabla^2 u\right] = 0$$

$$\nabla^2 u = 0$$

or $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

$$\frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial u}{\partial y} = C_1 y$$

or, $u = C_1 y + C_3$, which is linear,

At $y = 0$, $u = 0$, $\quad \therefore C_2 = 0$

$y = L$, $U = u$, $\quad U = C_1 L \quad$ or $\quad C_1 = U/L$

The velocity distribution in Couette flow is then given by

$$u(y) = \frac{U}{L} y$$

or, $$\frac{u}{U} = \frac{y}{L} \qquad \textit{Ans.}$$

The energy equation can be written as

$$k\nabla^2 T + \mu\phi = \rho c_p \frac{DT}{Dt}$$

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$$k\frac{\partial^2 T}{\partial y^2}+\mu\left(\frac{\partial u}{\partial y}\right)^2 = \rho c_p\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}\right)$$

$$k\frac{\partial^2 T}{\partial y^2}+\mu\left(\frac{U}{L}\right)^2 = 0$$

$$\frac{d^2T}{dy^2} = -\frac{\mu}{k}\left(\frac{U}{L}\right)^2$$

$$\frac{dT}{dy} = -\frac{\mu}{k}\left(\frac{U}{L}\right)^2 y + C_3$$

$$T = -\frac{\mu}{2k}\left(\frac{U}{L}\right)^2 y_2 + C_3 y + C_3$$

At $\quad y = 0,\ T = T_o, \quad \therefore\ C_4 = T_o$

At $\quad y = L,\ T = T_L, \quad \therefore\ T_L = -\frac{\mu}{2k}\left(\frac{U}{L}\right)^2 L^2 + C_3 L + T_o$

$$C_3 = \frac{T_L - T_o}{L} + \frac{\mu}{2k}\frac{U^2}{L}$$

$$\therefore \quad T(y) = -\frac{\mu}{2k}\left(\frac{U}{L}\right)^2 y^2 + \frac{T_L - T_o}{L}y + \frac{\mu}{2k}\frac{U^2}{L}y + T_o$$

or

$$T(y) = T_o + \frac{\mu}{2k}U^2\left(\frac{y}{L}-\frac{y^2}{L^2}\right)+(T_L - T_o)\frac{y}{L}$$

In dimensionless form,

$$\frac{T(y)-T_o}{T_L - T_o} = \frac{y}{L}+\frac{\mu}{2k}U^2\frac{(y/L - y^2/L^2)}{(T_L - T_o)}$$

$$= \frac{y}{L}\left[1+\frac{\mu U^2}{2k(T_L - T_o)}\left(1-\frac{y}{L}\right)\right]$$

Let, $\quad \frac{y}{L} = \eta,\ \Pr = \frac{\mu c_p}{k},\ E = \text{Eckert number} = \frac{U^2}{c_p\,(T_L - T_o)}$

and $\quad \theta(\eta) = \frac{T(y)-T_o}{T_L - T_o};\ \theta(\eta) = \eta\left[1+\frac{1}{2}\Pr.E(1-\eta)\right]$

This is the temperature distribution in Couette flow. Because of viscous dissipation, the maximum temperature occurs in the fluid and heat transfer occurs both to the hot and cold plates. For Pr. $E = 0$, there is no flow ($U = 0$) and hence there is no viscous dissipation, and the temperature distribution is linear. The surface heat fluxes can be obtained by Fourier's law:

$$q(y) = -k\frac{dT}{dy} = -k\left[\frac{\mu}{2k}U^2\left(\frac{1}{L}-\frac{1}{L^2}2y\right)+\frac{T_L - T_o}{L}\right]$$

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$$q_{y=0} = -k\frac{\mu}{2k}\frac{U^2}{L} - \frac{(T_L - T_o)k}{L} = -\frac{\mu U^2}{2L} - \frac{k}{L}(T_L - T_o) \text{ Bottom plate}$$

$$q_{y=L} = k\frac{\mu}{2k}\frac{U^2}{L} - \frac{k}{L}(T_L - T_o) = \frac{\mu U^2}{2L} - \frac{k}{L}(T_L - T_o) \text{ Top plate}$$

These are the heat fluxes.

**Example 4.2** A heavy lubricating oil ($\mu$ = 0.8 N s/m$^2$, $k$ = 0.15 W/mK) flows in the clearance between a shaft and its bearing. If the bearing and shaft are kept at 10°C and 30°C respectively and the clearance between them is 2 mm, determine the maximum temperature in the oil and the heat flux to the plates for a velocity $U$ = 6 m/s.

**Solution** Because of small clearance between the shaft and its bearing, the flow between them may be assumed as Couette flow. The surface heat fluxes are

$$q(0) = -\frac{\mu U^2}{2L} - \frac{k}{L}(T_L - T_o)$$

$$= -\frac{0.8\times 6^2}{2\times 2\times 10^{-3}} - \frac{0.15}{2\times 10^{-3}}(30-10)$$

$$= -7200 - 1500 = -8700 \text{ W/m}^2 = -8.7 \text{ kW/m}^2$$

$$q(L) = \frac{\mu U^2}{2L} - \frac{k}{L}(T_L - T_o) = 7200 - 1500$$

$$= 5700 \text{ W/m}^2 = 5.7 \text{ kW/m}^2$$

The maximum temperature in the oil will occur where $dT/dy = 0$

$$-\frac{\mu}{k}\left(\frac{U}{L}\right)^2 y + C_3 = 0$$

$$\frac{\mu}{k}\left(\frac{U}{L}\right)^2 y = \frac{T_L - T_o}{L} + \frac{\mu}{2k}\frac{U^2}{L}$$

$$y = \frac{T_L - T_o}{L}\frac{kL^2}{\mu U^2} + \frac{\mu}{2k}\frac{U^2}{L}\frac{kL^2}{\mu U^2}$$

$$= \left[\frac{k}{\mu U^2}(T_L - T_o) + \frac{1}{2}\right] L$$

$$y = \left[\frac{0.15}{0.8\times 36}(30-10) + \frac{1}{2}\right] L = 0.604\, L$$

$$T_{\max} = T_o + \frac{\mu}{2k}U^2\left(\frac{y}{L} - \frac{y^2}{L^2}\right) + \frac{T_L - T_o}{L}y$$

$$= 10 + \frac{0.8}{2\times 0.15}\times 36\,[0.604 - (0.604)^2] + (30-10)\,0.604$$

$$= 10 + 22.944 + 12.8 = 45.74°\text{C} \quad \textit{Ans.}$$

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**Example 4.3** Assume a linear temperature profile $T = C + dy$ for flow of a fluid over a flat plate.

(a) Apply the appropriate boundary condition and express $T$ in terms of $\delta_t$, $T_w$ and $T_\infty$.

(b) Assume a linear velocity profile

$$u = a + by$$

Obtain an expression for $\delta/\delta_t$ as a function of Prandtl number.

(c) Obtain an expression for $Nu_x$.

**Solution** Writing the energy balance equation for the control volume (Fig. Ex. 4.3.1),

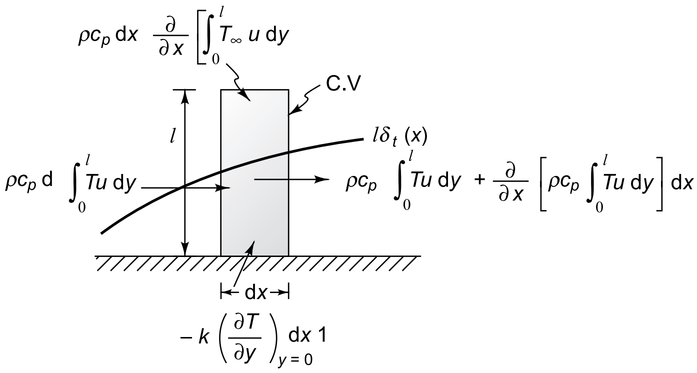


**Fig.** Ex. 4.3.1

$$\rho c_p \left(\frac{\partial}{\partial x}\int_0^1 Tu \, dy\right) dx - \rho c_p \, dx \frac{\partial}{\partial x}\int_0^1 T_\infty u \, dy + k\left(\frac{\partial T}{\partial y}\right)_{y=0} dx = 0$$

Since for $y \geq \delta_t$, the integrand is zero. Therefore, the energy equation can be written as

$$\frac{\partial}{\partial x}\int_0^{\delta_t} (T_\infty - T)u \, dy = \alpha\left(\frac{\partial T}{\partial y}\right)_{y=0} \quad (1)$$

The temperature profile is given to be

$$T = C + dy$$

$$\left(\frac{\partial T}{\partial y}\right)_{y=0} = d$$

At $\quad y = 0, \; T = T_w = C$

At $\quad y = \delta_t, \; T = T_\infty$

$$T_\infty = T_w + d\,\delta_t$$

$$d = (T_\infty - T_w)/\delta_t$$

The temperature distribution is then given by

$$T = T_w + \frac{T_\infty - T_w}{\delta_t} y$$

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or, $$\frac{T - T_w}{T_\infty - T_w} = \frac{y}{\delta_t} \quad (2) \quad \textit{Ans.}\ (a)$$

$$\frac{1}{T_\infty - T_w}\left(\frac{\partial T}{\partial y}\right)_{y=0} = \frac{1}{\delta_t}$$

$$\left(\frac{\partial T}{\partial y}\right)_{y=0} = \frac{T_\infty - T_w}{\delta_t} \tag{2}$$

Substituting in Eq. (1)

$$u_\infty \frac{\partial}{\partial x}\int_0^{\delta_t}[(T_\infty - T_w) - (T - T_w)]\frac{u}{u_\infty}\,\mathrm{d}y = \alpha\frac{T_\infty - T_w}{\delta_t}$$

$$u_\infty \frac{\partial}{\partial x}\int_o^{\delta}\left(1 - \frac{T - T_w}{T_\infty - T_w}\right)\frac{u}{u_\infty}\,\mathrm{d}y = \frac{\alpha}{\delta_t} \tag{3}$$

The velocity profile has been given to be

$$u = a + by$$

At $y = 0,\ u = 0, \quad \therefore a = 0$

$y = \delta,\ u = u_\infty = b\delta$

$\therefore \quad b = u_\infty/\delta$

$$\therefore \quad u = \frac{u_\infty}{\delta}y$$

or, $$\frac{u}{u_\infty} = \frac{y}{\delta} \tag{4}$$

Substituting Eqs (2) and (4) in Eq. (3),

$$u_\infty \frac{\partial}{\partial x}\int_o^{\delta_t}\left(1 - \frac{y}{\delta_t}\right)\frac{y}{\delta}\,\mathrm{d}y = \frac{\alpha}{\delta_t}$$

Let $\zeta = \delta_t/\delta$, so that $\delta_t = \zeta\delta$

$$u_\infty \frac{\partial}{\partial x}\int_o^{\delta_t}\left(1 - \frac{y}{\delta_t}\right)\frac{\zeta y}{\delta_t}\,\mathrm{d}y = \frac{\alpha}{\delta_t}$$

or $$u_\infty \frac{\partial}{\partial x}\int_o^{\delta_t}\left(\frac{\zeta y}{\delta_t} - \frac{\zeta y^2}{\delta_t^2}\right)\mathrm{d}y = \frac{\alpha}{\delta_t}$$

or $$u_\infty \frac{\partial}{\partial x}\left(\frac{\zeta}{\delta_t}\frac{\delta_t^2}{2} - \frac{\zeta}{\delta_t^2}\frac{\delta_t^3}{3}\right) = \frac{\alpha}{\delta_t}$$

$$\frac{u_\infty}{6}\frac{\mathrm{d}}{\mathrm{d}x}(\zeta^2\delta) = \frac{\alpha}{\zeta\delta}$$

or $$\frac{u_\infty}{6}\left(\zeta^2\delta\frac{\mathrm{d}\delta}{\mathrm{d}x} + \delta 2\zeta\frac{\mathrm{d}\zeta}{\mathrm{d}x}\right) = \frac{\alpha}{\zeta\delta}$$

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or $$\frac{u_\infty}{6}\left(\zeta^3\delta\frac{d\delta}{dx}+2\zeta^2\delta^2\frac{d\zeta}{dx}\right)=\alpha \tag{5}$$

From the $x$-momentum equation, we obtain

$$\rho\frac{\partial}{\partial x}\int_o^\delta (u_\infty-u)u\,dy=\tau_w=\mu\left(\frac{\partial u}{\partial y}\right)_{y=0}$$

or $$\rho u_\infty^2\frac{\partial}{\partial x}\int_o^\delta\left(1-\frac{u}{u_\infty}\right)\frac{u}{u_\infty}dy=\mu\frac{u_\infty}{\delta}$$

or $$\rho u_\infty\frac{d}{dx}\int_o^\delta\left(\frac{y}{\delta}-\frac{y^2}{\delta^2}\right)dy=\frac{\mu}{\delta}$$

or $$\rho u_\infty\frac{d}{dx}\left(\frac{1}{\delta}\frac{\delta^2}{2}-\frac{1}{\delta^2}\cdot\frac{\delta^3}{3}\right)=\frac{\mu}{\rho}$$

or $$\frac{\rho u_\infty}{6}\frac{d\delta}{dx}=\frac{\mu}{\delta}$$

$$\delta\frac{d\delta}{dx}=\frac{6\mu}{\rho u_\infty}=\frac{6\nu}{u_\infty} \tag{6}$$

$$\int_o^\delta \delta\,d\delta=\frac{6\nu}{u_\infty}\int_o^\delta dx$$

or $$\frac{\delta^2}{2}=\frac{6\nu x}{u_\infty}$$

$$\frac{\delta^2}{x^2}=\frac{12\nu}{u_\infty x}=\frac{12}{\mathrm{Re}_x}$$

$$\frac{\delta}{x}=\frac{3.464}{(\mathrm{Re}_x)^{1/2}} \tag{7}$$

Substituting Eq. (6) in Eq. (5),

$$\frac{u_\infty}{6}\left(\zeta^3\frac{6\nu}{u_\infty}+2\zeta^2\frac{12\nu x}{u_\infty}\frac{d\zeta}{dx}\right)=\alpha$$

$$\zeta^3+4\zeta^2x\frac{d\zeta}{dx}=\frac{1}{\mathrm{Pr}} \tag{8}$$

or $$\zeta^3=-\frac{4}{3}x\frac{d\zeta^3}{dx}=\frac{1}{\mathrm{Pr}}$$

Putting $$\zeta^3=y,\ y+\frac{4}{3}x\frac{dy}{dx}=\frac{1}{\mathrm{Pr}}$$

Particular integral: $y=\dfrac{1}{\mathrm{Pr}}$

Complementary function: $y+\dfrac{4}{3}x\dfrac{dy}{dx}=0$

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or
$$\frac{dy}{y} = -\frac{3}{4}\frac{dx}{x}$$

$$y = Cx^{-3/4}$$

$$\therefore \qquad \zeta^3 = Cx^{-3/4} + \frac{1}{\text{Pr}} \tag{9}$$

Let the portion $x_0$ of the plate be unheated (Fig. Ex. 4.3.2) so that the thermal boundary layer starts from $x = x_0$.

At $\quad x = x_0,\ \delta_t = 0,\ \therefore\ \zeta = 0$

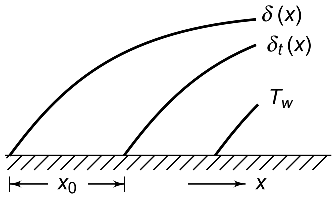


**Fig.** *Ex. 4.3.2*

Substituting in Eq. (9)

$$0 = Cx_0^{-3/4} + \frac{1}{\text{Pr}}$$

$$C = -\frac{x_0^{3/4}}{\text{Pr}}$$

or
$$\zeta^3 = -\frac{x_0^{3/4}}{\text{Pr}}\,x^{-3/4} + \frac{1}{\text{Pr}}$$

$$= \frac{1}{\text{Pr}}\left[1 - \left(\frac{x_o}{x}\right)^{3/4}\right]$$

$$\eta = \text{Pr}^{-1/3}\left[1 - \left(\frac{x_o}{x}\right)^{3/4}\right]^{1/3}$$

If $\quad x_0 = \text{unheated portion} = 0$

$$\zeta = \text{Pr}^{-1/3}$$

or
$$\frac{\delta_t}{\delta} = \text{Pr}^{-1/3}$$

$$\frac{\delta}{\delta_t} = \text{Pr}^{1/3} \qquad \textit{Ans.}\ \text{(b)}$$

$$\frac{Q}{A} = -k\left(\frac{\partial T}{\partial y}\right)_{y=0} = -k\,\frac{T_\infty - T_w}{\delta_t} = \frac{k}{\delta_t}(T_w - T_\infty)$$

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$$h_c = \frac{k}{\delta_t} = \frac{k}{\delta\,\mathrm{Pr}^{-1/3}} = \frac{k\,(\mathrm{Re}_x)^{1/2}}{3.464x\,(\mathrm{Pr})^{-1/3}}$$

$$\mathrm{Nu}_x = \frac{h_c x}{k} = 0.288\,\mathrm{Re}_x^{1/2}\,\mathrm{Pr}^{1/3} \quad Ans.\ (c)$$

**Example 4.4** Air at a temperature of $T_\infty$ flows over a flat plate with a free stream velocity of $u_\infty$. The plate is maintained at a constant temperature of $T_w$. The velocity $u$ and temperature $T$ of air at any location are given by

$$\frac{u}{u_\infty} = \sin\frac{\pi}{2\delta} \text{ and } \frac{T - T_w}{T_\infty - T_w} = 2\left(\frac{y}{\delta_t}\right) - \left(\frac{y}{\delta_t}\right)^2$$

where $y$ is the distance measured from the plate along its normal, and $\delta$ and $\delta_t$ are the hydrodynamic and thermal boundary layer thicknesses, respectively. Find the ratio of heat transfer coefficient to shear stress at the plate surface using the following data:

$$\begin{aligned} u_\infty &= 10 \text{ m/s}, \delta/\delta_t = \mathrm{Pr}^{1/3}, T_w = 200°\mathrm{C}, T_\infty = 50°\mathrm{C}, \\ \mu_{(\mathrm{air})} &= 2.5 \times 10^{-5} \text{ kg/ms}, k_{(\mathrm{air})} = 0.04 \text{ W/mK}, \\ c_p(\mathrm{air}) &= 1.0 \text{ kJ/kg K}. \end{aligned}$$

**Solution** Shear stress at the wall

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0}$$

Velocity profile

$$u/u_\infty = \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)$$

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = u_\infty\left(\cos\frac{\pi}{2}\frac{y}{\delta}\right)_{y=0}\frac{\pi}{2\delta} = \frac{u_\infty \pi}{2\delta}$$

$$\therefore \quad \tau_w = \mu\frac{\pi u_\infty}{2\delta} \tag{1}$$

Temperature distribution

$$\frac{T - T_w}{T_\infty - T_w} = 2\left(\frac{y}{\delta_t}\right) - \left(\frac{y}{\delta_t}\right)^2$$

$$\frac{1}{T_\infty - T_w}\left(\frac{\partial T}{\partial y}\right)_{y=0} = \frac{2}{\delta_t}$$

$$\left(\frac{\partial T}{\partial y}\right)_{y=0} = \frac{2\,(T_\infty - T_w)}{\delta_s}$$

$$h(T_w - T_\infty) = -k\left(\frac{\partial T}{\partial y}\right)_{y=0} = k\frac{2\,(T_w - T_\infty)}{\delta_t}$$

$$h = \frac{2k}{\delta_t} \tag{2}$$

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From Eqs (1) and (2),

$$\frac{h}{\tau_w} = \frac{2k}{\delta_t}\frac{2\delta}{\mu\pi u_\infty} = \frac{4k}{\mu\pi u_\infty}\mathrm{Pr}^{1/3}$$

This is the desired expression for $h/\tau_w$.

$$\mathrm{Pr} = \frac{\mu c_p}{k} = \frac{2.5\times10^{-5}\times1}{0.04\times10^{-3}} = 0.625$$

$$\frac{h}{\tau_w} = \frac{4\times0.04\times(0.625)^{1/3}}{2.5\times10^{-5}\times10\pi}\ \frac{\mathrm{W}}{\mathrm{mk}}\ \frac{\mathrm{ms}}{\mathrm{kg}}\ \frac{\mathrm{s}}{\mathrm{m}}\ \frac{\mathrm{Nm}}{\mathrm{sW}}$$

$$= 174.18\ \mathrm{ms^{-1}\,K^{-1}} \quad \textit{Ans.}$$

**Example 4.5** Given: Nitrogen gas at 0°C is flowing over a 1.2 m long, 2 m wide plate maintained at 80°C with a velocity of 2.5 m/s. For nitrogen, $\rho$ = 1.142 kg/m$^3$, $c_p$ = 1.04 kJ/kgK, $\nu$ = 15.63 × 10$^{-6}$ m$^2$/s and $k$ = 0.0262 W/mK. To find: (a) The average heat transfer coefficient and (b) the total heat transfer from the plate.

***Solution*** The critical Reynolds number for flow over a flat plate is 500,000, so that

$$\mathrm{Re}_{xc} = \frac{u_\infty x_c}{\nu} = 500{,}000$$

$$\therefore \quad x_c = \frac{500{,}000\times15.63\times10^{-6}}{2.5} = 3.126\ \mathrm{m} = 312.6\ \mathrm{cm}$$

Since the plate length is 120 cm in flow direction, laminar flow persists in the entire length of the plate, for which

$$\mathrm{Nu}_m = 0.664\,\mathrm{Re}_L^{1/2}\,\mathrm{Pr}^{1/3}$$

where $$\mathrm{Re}_L = \frac{u_\infty L}{\nu} = \frac{2.5\times1.2}{15.63\times10^{-6}} = 191{,}938.6$$

and $$(\mathrm{Re}_L)^{1/2} = 438.1$$

$$Pr = \frac{c_p\mu}{k} = \frac{1.04\times15.63\times10^{-6}\times1.142}{0.0262\times10^{-3}}$$

$$= 0.708$$

$$\mathrm{Pr}^{1/3} = (0.708)^{1/3} = 0.89$$

$$\therefore \quad \mathrm{Nu}_m = 0.664\times438.1\times0.89 = 258.9 = \frac{h_m L}{k}$$

$$\therefore \quad h_m = \frac{258.1\times0.0262}{1.2} = 5.653\ \mathrm{W/m^2K}\ \textit{Ans.}\ \text{(a)}$$

Total rate of heat transfer from the plate

$$Q = h_m(Lb)\,(T_\omega - T_\infty) = 5.653\times1.2\times2\times(80-0)$$

$$= 1085.4\ \mathrm{W} = 1.085\ \mathrm{kW}\ \textit{Ans.}\ \text{(b)}$$

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**Example 4.6** Air at 20°C and a pressure of 1 bar is flowing over a flat plate at a velocity of 3 m/s. If the plate is 280 mm wide and at 56°C, estimate the following quantities at $x$ = 280 mm, given that the properties of air at the bulk mean temperature of 38°C are: $\rho$ = 1.1374 kg/m$^3$, $k$ = 0.02732 W/mK, $c_p$ = 1.005 kJ/kgK, and $\nu = 16.768 \times 10^{-6}$ m$^2$/s:

(i) boundary layer thickness, (ii) local friction coefficient, (iii) average friction coefficient, (iv) shearing stress due to friction, (v) thickness of thermal boundary layer, (vi) local convective heat transfer coefficient, (vii) average convective heat transfer coefficient, (viii) rate of heat transfer by convection, (ix) total drag force on the plate, and (x) local mass flow rate through the boundary.

**Solution** $u_\infty = 3$ m/s, $x = 280$ mm $= 0.28$ m, $\rho = 1.1374$ kg/m$^3$,

$k = 0.02732$ W/mK, $c_p = 1.005$ kJ/kgK, $\nu = 16.768 \times 10^{-6}$ m$^2$/s,

$$\text{Pr} = \frac{c_p \mu}{k} = \frac{1.005 \times 16.768 \times 10^{-6} \times 1.1374 \times 10^3}{0.02732} = 0.7$$

We are to confirm first whether the flow is laminar or turbulent. At $x = 0.28$ m,

$$\text{Re}_x = \frac{u_\infty x}{\nu} = \frac{3 \times 0.28}{16.768 \times 10^{-6}} = 5 \times 10^4$$

Since $\text{Re}_x < 5 \times 10^5$, the flow is laminar throughout.

(i) Boundary layer thickness

$$\delta = \frac{5x}{\sqrt{\text{Re}_x}} = \frac{5 \times 0.28}{\sqrt{5 \times 10^4}} = 0.00626 \text{ m}$$

$$= 6.26 \text{ mm} \quad \textit{Ans.}$$

(ii) Local friction coefficient

$$c_{fx} = \frac{0.664}{\sqrt{\text{Re}_x}} = \frac{0.664}{\sqrt{5 \times 10^4}}$$

$$= 0.002969 \quad \textit{Ans.}$$

(iii) Average friction coefficient

$$\bar{c}_f = \frac{1.328}{\sqrt{\text{Re}_L}} = \frac{1.328}{\sqrt{5 \times 10^4}} = 0.005938 \quad \textit{Ans.}$$

(iv) Shearing stress due to friction

$$\tau_w = c_{fx} \frac{\rho u_\infty^2}{2} = 0.002969 \times \frac{1.1374 \times 3^2}{2}$$

$$= 0.01519 \text{ N/m}^2 \quad \textit{Ans.}$$

(v) Thickness of thermal boundary layer

$$\delta_{\text{th}} = \frac{8}{\text{Pr}^{1/3}} = \frac{0.00626}{(0.7)^{1/3}} = 0.00705 \text{ m} = 7.05 \text{ mm} \quad \textit{Ans.}$$

(vi) Local convective heat transfer coefficient.

$$h_x = 0.332 \frac{k}{x} (\text{Re}_x)^{1/2} (\text{Pr})^{1/3}$$

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$$= 0.332 \times \frac{0.02732}{0.28} \times (5 \times 10^4)^{1/2} (0.7)^{1/3}$$
$$= 6.43 \text{ W/m}^2 \text{ K} \quad \textit{Ans.}$$

(vii) Average convective heat transfer coefficient

$$\bar{h} = 0.664 \frac{k}{L} (\text{Re}_L)^{1/2} (\text{Pr})^{1/3}$$
$$= 0.664 \times \frac{0.02732}{0.28} \times (5 \times 10^4)^{1/2} (0.7)^{1/3}$$
$$= 12.86 \text{ W/m}^2\text{K} \quad \textit{Ans.}$$

(viii) Rate of heat transfer

$$Q = \bar{h} \cdot A\,(T_\omega - T_\infty)$$
$$= 12.86 \times 0.28 \times 0.28 \times (56 - 20)$$
$$= 36.29 \text{ W} \quad \textit{Ans.}$$

(ix) Total drag force on the plate

$$F_D = \tau_\omega \times A$$
$$= 0.01519 \times 0.28 \times 0.28$$
$$= 0.00119 \text{ N} \quad \textit{Ans.}$$

(x) Total mass flow through the boundary

$$\omega = \frac{5}{8} \rho u_\infty (\delta_2 - \delta_1)$$
$$= \frac{5}{8} \times 1.1374 \times 3 \times (0.00626 - 0)$$
$$= 0.01335 \text{ kg/s} \quad \textit{Ans.}$$

**Example 4.7** Air at 20°C and at atmospheric pressure flows at a velocity of 4.5 m/s past a flat plate with a sharp leading edge. The entire plate surface is maintained at a temperature of 60°C. Assuming that the transition occurs at a critical Reynolds number of $5 \times 10^5$, find the distance from the leading edge at which the boundary layer changes from laminar to turbulent. At the location, calculate the following:

(i) Thickness of hydrodynamic boundary layer,
(ii) Thickness of thermal boundary layer,
(iii) Local and average convective heat transfer coefficients,
(iv) Heat transfer rate from both sides per unit width of plate,
(v) Mass entrainment in the boundary layer,
(vi) Skin friction coefficient.

Assume cubic velocity profile and approximate method. Thermophysical properties of air at mean film temperature of 40°C are: $\rho$ = 1.128 kg/m$^3$, $\nu = 16.96 \times 10^{-6}$ m$^3$/s, $k$ = 0.02755 W/mK and Pr = 0.7.

**Solution** At the transition point,

$$\text{Re}_{x_c} = \frac{u_\infty x_c}{\nu}$$

$$\therefore \quad x_c = \frac{5 \times 10^5 \times 16.96 \times 10^{-6}}{4.5} = 1.88 \text{ m}$$

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(i) Thickness of hydrodynamic boundary layer

$$\delta = \frac{4.64\,x}{\sqrt{\mathrm{Re}_{x_c}}} = \frac{4.64 \times 1.88}{\sqrt{5 \times 10^5}}$$

$$= 0.01234 \text{ m or } 12.34 \text{ mm} \quad \textit{Ans. (i)}$$

(ii) Thickness of thermal boundary layer

$$\delta_{\mathrm{th}} = \frac{0.975\,\delta}{(\mathrm{Pr})^{1/3}} = \frac{0.975 \times 0.01234}{(0.7)^{1/3}}$$

$$\therefore \quad \delta_{\mathrm{th}} = 0.01355 \text{ m or } 13.55 \text{ mm} \quad \textit{Ans. (ii)}$$

(iii) $\mathrm{Nu}_{x_c} = 0.332\,(\mathrm{Re}_{x_c})^{1/2}\,(\mathrm{Pr})^{1/3}$

$$= 0.332\,(5 \times 10^5)^{1/2}\,(0.7)^{1/3} = 208.34$$

$$\therefore \quad h_c = \frac{\mathrm{Nu}_c \times k}{x_c} = \frac{208.34 \times 0.02755}{1.88}$$

$$= 3.05 \text{ W/m}^2\text{ K} \quad \textit{Ans. (iii)}$$

Average heat transfer coefficient

$$\bar{h} = 2\,h_c = 2 \times 3.05 = 6.1 \text{ W/m}^2\text{K} \quad \textit{Ans. (iii)}$$

(iv) Heat transfer from both sides per unit width of the plate

$$Q = \bar{h}\,(2A_s)\,\Delta T = 6.1 \times 2 \times (1.88 \times 1) \times (60 - 20)$$

$$= 917.4 \text{ W} \quad \textit{Ans. (iv)}$$

(v) Maximum entrainment in the boundary layer

$$\dot{m} = \frac{5}{8}\,\rho u(\delta_2 - \delta_1)$$

$$= \frac{5}{8} \times 1.128 \times 4.5\,(0.01234 - 0)$$

$$= 0.039 \text{ kg/s} \quad \textit{Ans. (v)}$$

(vi) Skin friction coefficient

$$C_{f_x} = \frac{0.646}{\sqrt{\mathrm{Re}_{x_c}}} = \frac{0.646}{\sqrt{5 \times 10^5}} = 9.136 \times 10^{-4} \quad \textit{Ans. (vi)}$$

**Example 4.8** *Given:* Water at 10°C flows over a flat plate (at 90°C) measuring 1 m × 1 m, with a velocity of 2 m/s. Properties of water at 50°C are $\rho$ = 988.1°C, $\nu = 0.556 \times 10^{-6}$ m²/s, Pr = 3.54 and $k$ = 0.648 W/mK.
*To find:* (a) The length of plate over which the flow is laminar, (b) the rate of heat transfer from the entire plate.

**Solution** Up to $\mathrm{Re}_c$ = 500,000, the flow is laminar and beyond this value, the flow is turbulent after a transition section. Therefore,

$$\mathrm{Re}_c = 500{,}000 = \frac{u_\infty x_c}{\nu}$$

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$$x_c = \frac{500,000 \times 0.556 \times 10^{-6}}{2} = 0.139 \text{ m}$$

The length of plate up to which the flow is laminar is 0.139 m. *Ans*. (a)

*Laminar Part:*

$$\text{Nu}_c = 0.664\,(\text{Re}_c)^{1/2}\,(\text{Pr})^{1/3}$$
$$= 0.664\,(500,000)^{1/2}\,(3.54)^{1/3}$$
$$= 715.08 = \frac{h_m x_c}{k}$$
$$h_m = \frac{715.08 \times 0.648}{0.139} = 3333.6 \text{ W/m}^2\text{K}$$
$$Q_{\text{laminar}} = h_m A(T_\omega - T_\infty)$$
$$= 3333.6 \times 0.139 \times 1 \times (90 - 10)$$
$$= 37069.6 \text{ W} = 37.07 \text{ kW}$$

*Turbulent Part*

$$\frac{h_m(L - x_c)}{k} = 0.036[(\text{Re}_L)^{0.8} - (\text{Re}_C)^{0.8}]\text{Pr}^{1/3}$$

where $\text{Re}_L = \frac{U_\infty L}{\nu} = \frac{2 \times 1 \times 10^6}{0.556} = 3.597 \times 10^6$

and $(\text{Re}_L)^{0.8} = 175,692$

$$(\text{Re}_c)^{0.8} = (500,000)^{0.8} = 36239$$
$$\text{Pr}^{1/3} = (3.54)^{1/3} = 1.523$$

$\therefore$ $$\frac{h_m(1 - 0.139)}{0.648} = 0.036\,(175692 - 36,239) \times 1.523$$
$$= 7645.93$$

$\therefore$ $$h_m = 5754.4 \text{ W/m}^2\text{K}$$
$$Q_{\text{turbulent}} = 5754.4 \times 0.861 \times 1 \times (90 - 10)$$
$$= 396365 \text{ W} = 396.365 \text{ kW}$$

$\therefore$ $$Q_{\text{total}} = Q_{\text{lam.}} + Q_{\text{turb.}}$$
$$= 37.07 + 396.37$$
$$= 433.44 \text{ kW}$$ *Ans*. (b)

**Example 4.9** Atmospheric air flows inside a heated thin-walled 25 mm diameter tube with a velocity of 0.5 m/s. Heating can be done either by condensing steam on the outer surface of the tube, thus maintaining a uniform surface temperature, or by electric resistance heating, thus maintaining a uniform surface heat flux. Calculate the heat transfer coefficient for both of these heating conditions. Assume air properties of 350 K given as: $\nu = 20.76 \times 10^{-6}$ m$^2$/s and $k = 0.03$ W/mK and the flow as hydrodynamically and thermally developed.

**Solution** Reynolds number, $\text{Re}_d = \frac{u_m d}{\nu}$

$$= \frac{0.5 \times 0.025}{20.76 \times 10^{-6}} = 602$$

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Thus, the flow is laminar. In the hydrodynamically and thermally developed flow for

(a) Constant wall temperature, the heat transfer coefficient (by condensing steam), using Eq. (4.185),

$$h = 3.66 \frac{k}{d} = 3.66 \times \frac{0.03}{0.025}$$
$$= 4.39 \text{ W/m}^2\text{K} \quad \textit{Ans.}$$

(b) Constant heat flux (by electric resistance heating) using Eq. (4.184),

$$h = 4.364 \frac{k}{d} = 4.364 \times \frac{0.03}{0.025}$$
$$= 5.24 \text{ W/m}^2\text{K} \quad \textit{Ans.}$$

**Example 4.10** Water flows through a 20 mm ID at a rate of 0.01 kg/s entering at 10°C. The tube is wrapped from outside by an electric heating element that produces a uniform flux of 15 kW/m$^2$. If the exit temperature of water is 40°C, estimate (a) the Reynolds number, (b) the heat transfer coefficient, (c) the length of pipe needed, (d) the inner tube surface temperature at exit, (e) the friction factor, (f) the pressure drop in the tube, and (g) the pumping power required if the pump efficiency is 60%. Neglect entrance effects. Properties of water at mean temperature of 25°C are: $\rho$ = 997 kg/m$^3$, $c_p$ = 4180 J/kgK, m = 910 × 10$^{-6}$ Ns/m$^2$, and $k$ = 0.608 W/mK.

**Solution** Reynolds number is

$$\text{Re}_d = \frac{\rho u_m d}{\mu} = \frac{4\dot{m}}{\pi d \mu}$$
$$= \frac{4 \times 0.01}{\pi \times (0.02) \times 910 \times 10^{-6}} \cong 700 \quad \textit{Ans. (a)}$$

Thus, the flow is laminar.

Since the thermal boundary layer is of uniform heat flux, $\text{Nu}_d = 4.364 = \dfrac{hd}{k}$

$$\therefore \quad h = 4.364 \times \frac{0.608}{0.02} = 132.5 \text{ W/m}^2\text{K} \quad \textit{Ans.} \text{ (b)}$$

By energy balance,

$$q'' \pi d L = \dot{m}\, c_p\, (T_o - T_i)$$
$$\therefore \quad L = \frac{\dot{m} c_p\, \Delta T}{\pi d q''}$$
$$= \frac{0.01 \times 4180 \times 30}{\pi \times 0.02 \times 15{,}000} = 1.33 \text{ m} \quad \textit{Ans.} \text{ (c)}$$

Since $L/d = \dfrac{1.33}{0.02} = 66.5$ and $0.05\,\text{Re}_d = 33.5$, entrance effects are not significant according to Eq. (4.162).

Again, $q'' = h\,(T_w - T_b)_e$

$$15000 = 132.5\,(T_w - 40)$$
$$\therefore \quad T_w = \frac{15000}{132.5} + 40 = 153.2°\text{C} \quad \textit{Ans.} \text{ (d)}$$

Friction factor, $f = \dfrac{64}{\text{Re}_d} = \dfrac{64}{700} = 0.0915 \quad \textit{Ans.}$ (e)

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Water velocity, $u_m = \dfrac{4\dot{m}}{\rho \pi d^2} = \dfrac{4 \times 0.01}{997 \times \pi (0.02)^2} = 0.032$ m/s

Pressure drop in the tube

$$\Delta p = \frac{fL}{d}\frac{\rho u_m^2}{2}$$

$$= 0.0915 \times 66.5 \times \frac{997 \times (0.032)^2}{2}$$

$$= 3.1 \text{ N/m}^2 \quad \textit{Ans.} \text{ (f)}$$

Pumping power $= \dfrac{\dot{m}\Delta p}{\rho \eta_p} = \dfrac{0.01 \times 3.1}{997 \times 0.6}$

$$= 5.18 \times 10^{-5} \text{ W} \quad \textit{Ans.} \text{ (g)}$$

**Example 4.11** Water is heated while flowing through a 1.5 cm × 3.5 cm rectangular cross-section tube at a velocity of 1.2 m/s. The entering temperature of the water is 40°C, and the tube wall is maintained at 85°C. Determine the length of the tube required to raise the temperature of water to 70°C. Properties of water at the mean bulk temperature of 55°C are: $\rho = 985.5$ kg/m$^3$; $c_p = 4.18$ kJ/kg K, $\nu = 0.517 \times 10^{-6}$ m$^2$/s, $k = 0.654$ W/m K and Pr = 3.26.

**Solution** The hydraulic/equivalent diameter of the duct (Fig. Ex. 4.7)

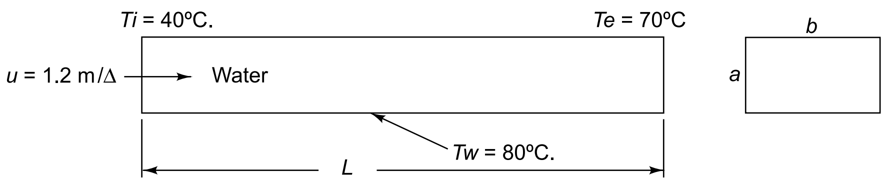


**Fig.** Ex. 4.7

$$D_e = \frac{4A}{P} = \frac{4ab}{2(a+b)} = \frac{2ab}{a+b} = \frac{2 \times 1.5 \times 3.5}{1.5 + 3.5} = 2.1 \text{ cm}$$

$$\text{Re}_d = \frac{u_m D_e}{\nu} = \frac{1.2 \times 2.1 \times 10^{-2}}{0.517 \times 10^{-6}} = 48{,}740$$

$$(\text{Re}_d)^{0.8} = (40{,}740)^{0.8} = 5627.4$$

$$(\text{Pr})^{0.4} = (3.26)^{0.4} = 1.604$$

Using Dittus–Boelter equation, with water being heated,

$$\text{Nu}_d = 0.023 \text{ Re}_d^{0.8} \text{ Pr}^{0.4} = 0.023 \times 5627.4 \times 1.604$$

$$= 207.6 = \frac{h_c D_e}{k}$$

$$h_c = \frac{207.6 \times 0.654}{2.1 \times 10^{-2}} = 6465 \text{ W/m}^2 \text{ K} = 6.465 \text{ kW/m}^2 \text{ K}$$

Now, $h_c P \, dx \, (T_w - T) = \dot{m} c_p \, dT$

where d$T$ is the temperature rise of water in the differential length d$x$.

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$$\frac{h_c P}{\dot{m} c_p} \int_0^L dx = \int_{T_i}^{T_e} \frac{dT}{T_w - T}$$

$$\frac{h_c PL}{\dot{m} c_p} = \ln \frac{T_w - T_i}{T_w - T_e}$$

Therefore,

$$L = \frac{\dot{m} c_p}{h_c P} \ln \frac{T_w - T_i}{T_w - T_e} \qquad (1)$$

Now,

$$\dot{m} = \rho A u_m = 985.5 \times 1.5 \times 3.5 \times 10^{-4} \times 1.2$$
$$= 0.621 \text{ kg/s}$$
$$P = 2\,(a + b) = 2\,(1.5 + 3.5) = 10 \text{ cm} = 0.1 \text{ m}$$

On substitution in Eq. (1),

$$L = \frac{0.621 \times 4.18}{6.465 \times 0.1} \ln \frac{85 - 40}{85 - 70} = 4.41 \text{ m}$$

The length of tube required is 4.41 m *Ans.*

**Example 4.12** It was found during a test in which water flowed with a velocity of 2.44 m/s through a tube (2.54 cm inner diameter and 6.08 m long), that the head lost due to friction was 1.22 m of water. Estimate the surface heat transfer coefficient based on Reynolds analogy. Take $\rho$ = 998 kg/m$^3$ and $c_p$ = 4.187 kJ/kg K.

**Solution**

$$\Delta p = h\rho g = 1.22 \text{ m} \times 998 \frac{\text{kg}}{\text{m}^3} \times 9.81 \times 10^{-3}$$
$$= 11.944 \text{ kPa}$$

$$\Delta p = \frac{fL}{D} \frac{\rho u_m^2}{2}$$

$$= \frac{f \times 6.08}{2.54 \times 10^{-2}} \cdot \frac{998 \times (2.44)^2}{2} = 11{,}944 \text{ N/m}^2$$

$\therefore$

$$f = \frac{11{,}944 \times 2.54 \times 10^{-2} \times 2}{6.08 \times 998 \times (2.44)^2} = 0.0168$$

By Reynolds analogy,

$$\text{St}_d = \frac{h}{\rho c_p u_m} = \frac{f}{8}$$

$\therefore$

$$h = \frac{0.0168}{8} \times 998 \times 4.18 \times 2.44$$
$$= 21.38 \text{ kW/m}^2 \text{ K} \quad \textit{Ans.}$$

**Example 4.13** *Given:* Atmospheric pressure air at 100°C enters a 0.04 m dia 2 m long tube with a velocity of 9 m/s. A 1 kW electric heater wound on the outer surface of the tube provides a uniform heat flux to the tube.

*To find:* (a) The mass flow rate of air, (b) the exit temperature of air, and (c) the wall temperature of tube at outlet.

**Solution** Density of air at 100°C,

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$$\rho = \frac{p}{RT} = \frac{101.325}{0.287 \times 373} = 0.946 \text{ kg/m}^3$$

Mass flow rate of air, $\dot{m} = \rho A u_m$

$$= 0.946 \times \frac{\pi}{4} \times (0.04)^2 \times 9$$

$$= 0.0107 \text{ kg/s} \quad \textit{Ans.} \text{ (a)}$$

Now, $$Q = \dot{m} c_p (T_e - T_i) = 1 \text{ kW}$$

$$T_c = 100 + \frac{1}{0.0107 \times 1.005} = 193°\text{C} \quad \textit{Ans.} \text{ (b)}$$

Mean air temperature, $T_m = \dfrac{T_e + T_i}{2} = \dfrac{293}{2}$

$$= 146.5°\text{C}$$

At 146.5ºC, the properties of air taken from the Appendix by interpolation are $\rho$ = 0.84 kg/m$^3$, Pr = 0.683, $k$ = 0.026 W/mK and $\nu = 28.8 \times 10^{-6}$ m$^2$/s

$\text{Re}_d = u_m D/\nu = 9 \times 0.04/28.8 \times 10^{-6} = 12{,}500$

Using Dittus – Boelter equation

$$\begin{aligned}\text{Nu}_d &= 0.023\,(\text{Re}_d)^{0.8}\,(\text{Pr})^{0.4}\\ &= 0.023\,(12{,}500)^{0.8}\,(0.683)^{0.4}\\ &= 0.023 \times 1894.6 \times 0.859\\ &= 37.43\end{aligned}$$

$$\therefore \quad h = \frac{37.43 \times 0.026}{0.04} = 24.33 \text{ W/m}^2\text{K}$$

Since heat flux is uniform throughout the length of the tube,

$$Q = hA\,(T_{w_e} - T_e)$$

where $T_{w_e}$ is the exit wall temperature and $T_e$ is the exit air temperature.

$$\therefore \quad 1 \text{ kW} = 24.33 \times 10^{-3} \text{ kW/m}^2\text{K} \times \pi \times 0.04 \times 2 \text{ m}^2 (T_{w_e} - 193)\text{K}$$

$$\therefore \quad T_{w_e} = 193 + \frac{10^4}{2.433 \times \pi \times 8} = 356.5°\text{C}$$

The exit wall temperature is 356.5ºC. *Ans.* (c)

**Example 4.14** *Given:* Lubricating oil ($\rho$ = 865 kg/m$^3$, $k$ = 0.14 W/mK, $c_p$ = 1.78 kJ/kgK, and $\nu = 9 \times 10^{-6}$ m$^2$/s) at 60°C enters a 1 cm dia tube with a velocity of 3.5 m/s. $T_w$ = 30°C, constant.
*To find:* The tube length required to cool the oil to 45°C.

**Solution** Reynolds number, $\text{Re}_d = \dfrac{u_m D}{\nu} = \dfrac{3.5 \times 0.01}{9 \times 10^{-6}}$

$$= 3888.9$$

Prandtl number, $\text{Pr} = \dfrac{c_p \mu}{k} = \dfrac{1.78 \times 9 \times 10^{-6} \times 865}{0.14 \times 10^{-3}}$

$$= 98.98$$

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Using Dittus–Boelter equation

$$\text{Nu}_d = 0.023\,(\text{Re}_d)^{0.8}\,(\text{P}_r)^{0.3}$$
$$= 0.023\,(3888.9)^{0.8}\,(98.98)^{0.3}$$
$$= 0.023 \times 744.5 \times 3.97 = 67.98 = \frac{hD}{K}$$
$$\bar{h} = \frac{67.98 \times 0.14}{0.01} = 951.72\ \text{W/m}^2\text{K}$$

For an isothermal tube surface (Eq. 4.187),

$$\frac{\bar{h}\pi DL}{\dot{m}c_p} = \ln\frac{T_w - T_i}{T_w - T_e}$$

where $\dot{m} = \rho A u_m = 865 \times \frac{\pi}{4} \times (0.01)^2 \times 3.5$

$$= 0.238\ \text{kg/s}$$

$$\therefore \quad L = \frac{0.238 \times 1.78}{\pi \times 0.01 \times 0.951} \ln\frac{30-60}{30-45}$$
$$= 14.18\ \text{m} \quad \textit{Ans.}$$

**Example 4.15** *Given:* Air flows at 30 m/s through a 2 cm dia 1 m long tube entering at 20°C and 101.3 kPa. The pressure loss in the tube is 8 cm of water gauge.
*To determine:* The amount of heat transferred from the tube wall to the air when the wall temperature is maintained constant at 95°C.

**Solution** Pressure drop of air in the tube

$$\Delta p = h\rho g = 0.08 \times 1000 \times 9.81 = 784.8\ \text{N/m}^2$$
$$= 0.785\ \text{kPa}$$

By Reynolds analogy, from (Eq. 4.201),

$$q_w = \tau_w c_p \frac{T_w - T_B}{U_m}$$

Multiplying both sides by the tube wall surface

$$Q_w = R\frac{c_p}{U_m}(T_w - T_B)$$

where $R$ = flow resistance = $\tau_w \cdot A$ in newtons.
Power necessary to make the fluid flow

$$\dot{W} = \dot{V}\Delta P = Ru_m$$

where $\dot{V}$ = Volume flow rate $= \frac{\pi}{4} D^2 u_m$

and $R = \tau_w A = \Delta p \frac{\pi}{4} D^2 = 0.785 \times \frac{\pi}{4} \times (0.02)^2 = 2.466 \times 10^{-4}\ \text{kN}$

$$= 0.2466\ \text{N}$$

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$$Q_w = 0.2466 \times .005 \times \frac{T_w - T_B}{30} = 8.26 \times 10^{-3} (T_w - T_B) \quad (1)$$

$$\rho = \frac{p}{RT} = \frac{101.325}{0.287 \times 293} = 1.2 \text{ kg/m}^3$$

$$\dot{m} = 1.2 \times \frac{\pi}{4} (0.02)^2 \times 30 = 0.0113 \text{ kg/s}$$

$$Q = \dot{m} c_p (T_e - T_i) = 0.0113 \times 1.005 \, (T_e - 20)$$

$$= 0.011356 \, (T_e - 20) \quad (2)$$

From Eqs (1) and (2) above,

$$8.26 \times 10^{-3} \left( 95 - \frac{T_e + 20}{2} \right) = 0.011356 (T_e - 20)$$

$$95 - \frac{T_e}{2} - 10 = 1.375 \, T_e - 27.5$$

$$\therefore \qquad T_e = 60°\text{C}$$

Rate of heat transfer, $Q = 0.011356 \, (60 - 20) \times 10^3$

$= 454$ W *Ans.*

Also, from Eq. (1),

$$Q = 8.26 \times 10^{-3} \left( 95 - \frac{60 + 20}{2} \right)$$

$= 454.3$ W *Ans.*

**Example 4.16** A refrigerated truck is travelling at 80 km/h on a desert highway where the air temperature is 60°C. The body of the truck may be idealised as a rectangular box, 3.2 m wide, 2.1 m high and 6.6 m long, at a surface temperature of 10°C. Assume that the heat transfer from the front and back of the truck may be neglected, that the stream does not separate from the surface and that the boundary layer is turbulent over the whole surface. Calculate (a) the rate of heat transfer to the four surfaces, (b) the required tonnage of the refrigeration unit and (c) the power required to overcome the resistance acting on the four surfaces. Properties of air at 35°C are: $\rho = 1.147$ kg/m$^3$, $k = 0.0271$ W/m K, $\nu = 16.48 \times 10^{-6}$ m$^2$/s and $c_p = 1.005$ kJ/kg K.

**Solution** *Given:* $u_\infty = 80$ km/h $= 22.22$ m/s

$T_\infty = 60°$C, $T_w = 10°$C

$$\text{Re}_L = \frac{u_\infty L}{\nu} = \frac{22.22 \times 6.6}{16.48 \times 10^{-6}} = 8.9 \times 10^6$$

$$\text{Pr} = \frac{\nu \rho c_p}{k} = \frac{16.48 \times 10^{-6} \times 1.147 \times 1.005}{0.0271 \times 10^{-3}} = 0.7$$

Since the boundary layer is turbulent throughout the length

$$\text{Nu}_m = 0.036 \, (\text{Re}_L)^{0.8} \, \text{Pr}^{1/3}$$

$$h_m = 0.036 \times (8.9 \times 10^6)^{0.8} \, (0.7)^{1/3} \times \frac{0.0271}{6.6}$$

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$$= 53.61 \times 0.89 = 47.7 \text{ W/m}^2 \text{ K}$$

$$Q = h_m A\,(T_w - T_\infty)$$
$$= 47.7 \times 6.6 \times 2\,(2.1 + 3.2)\,(60 - 10) \times 10^{-3}$$
$$= 166.85 \text{ kW} \quad \textit{Ans.} \text{ (a)}$$

$$\text{Tonnage} = \frac{166.85}{3.89} = 42.9 \text{ tons of refrigeration} \quad \textit{Ans.} \text{ (b)}$$

(c)
$$\bar{c}_f = 0.072\,(\text{Re}_L)^{-0.2} = \frac{0.072}{(8.9 \times 10^6)^{0.2}}$$
$$= 2.934 \times 10^{-3}$$

$$\tau_w = \bar{c}_f \frac{\rho u_\infty^2}{2} = \frac{2.934 \times 10^{-3} \times 1 \times 1.147 \times (22.22)^2}{2}$$
$$= 0.83 \text{ N/m}^2$$

Drag force
$$R = \tau_w \times \text{Area} = 0.83 \times 6.6 \times 2\,(3.2 + 2.1)$$
$$= 58.1 \text{ N}$$

Power required
$$= R u_\infty = 58.1 \times 22.22 \times 10^{-3}$$
$$= 1.29 \text{ kW} \quad \textit{Ans.} \text{ (c)}$$

**Example 4.17** Air at 27°C and 1 atm flows over a flat plate at a velocity of 3 m/s. Calculate the boundary layer thickness at distances of 25 and 45 cm from the leading edge of the plate. Calculate the mass flow which enters the boundary layer between $x$ = 25 cm and $x$ = 45 cm. The viscosity of air at 27°C is $1.85 \times 10^{-5}$ kg/ms. Assume unit depth in z-direction.

**Solution**

$$\rho = \frac{p}{RT} = \frac{1.01325}{0.287 \times 300} = 1.177 \text{ kg/m}^3$$

At
$$x = 25 \text{ cm}, \text{Re}_x = \frac{1.177 \times 3 \times 0.25}{1.85 \times 10^{-5}} = 47{,}716$$

At
$$x = 45 \text{ cm}, \text{Re}_x = \frac{1.177 \times 3 \times 0.45}{1.85 \times 10^{-5}} = 85{,}889$$

$$\frac{\delta}{x} = \frac{4.64}{(\text{Re}_x)^{1/2}}$$

At
$$x = 25 \text{ cm}, \delta = \frac{4.64 \times 0.25}{(47{,}716)^{1/2}} \times 10^3 = 5.31 \text{ mm}$$

At
$$x = 45 \text{ cm}, \delta = \frac{4.64 \times 0.45 \times 10^3}{(85{,}889)^{1/2}} = 7.125 \text{ mm} \quad \textit{Ans.}$$

At any $x$-position the mass flow in the boundary layer is given by

$$\int_0^\delta \rho u \, dy$$

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where the velocity is given by

$$= \frac{u}{u_\infty} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$$

Mass flow $$= \int_0^\delta \rho u_\infty \left(\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\frac{y^3}{\delta^3}\right) \text{dy}$$

$$= \rho u_\infty \left(\frac{3}{2\delta}\frac{\delta^2}{2} - \frac{1}{2\delta^3}\frac{\delta^4}{4}\right)$$

$$= \frac{5}{8}\rho u_\infty \delta$$

Mass flow entering the boundary layer between $x = 25$ cm and $x = 45$ cm

$$= \frac{5}{8}\rho u_\infty (\delta_{45} - \delta_{25})$$

$$= \frac{5}{8} \times 1.177 \times 3 \times (7.125 - 5.31) \times 10^{-3}$$

$$= 4 \times 10^{-3} \text{ kg/s} \quad \textit{Ans.}$$

**Example 4.18** For the flow system in Example 4.17 assume that the plate is heated over its entire length to a temperature of 70°C. Calculate the heat transferred in (a) the first 25 cm of the plate and (b) the first 45 cm of the plate. Given are the properties of air at 48.5°C: $k$ = 0.02749 W/m K, $\nu$ = 17.36 × $10^{-6}$ m$^2$/s, $c_p$ = 1.006 kJ/kg K and Pr = 0.7.

**Solution**

$$\text{Re}_x \text{ at 25 cm} = \frac{u_\infty x}{\nu} = \frac{3 \times 0.25}{17.36 \times 10^{-6}} = 43{,}200$$

$$\text{Re}_x \text{ at 45 cm} = \frac{3 \times 0.45}{17.36 \times 10^{-6}} = 77{,}765$$

$$\text{Nu}_x = \frac{h_x x}{k} = 0.332\,(\text{Re}_x)^{0.5}\,(\text{Pr})^{1/3}$$

$$= 0.332\,(43{,}200)^{0.5}\,(0.7)^{1/3}$$

$$= 0.332 \times 207.85 \times 0.89$$

$$= 61.4$$

$$h_x = \frac{61.4 \times 0.02749}{0.25} = 6.75 \text{ W/m}^2\text{ K}$$

$$h = 2h_x = 2 \times 6.75 = 13.5 \text{ W/m}^2\text{ K}$$

For $z = 1$ m, $Q_{25\text{ cm}} = \bar{h}A\,(T_w - T_\infty) = 13.5 \times (0.25 \times 1)\,(70 - 27)$

$= 145.1$ W *Ans.*

For $x = 45$ cm,

$$\text{Nu}_x = 0.332\,(77{,}765)^{0.5}\,(0.7)^{1/3} = 82.4 = \frac{h_x \times 0.45}{0.02749}$$

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$$h_x = 5.034 \text{ W/m}^2\text{K}$$

$$\therefore \quad \bar{h} = 10.07 \text{ W/m}^2\text{K}$$

$$Q_{45 \text{ cm}} = 10.07 \times (0.45 \times 1) \times (70 - 27) = 194.85 \text{ W} \quad \textit{Ans.}$$

**Example 4.19** For the flow system in Example 4.18 estimate the drag force exerted on the first 45 cm of the plate using the analogy between fluid friction and heat transfer and also heat transfer in the plate from $x$ = 25 cm to $x$ = 45 cm.

**Solution** For the 45 cm length, $\overline{\text{St}} = \dfrac{\bar{h}}{\rho c_p u_\infty}$

where, $\bar{h} = 10.07 \text{ W/m}^2 \text{ K}$

$$\rho = \frac{101.325}{0.287\,(273 + 48.5)} = 1.098 \text{ kg/m}^3$$

$$\overline{\text{St}} = \frac{10.07 \times 10^{-3}}{1.098 \times 1.006 \times 3} = 3.04 \times 10^{-3}$$

Colburn's $j$-factor is

$$\overline{\text{St}}\,\text{Pr}^{2/3} = \frac{\bar{C}_f}{2}$$

$$3.04 \times 10^{-3} \times (0.7)^{2/3} = \frac{\bar{C}_f}{2}$$

$$\bar{C}_f = 4.793 \times 10^{-3}$$

Average shear stress $\tau_w = C_f \dfrac{\rho u_\infty^2}{2}$

$$= 4.793 \times 10^{-3} \times \frac{1.098 \times 9}{2}$$

$$= 23.68 \times 10^{-3} \text{ N/m}^2$$

Total drag force $F = 23.68 \times 10^{-3} \times 0.45 \times 1$

$$= 10.66 \times 10^{-3} \text{ N} = 10.66 \text{ mN} \quad \textit{Ans.}$$

$$h_{x = 45 \text{ cm}} = \frac{82.4 \times 0.02749}{0.45} = 5.034 \text{ W/m}^2 \text{ K}$$

$$\bar{h} = 2 \times 5.034 = 10.07 \text{ W/m}^2 \text{ K}$$

$$Q_{45 \text{ cm}} = 10.07 \times 0.45 \times (70 - 27) = 194.85 \text{ W} \quad \textit{Ans.}$$

Heat transfer in the length between $x$ = 45 cm and $x$ = 25 cm = 194.85 – 145.1 = 49.75 W.

**Example 4.20** *Given:* Engine, oil at 25°C flows over a 30 cm long 20 cm wide plate at 1.5 m/s, which is heated to a uniform temperature of 55°C. At 40°C for the engine oil, $\rho$ = 876 kg/m$^3$, $\nu = 24 \times 10^{-5}$ m$^2$/s, $k$ = 0.144 W/mK and $Pr$ = 2870.
*To determine:* The rate of heat transfer to the oil from the plate.

**Solution** Reynolds number, $\text{Re}_L = \dfrac{u_\infty L}{\nu}$

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$$\text{Re}_L = \frac{1.5 \times 0.3}{24 \times 10^{-5}} = 1875$$

The flow is laminar throughout the flow over the plate. For laminar flow on an isothermal plate when the Prandtl number of fluid is high, the following equation (Eq. 4.150) can be used.

$$\text{Nu}_x = \frac{0.3387(\text{Re}_x)^{1/2}\,\text{Pr}^{1/3}}{\left[1+\left(\dfrac{0.0468}{\text{Pr}}\right)^{2/3}\right]^{1/4}}$$

$$\text{Nu}_x = \frac{0.3387(1875)^{1/2}(2870)^{1/3}}{\left[1+\left(\dfrac{0.0468}{2870}\right)^{2/3}\right]^{1/4}}$$

$$= \frac{14.67 \times 14.17}{1} = \frac{h_x x}{k}$$

$$\therefore \quad h_x = \frac{207.87 \times 0.144}{0.3} = 99.78 \text{ W/m}^2\text{K}$$

$$h_m = 2h_x = 2 \times 99.78 = 199.6 \text{ W/m}^2\text{K}$$

$$\therefore \quad Q = h_m A\,(T_w - T_m) = 199.6 \times 0.3 \times 0.2 \times (55 - 25)$$

$$= 359.28 \text{ W} \quad \textit{Ans.}$$

**Example 4.21** Air at 20°C and 1 atm flows over a flat plate at 40 m/s. The plate is 80 cm long and is maintained at 60°C. Assuming unit depth in z-direction, calculate the heat transfer rate from the plate. Properties of air at 40°C are : Pr = 0.7, $k$ = 0.02723 W/m K, $c_p$ = 1.007 kJ/kg K and $\mu$ = 1.906 × $10^{-5}$ kg/ms.

**Solution**

$$T_f = \frac{20+60}{2} = 40°\text{C} = 313 \text{ K}$$

$$\rho = \frac{p}{RT} = \frac{101.325}{0.287 \times 313} = 1.128 \text{ kg/m}^3$$

$$\text{Re}_L = \frac{\rho u_\infty L}{\mu} = \frac{1.128 \times 40 \times 0.80}{1.906 \times 10^{-5}} = 1.89 \times 10^6$$

The boundary layer is turbulent, since $\text{Re}_L > 5 \times 10^5$

$$\overline{\text{Nu}}_L = \frac{\bar{h} L}{k} = (0.036\ \text{Re}_L^{0.8} - 871)\ \text{Pr}^{1/3}$$

$$= [0.036\ (1.89 \times 10^6)^{0.8} - 871]\ (0.7)^{1/3}$$

$$= 2908.83 \times 0.89 = 2588.86$$

$$\bar{h} = \frac{2588.86 \times 0.02723}{0.8} \quad 88.12 \text{ W/m}^2\text{ K}$$

$$Q = \bar{h}A(T_w - T_\infty) = 88.12 \times (0.8 \times 1)\ (60 - 20)$$

$$= 2820 \text{ W} = 2.82 \text{ kW} \quad \textit{Ans.}$$

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**Example 4.22** Air at 2 atm and 200°C is heated as it flows at a velocity of 12 m/s through a tube with a diameter of 3 cm. A constant heat flux condition is maintained at the wall and the wall temperature is 20°C above the air temperature all along the length of the tube. Calculate (a) the heat transfer per unit length of tube and (b) the increase in bulk temperature of air over a 4 m length of the tube. Properties of air at 200°C are Pr = 0.681, $\mu = 2.57 \times 10^{-5}$ kg/ms, $k$ = 0.0386 W/m K and $c_p$ = 1.025 kJ/kg K.

**Solution**

$$\rho = \frac{p}{RT} = \frac{101.325 \times 2}{0.287 \times 473} = 1.493 \text{ kg/m}^3$$

$$\text{Re}_d = \frac{\rho u_m D}{\mu} = \frac{1.493 \times 12 \times 0.03}{2.57 \times 10^{-5}} = 20{,}914$$

$$(\text{Re}_d)^{0.8} = (20{,}914)^{0.8} = 2859.89$$

$$(\text{Pr})^{0.4} = (0.681)^{0.4} = 0.8575$$

Since the flow is turbulent, we use Dittus–Boelter equation:

$$\text{Nu}_d = \frac{hD}{k} = 0.023\,(\text{Re}_d)^{0.8}\,(\text{Pr})^{0.4}$$

$$= 0.023 \times 2859.89 \times 0.8575 = 56.404$$

$$\bar{h} = \frac{56.404 \times 0.0386}{0.03} = 72.57 \text{ W/m}^2\text{ K}$$

$$\frac{Q}{L} = h\pi D\,[T_w - T_b] = 72.57 \times \pi \times 0.03\,(20)$$

$$= 136.79 \text{ W/m}$$

$$\dot{m} = \rho \frac{\pi}{4} D^2 u_m = 1.493 \times \frac{\pi}{4} \times (0.03)^2 \times 12$$

$$= 0.012664 \text{ kg/s}$$

$$\dot{m} c_p \Delta T_b = 136.79 \times 4$$

$$\Delta T_b = \frac{136.79 \times 4}{0.012664 \times 1025} = 42.15\text{°C} \quad \textit{Ans.}$$

**Example 4.23** *Given:* Water ($\rho$ = 978 kg/m$^3$, $\mu = 4 \times 10^{-4}$ kg/m-s, $k$ = 0.664 W/mK, Pr = 2.54 at 70°C, $\mu_b = 5.55 \times 10^{-4}$ kg/m-s, $\mu_w = 2.81 \times 10^{-4}$ kg/m-s) enters at 40°C a 0.02 m dia tube having a relative roughness of 0.001 and a constant wall temperature of 90°C with a velocity of 3 m/s and leaves at 60°C.
*To determine:* The length of tube required using Petukhov's equation.

**Solution** Reynolds number, $\text{Re}_d = \dfrac{U_m D \rho}{\mu}$

$$\therefore \qquad \text{Re}_d = \frac{3 \times 0.02 \times 978}{4 \times 10^{-4}} = 146{,}700$$

From Moody's chart (Fig. 4.45) for $\text{Re}_d$ = 146,700 and relative roughness, $\varepsilon/D$ = 0.001, the friction factor is found to be 0.0218. Since $T_w > T_b$, in Petukhov's equation (Eq. 4.213), $n$ = 0.11, which becomes

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$$\text{Nu}_d = \frac{(f/8)\text{Re}_d\text{Pr}}{1.07 + 12.7(f/8)^{1/2}(\text{Pr}^{2/3} - 1)}\left(\frac{\mu_b}{\mu_w}\right)^n$$

$$= \frac{(0.0218/8)(146,700) \times (2.54)}{1.07 + 12.7\left(\frac{0.0218}{8}\right)^{1/2}(2.54^{2/3} - 1)}\left(\frac{5.55}{2.81}\right)^{0.11}$$

$$= 666.8 = \frac{hD}{k}$$

$$\therefore \qquad h = \frac{666.8 \times 0.664}{0.02} = 221.38 \text{ W/m}^2\text{K}$$

$$Q = h\pi L\,(T_w - T_b) = \dot{m}c_p\Delta T_b = \rho\pi r^2 u_m \cdot c_p\Delta T_b$$

$$\therefore \qquad L = \frac{978 \times \pi \times (0.01)^2 \times 2.54 \times 0.664 \times 3 \times (60 - 40)}{4.0 \times 10^{-4} \times 221.38 \times \pi \times 0.02 \times \left(90 - \frac{60 + 40}{2}\right)}$$

$$= 1.397 \text{ m } \textit{Ans.}$$

**Example 4.24** *Given:* Air at 1 atm, 27°C flow across a sphere of 0.015 m dia at a velocity of 5 m/s. A heater inside the sphere maintains the surface temperature at 77°C.
*To find:* The rate of heat transfer from the sphere.

**Solution** Reynolds number, $\text{Re}_d = \dfrac{U_\infty D}{\nu}$

$$\text{Re}_d = \frac{5 \times 0.015}{15.69 \times 10^{-6}} = 4780$$

For flow across a sphere with $\text{Re}_d$ in the range of 3500 and 7600 and Pr between 0.7 and 380, the following equation (Eq. 4.225) can be used.

$$\text{Nu}_d = 2 + (0.4\,\text{Re}_d^{1/2} + 0.06\,\text{Re}_d^{2/3})\,\text{Pr}^{0.4}\left(\frac{\mu_\infty}{\mu_w}\right)^{1/4}$$

$$= 2 + \left[0.4(4780)^{1/2} + 0.06(4780)^{2/3}\right](0.708)^{0.4}\left(\frac{1.8462}{2.075}\right)^{1/4}$$

$$= 2 + (27.66 + 17.07)\ 0.87 \times 0.97$$

$$= 39.75 = \frac{hD}{k}$$

$$h = \frac{39.75 \times 0.02624}{0.015} = 69.54 \text{ W/m}^2\text{K.}$$

$$\therefore \qquad Q = hA(T_w - T_\infty) = 69.54 \times \pi \times \frac{(0.015)^2}{4} \times (77 - 27)$$

$$= 2.46 \text{ W } \textit{Ans.}$$

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**Example 4.25** Hot air flowing through a metal pipe of 20 mm diameter is cooled at a constant rate per unit length of pipe. At a particular section (a), the air velocity in the centre of the pipe is found to be 2 m/s and the wall temperature, as measured by a thermocouple on the inner surface of the pipe, is 250°C. At a section (b), situated 1 m downstream from the section (a), the wall temperature is found to be 200°C. Estimate the mean air temperature at section (b).

**Solution** Assuming for the moment that the flow is laminar, the bulk velocity may be determined from the measured centre-line velocity by the equation

$$\frac{u}{U} = \left(1 - \frac{r^2}{r_0^2}\right)\frac{1}{2}$$

At $\quad r = 0, u = \frac{U}{2} = \frac{2\text{ m/s}}{2} = 1\text{ m/s}$

$$\text{Re}_d = \frac{u_m D\rho}{\mu} = \frac{1 \times 0.02 \times 0.706}{2.67 \times 10^{-5}} = 528$$

(Substituting property values for air at 500 K).

Since $\text{Re}_d < 2300$, the flow is laminar.

For constant heat flux boundary condition

$$\text{Nu}_d = \frac{hD}{k} = 4.364$$

At 500 K, $\quad k = 4.04 \times 10^{-5}\text{ kW/m}^2\text{ K}, c_p = 1.03\text{ kJ/kg K}.$

Mean heat flux through the pipe wall between sections (a) and (b) is

$$q = \frac{Q}{A} = \frac{\dot{m}c_p(T_b - T_a)}{\pi DL} = \frac{0.706 \times \frac{\pi}{4}(0.02)^2 \times 1 \times 1.03(-50)}{\pi \times 0.02 \times 1}$$

$$= -0.182\text{ kW/m}^2$$

Since $q$ is given as constant along the length of the pipe

$$q_a = q_b = q$$

Now, $\quad h = \text{Nu}_d \times \frac{k}{D} = 4.364 \times \frac{4.04 \times 10^{-5}}{0.02} \times 1000$

$$= 8.82\text{ W/m}^2\text{ K}$$

$$q = h\,(T_w - T_m)_b$$

where $T_w$ is the wall temperature and $T_m$ is the mean bulk temperature of air at section (b).

$$(T_m)_b = T_w - \frac{q}{h} = 200 - \frac{(-0.182)}{0.00882} = 221°\text{C} \quad \textit{Ans.}$$

At section (a), $T_m$ would have been (250 + 21) or 271°C. More accurate solution could be obtained if property values at the mean fluid temperature $\left(\frac{271+221}{2} = 246°\text{C}\right)$ over the length of the pipe were used, i.e. at 519 K.

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**Example 4.26** In a power plant feedwater is flowing through a duct of rectangular cross-section 8 cm × 4 cm and the wall temperature is maintained at 170°C throughout. The feedwater flows at a rate of 300 kg/min, enters at a temperature of 20°C and is heated to 150°C. Compare the heat transfer coefficients obtained using (a) Dittus–Boelter equation and (b) Sieder–Tate equation and estimate the required length of the duct. Properties of water at 105°C are Pr = 1.64, $\mu = 265 \times 10^{-5}$ kg/ms, $c_p = 4.226$ kJ/kg K, $k = 683 \times 10^{-6}$ kW/m K and $\mu_w$ at 170°C = $158 \times 10^{-6}$ kg/ms.

**Solution**

$D_h$ = Hydraulic diameter of the duct

$$= \frac{4A}{P} = \frac{4 \times 8 \times 4}{2(8+4)} = 5.333 \text{ cm}$$

$$\Delta T_{1m} = \frac{150 - 20}{\ln \dfrac{150}{20}} = 64.52°\text{C}$$

Property values of water were therefore selected at 170 – 64.52 ≈ 105°C.

$$\text{Re}_d = \frac{\rho u D}{\mu} = \frac{\dot{m} D_h}{A\mu} = \frac{300}{60} \times \frac{5.333 \times 10^{-2}}{32 \times 10^{-4} \times 265 \times 10^{-6}}$$

$$= 314{,}400$$

$$\text{Pr} = 1.64$$

(a) Dittus–Boelter equation

$$\text{Nu}_d = 0.023\,(\text{Re}_d)^{0.8}\,(\text{Pr})^{0.4}$$
$$= 0.023 \times (314,\,400)^{0.8}\,(1.64)^{0.4}$$
$$= 0.023 \times 2.5 \times 10^4 \times 1.219 = 700.8$$

$$h = \frac{700.8 \times 683 \times 10^{-6}}{5.333 \times 10^{-2}} = 8.975 \text{ kW/m}^2 \text{ K} \quad \textit{Ans.}$$

(b) Sieder–Tate equation

$$\text{Nu}_d = 0.027\ \text{Re}_d^{0.8}\,\text{Pr}^{0.33}\left(\frac{\mu}{\mu_w}\right)^{0.14}$$

$$= 0.027 \times 2.5 \times 10^4 \times 1.179\left(\frac{265 \times 10^{-6}}{158 \times 10^{-6}}\right)^{0.14}$$

$$= 855.7$$

$$h = \frac{855.7 \times 683 \times 10^{-6}}{5.333 \times 10^{-2}} = 10.96 \text{ kW/m}^2 \text{ K} \quad \textit{Ans.}$$

The latter is probably more accurate as it takes into account viscosity effects varying with temperature. Taking the latter value of $h$, i.e., 10.96 kW/m$^2$ K, the duct length has been calculated.

$$Q = \dot{m} c_p\,(\Delta T)_{\text{water}} = 5 \times 4.226 \times (150 - 20)$$
$$= 2747 \text{ kW}$$
$$Q = hA\Delta T_{1m}$$
$$2747 = 10.96 \times 2(0.4 + 0.08)L \times 64.52$$
$$L = 16.19 \text{ m.} \quad \textit{Ans.}$$

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**Example 4.27** A liquid metal flows at the rate of 4 kg/s through a constant heat flux 6 cm inner diameter tube in a nuclear reactor. The fluid at 200°C is to be heated with the tube wall 40°C above the fluid temperature. Determine the length of the tube required for 25°C rise in bulk fluid temperature, using the following properties: $\rho = 7.7 \times 10^3$ kg/m$^3$, $\nu = 8 \times 10^{-8}$ m$^2$/s, $c_p$ = 130 J/kg°C, $k$ = 12 W/mK and Pr = 0.011.

**Solution**

$$\text{Re}_d = \frac{u_m D}{\nu} = \frac{\dot{m}}{\rho A}\frac{D}{\nu} = \frac{4 \times 0.06}{7.7 \times 10^3 \times \frac{\pi}{4}(0.06)^2 \times 8 \times 10^{-8}}$$
$$= 137796$$

For fully developed turbulent flow in tubes with uniform heat flux

$$\text{Nu}_d = 0.625\,(\text{Re}_d\,\text{Pr})^{0.4}$$
$$= 0.625\,(137796 \times 0.011)^{0.4} = 11.7$$
$$h_c = \frac{11.7 \times 12}{0.06} = 2340 \text{ W/m}^2\text{ K}$$
$$Q = \dot{m}c_p\,\Delta T = 4 \times 0.13 \times 25 = 13 \text{ kW}$$
$$A = \frac{Q}{h_c\,(T_w - T_b)} = \frac{13 \text{ kW}}{2.34 \text{ kW/m}^2\text{K} \times 40\text{K}} = 0.139 \text{ m}^2$$
$$L = \frac{A}{\pi D} = \frac{0.139}{\pi \times 0.06} = 0.7374 \text{ m} \quad \textit{Ans.}$$

**Example 4.28** Atmospheric air at 250 K and a free stream velocity 32 m/s flows across (a) a circular cylinder and (b) a sphere, each of diameter 2.5 cm. The surface of each is maintained at a uniform temperature of 350 K. You may use Whitaker's correlations

$$\text{Nu}_m = (0.4\,\text{Re}^{0.5} + 0.06\,\text{Re}^{2/3})\,\text{Pr}^{0.4}\left(\frac{\mu}{\mu_w}\right)^{0.25}$$

for cylinder

$$\text{Nu}_m = 2 + (0.4\,\text{Re}^{0.5} + 0.06\,\text{Re}^{2/3})\,\text{Pr}^{0.4}\left(\frac{\mu}{\mu_w}\right)^{0.25}$$

for sphere

Calculate for each (i) the average heat transfer coefficient, (ii) the heat transfer rate per metre length of cylinder, and the sphere and (iii) the drag force acting per metre length of the cylinder. Properties of air at 300 K are,

$k$ = 0.0262 W/m K, $\nu = 16.84 \times 10^{-6}$ m$^2$/s, Pr = 0.708,
$\mu = 1.983 \times 10^{-5}$ kg/ms and $\rho$ = 1.177 kg/m$^3$.

**Solution**

(a) Cylinder

$$\text{Re}_d = \frac{u_\infty D}{\nu} = \frac{32 - 0.025}{16.84 \times 10^{-6}} = 47{,}506$$
$$\text{Nu}_m = [0.4\,(47506)^{0.5} + 0.06\,(47{,}506)^{2/3}]\,(0.708)^{0.4}\ 1$$
$$= (87.18 + 78.98)\,0.87 = 144.56$$

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$$h_m = \frac{144.56 \times 0.0262}{0.025} = 151.5 \text{ W/m}^2\text{ K} \quad \textit{Ans.}$$

$$Q = h_m \pi DL (T_w - T_\infty)$$

$$= 151.5 \times \pi \times 0.0251 \times 1 (350 - 250)$$

$$= 1190 \text{ W/m} \quad \textit{Ans.}$$

Drag coefficient $\overline{C}_f = 1.1$ (From Fig. 4.52)

Drag force,

$$F = \overline{C}_f LD \frac{\rho u_\infty^2}{2} = 1.1 \times 1 \times 0.025 \times \frac{1.177 \times (32)^2}{2}$$

$$= 16.5 \text{ N} \quad \textit{Ans.}$$

(b) Sphere

$$\text{Re}_d = \frac{u_\infty D}{\nu} = \frac{32 \times 0.025}{16.84 \times 10^{-6}} = 47{,}506$$

$$\text{Nu}_m = 2 + 144.56 = 146.56$$

$$h_m = \frac{146.56 \times 0.0262}{0.025} = 153.6 \text{ W/m}^2\text{K} \quad \textit{Ans.}$$

$$Q = h_m \pi D^2 (T_w - T_\infty)$$

$$= 153.6 \times \pi \times (0.025)^2 \ (350 - 250)$$

$$= 30.15 \text{ W} \quad \textit{Ans.}$$

From Fig. 4.52 the mean drag coefficient $\overline{C}_f = 0.45$,

$$\therefore \quad F = C_f \frac{1}{4} \pi D^2 \frac{\rho u_\infty^2}{2} = 0.45 \times \frac{1}{4} \pi \times (0.025)^2 \times \frac{1.177(32)^2}{2}$$

$$= 0.133 \text{ N} \quad \textit{Ans.}$$

**Example 4.29** *Given:* Air is heated by passing it through a 0.025 m bore copper tube, maintained at 280°C, from 15°C to 270°C and at a mean velocity of 30 m/s.
*To find:* The length of the tube and the pumping power required, by taking $f = 0.3164/(\text{Re}_d)^{1/4}$ and properties of air at the mean film temperature.

**Solution** The mean film temperature $T^*$

$$= \frac{T_w + T_B}{2} = \frac{1}{2}\left[280 + \frac{15 + 270}{2}\right] = 211.25°\text{C} = 484.4 \text{ K.}$$

At 484.4 K, the properties of air are taken from Table A-4 in the Appendix, as given below:

$$\nu = 3.591 \times 10^{-5} \text{ m}^2\text{/s, Pr} = 0.681,\ k = 3.938 \times 10^{-5} \text{ kW/mK,}$$

$$c_p = 1.027 \text{ kJ/kgK and } \rho = 0.73 \text{ kg/m}^3.$$

$$\text{Re}_d = \frac{u_m D}{\nu} = \frac{30 \times 0.025}{3.591 \times 10^{-5}} = 20{,}900$$

$$\therefore \quad \text{Friction factor, } f = \frac{0.3164}{(20{,}900)^{1/4}} = 0.0263$$

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By Reynolds analogy,

$$\mathrm{St}_d = \frac{\mathrm{Nu}_d}{\mathrm{Re}_d\,\mathrm{Pr}} = \frac{f}{8} = \frac{0.0263}{8} = 0.00329$$

$$\mathrm{Nu}_d = 0.00329 \times 20{,}900 \times 0.681 = 46.8$$

$$\therefore \qquad h = \frac{46.8 \times 3.938 \times 10^{-5}}{0.025} = 0.0737 \text{ kW/m}^2\text{K}$$

Mass flow rate, $\dot{m} = \frac{\pi}{4} \times (0.025)^2 \times 30 \times 0.73$

$$= 0.01075 \text{ kg/s}$$

Heat received by air,

$$Q = \dot{m}c_p(Ta_2 - T_{a_1})$$

$$= 0.01075 \times 1.027 \times (270 - 15)$$

$$= 2.815 \text{ kW}$$

Again, $Q = hA\,\Delta T_{\mathrm{lm}}$

where, $\Delta T_{\mathrm{lm}} = \dfrac{\Delta T_1 - \Delta T_e}{\ln(\Delta T_1/\Delta T_2)} = \dfrac{(280-15)-(280-270)}{\ln\dfrac{280-15}{280-270}}$

$$= \frac{255}{3.277} = 77.8^\circ\text{C} = 77.8\text{K}$$

$$\therefore \qquad Q = 2.815 = 0.0737 \times A \times 77.8$$

$$\therefore \qquad A = 0.49 \text{ m}^2 = \pi DL$$

$$\therefore \qquad L = \text{Tube length} = \frac{0.49}{\pi \times 0.025}$$

$$= 6.24 \text{ m } \textit{Ans.}$$

Pumping power, $\dot{W} = \tau_w \pi DL \cdot u_m = \Delta P \cdot \frac{\Delta}{4} D^2 \cdot u_m$

$$\therefore \qquad \dot{W} = \frac{fL}{D}\frac{\rho u_m^2}{2}\cdot\frac{\pi}{4}D^2 \cdot u_m$$

$$= \frac{f}{8} f u_m^3 LD\pi$$

$$= 0.00329 \times 0.73 \times (30)^3 \times 6.24 \times 0.025 \times \pi$$

$$= 31.78 \text{ W } \textit{Ans.}$$

**Example 4.30** In a 25 mm diameter tube the pressure drop per metre length is 0.0002 bar at a section where the mean velocity is 24 m/s and the mean specific heat of the gas is 1.13 kJ/kg K. Calculate the heat transfer coefficient.

**Solution**

$$\tau_w \pi D = \Delta p \frac{\pi}{4} D^2$$

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$$\tau_w = \Delta p \frac{D}{4} = 0.0002 \times 10^5 \times \frac{0.025}{4} = 0.125 \text{ N/m}^2$$

$$\Delta p = \frac{fL}{D}\frac{\rho u_m^2}{2}$$

$$f = \frac{0.0002 \times 10^5 \times 2 \times 0.025}{\rho \times (24)^2} = \frac{1}{576\rho}$$

$$\text{St} = \frac{f}{8} = \frac{1}{8 \times 576\rho} = \frac{h}{\rho c_p u_m}$$

$$h = \frac{1.13 \times 24}{8 \times 576} \times 1000 = 5.885 \text{ W/m}^2\text{ K.} \quad \textit{Ans.}$$

**Example 4.31** The crankcase of an IC engine measuring 0.8 m × 0.2 m may be assumed as a flat plate. The engine runs at a speed of 25 m/s and the crankcase is cooled by the air flowing past it at the same speed. Calculate the heat loss from the crank surface maintained at 85°C to the ambient air at 15°C. Due to road induced vibration, the boundary layer becomes turbulent from the leading edge itself.

**Solution** $u = 25$ m/s, $T_\omega = 85°\text{C}$, $T_a = 15°\text{C}$, $L = 0.8$ m, $B = 0.2$ m.

$$\text{Properties of air at } T_f = \frac{T_\omega + T_a}{2} = \frac{85 + 15}{2} = 50°\text{C} = 50°\text{C}$$

$$v = 17.95 \times 10^{-6} \text{ m}^2\text{/s}, k = 0.02824 \text{ W/mK}, \text{Pr} = 0.698$$

$$\therefore \quad \text{Re}_L = \frac{uL}{v} = \frac{25 \times 0.8}{17.95 \times 10^{-6}} = 1.114 \times 10^6$$

Since $\text{Re}_L > 5 \times 10^5$, the flow is turbulent.

$$\therefore \quad \text{Nu}_L = \frac{hL}{k} = 0.036(\text{Re}_L)^{0.8} (\text{Pr})^{1/3}$$

$$= 0.036\,(1.114 \times 10^6)^{0.8}\,(0.698)^{0.333}$$

$$= 0.036 \times 68787.3 \times 0.887$$

$$= 2196.5$$

$$\therefore \quad \bar{h} = \frac{2196.5 \times 0.02824}{0.8} = 77.53 \text{ W/m}^2\text{K}$$

Heat loss from the crankcase

$$Q = \bar{h}BL(T_w - T_a)$$

$$= 77.53 \times 0.2 \times 0.8\,(85 - 15)$$

$$= 868.34 \text{ W} \quad \textit{Ans.}$$

**Example 4.32** A square plate maintained at 95°C experiences a force of 10.5 N when forced air at 25°C flows over it at a velocity of 30 m/s. Assuming the flow to be turbulent and using Colburn analogy calculate (a) the heat transfer coefficient and (b) the heat loss from the plate surface. Properties of air at 60°C are:
$\rho = 1.06$ kg/m$^3$, $c_p = 1.005$ kJ/kgK, $v = 18.97 \times 10^{-6}$ m$^2$/s, Pr = 0.696.

**Solution** The drag force for turbulent flow

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$$F = \bar{c}_f \times \frac{1}{2}\rho A u^2 = \frac{0.072}{(\text{Re}_L)^{0.2}} \times \frac{1}{2}\rho A u^2$$

$$= \frac{0.072}{(25 \times L / 18.97 \times 10^{-6})^{0.2}} \times \frac{1}{2} \times 1.06 \times L^2 \times (30)^2$$

$$10.5 = \frac{34.34\, L^2}{16.75\, L^{0.2}} = 2.05\, L^{1.8}$$

$$\therefore \qquad L = \left(\frac{10.5}{2.05}\right)^{\frac{1}{1.8}} = 2.478 \text{ m}$$

$$\text{Re}_L = \frac{uL}{v} = \frac{30 \times 2.478}{18.97 \times 10^{-6}} = 3.919 \times 10^6$$

Average skin friction coefficient

$$\bar{c}_f = \frac{0.072}{(\text{Re}_L)^{0.2}} = 3.457 \times 10^{-3}$$

By Colburn analogy,

$$\text{St Pr}^{2/3} = \frac{\bar{c}_f}{2}$$

$$\frac{\bar{h}}{\rho c_p u} (\text{Pr})^{2/3} = \frac{\bar{c}_f}{2}$$

$$\bar{h} = \frac{1.06 \times 1.005 \times 10^3 \times 30}{(0.696)^{0.667}} \times \frac{3.457 \times 10^{-3}}{2} = \frac{0.0552 \times 10^3}{0.7852} = 70.35 \text{ W/m}^2\text{K} \quad \textit{Ans.}$$

$$Q = 70.35 \times (2.478)^2 \times (95 - 25)$$

$$= 30238 \text{ W} = 30.238 \text{ kW} \quad \textit{Ans.}$$

**Example 4.33** Air at 20°C flows past a 800 mm long plate at a velocity of 45 m/s. If the surface of the plate is maintained at 300°C, determine (a) the heat transferred from the entire plate length to air taking into consideration both laminar and turbulent portions of the boundary layer, (b) the percentage error if the boundary layer is assumed to be turbulent from the leading edge of the plate. Assume unit width of the plate. Take the properties of air at 160°C as: $k$ = 0.03638 W/mK, $v = 30.08 \times 10^{-6}$ m$^2$/s, Pr = 0.682.

**Solution**

$$L = 0.8 \text{ m}, u = 45 \text{ m/s}, T_\omega = 300°\text{C}, T_\infty = 20°\text{C}.$$

$$\text{Re}_L = \frac{uL}{v} = \frac{45 \times 0.8}{30.08 \times 10^{-6}} = 1.197 \times 10^6$$

$$\text{Re}_c = 5 \times 10^5 = \frac{45 \times x_c}{30.08 \times 10^{-6}}$$

$$x_c = 0.3342 \text{ m}$$

Laminar boundary layer region:

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$$\bar{h} = 0.664 \frac{k}{x_c} (\text{Re}_c)^{0.5} (\text{Pr})^{0.333}$$

$$= 0.664 \times \frac{0.03638}{0.3342} \times (5 \times 10^5)^{0.5} (0.682)^{0.333}$$

$$= 51.11 \times 0.88 = 45 \text{ W/m}^2\text{K}$$

$$\therefore \qquad Q_{\text{lam}} = 45 \times (0.3342 \times 1) \times (300 - 20) = 42.10 \text{ W}$$

Turbulent boundary layer region:

$$\bar{h} = 0.036 \frac{k}{L - x_c} [(\text{Re}_L)^{0.8} - (\text{Re}_c)^{0.8}] \text{Pr}^{0.333}$$

$$= 0.036 \frac{0.03638}{0.8 - 0.3342} [(1.197 \times 10^6)^{0.8} - (5 \times 10^5)^{0.8}] (0.682)^{0.333}$$

$$\bar{h} = 2.474 \times 10^{-3} (72857.68 - 36238.98)$$

$$= 90.6 \text{ W/m}^2\text{K}$$

$$Q_{\text{turb.}} = 90.6 \times (0.8 - 0.3342) \times 280$$

$$= 11816.41 \text{ W}$$

$\therefore$ Total heat transfer, $Q$

$$= Q_{\text{lam}} + Q_{\text{turb.}}$$

$$= 4210 + 11816.41$$

$$= 16026.41 \text{ W} \quad \textit{Ans.}$$

If the boundary layer is turbulent from the leading edge itself,

$$\bar{h} = 0.036 \frac{k}{L} \text{Re}_L^{0.8} \text{Pr}^{0.333}$$

$$= 0.036 \frac{0.03638}{0.8} (1.197 \times 10^6)^{0.8} (0.682)^{0.333}$$

$$= 105 \text{ W/m}^2\text{K}$$

$$\therefore \qquad (Q_{\text{total}})_{\text{turb.}} = 105 \times 0.8 \times 1 \times 280 = 23520 \text{ W} = 23.52 \text{ kW}$$

$$\therefore \quad \text{Percentage error} = \frac{23.52 - 16.026}{16.026} \times 100$$

$$= 46.76\% \quad \textit{Ans.}$$

**Example 4.34** Water flows at a velocity of 12 m/s in a straight tube of 60 mm diameter. The tube surface temperature is maintained at 70°C and the flowing water is heated from the inlet temperature of 15°C to an outlet temperature of 45°C. Taking the physical properties of water at the mean bulk temperature of 30°C as $\rho$ = 995.7 kg/m$^3$, $c_p$ = 4.174 kJ/kgK, $k$ = 61.718 × 10$^{-2}$ W/mK, $\nu$ = 0.805 × 10$^{-6}$ m$^2$/s and Pr = 5.42, calculate (a) the heat transfer coefficient from the tube surface to the water, (b) the heat transferred and (c) the length of the tube.

**Solution** Reynolds number, $\text{Re}_d = \frac{u_m d}{\nu} = \frac{12 \times 0.06}{0.805 \times 10^{-6}}$

$$= 0.894 \times 10^6$$

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Since $\text{Re}_d > 2300$, the flow is turbulent. Using the Dittus–Boelter equation

$$\text{Nu}_d = \frac{hd}{k} = 0.023\,(\text{Re}_d)^{0.8}\,(\text{Pr})^{0.4}$$

$$\therefore \quad \frac{h \times 0.06}{0.61718} = 0.023\,(894000)^{0.8}\,(5.42)^{0.4}$$

$$= 0.023 \times 57685.95 \times 1.966$$

$$= 2608.54$$

$$\therefore \quad \bar{h} = 26832.32 \text{ W/m}^2\text{K} \quad \textit{Ans.}\text{ (a)}$$

Heat transferred, $Q = \dot{m}c_p\,(T_2 - T_1)$

$$= \rho \frac{\pi}{4} d^2 u_m c_p (T_2 - T_1)$$

$$= 995.7 \times \frac{\pi}{4}\,(0.06)^2 \times 12 \times 4.174 \times 10^3 \times (45 - 15)$$

$$= 4230355 \text{ W} \quad \textit{Ans.}\text{ (b)}$$

Now, $Q = \bar{h}\pi dL\,(T_w - T_b)$

$$4230355 = 26832.32 \times \pi \times 0.06 \times L\,(70 - 30)$$

$$\therefore \quad L = 20.91 \text{ m} \quad \textit{Ans.}\text{ (c)}$$

**Example 4.35** Water at 20°C flowing at the rate of 0.015 kg/s enters a 25 mm ID tube which is maintained at a temperature of 90°C. Assuming hydrodynamically and thermally fully developed flow determine the heat transfer coefficient and the tube length required to heat the water to 70°C. Given: water properties at 20°C: $\rho$ = 1000.5 kg/m$^3$, $c_p$ = 4181.8 J/kgK, $\nu = 1.006 \times 10^{-6}$ m$^3$/s; properties of water at 45°C: $\rho$ = 992 kg/m$^3$, $c_p$ = 4180 J/kgK, $k$ = 0.638 W/mK, $\nu = 0.613 \times 10^{-6}$ m$^2$/s; the average Nusselt number for the tube $\text{Nu}_d = 3.657$.

**Solution** Given: $T_1 = 20°\text{C}$, $T_2 = 70°\text{C}$, $T_\omega = 90°\text{C}$, $\dot{m} = 0.015$ kg/s, $d = 0.025$ m. Considering the properties of water at the mean bulk temperature, $\frac{20 + 70}{2}$ or 45°C,

$$\text{Nu}_d = 3.657 = \frac{hd}{k}$$

$$\therefore \quad = \frac{3.657 \times 0.638}{0.025} = 93.33 \text{ W/m}^2\text{K} \quad \textit{Ans.}$$

$$Q = \dot{m}c_p(T_2 - T_1) = 0.015 \times 4180\,(70 - 20)$$

$$= 3135 \text{ W}$$

Taking the log-mean temperature difference (Fig. Ex. 4.35),

$$\Delta T_{\text{l.m}} = \frac{70 - 20}{\ln \frac{70}{20}} = \frac{50}{\ln 3.5}$$

$$= 39.9°\text{C}$$

$$Q = hA\,\Delta T_m$$

$$3135 = 93.33 \times \pi \times 0.025 \times L \times 39.9$$

$$\therefore \quad L = 10.72 \text{ m} \quad \textit{Ans.}$$

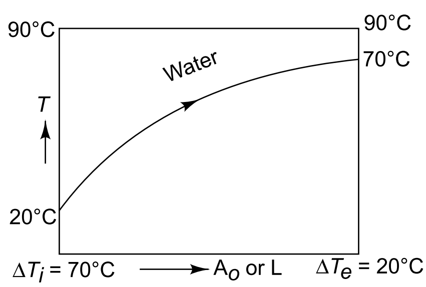


**Fig.** *Ex. 4.35*

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**Example 4.36** Liquid mercury flows at a rate of 1.6 kg/s through a copper tube of 20 mm diameter. The mercury enters the tube at 15°C and leaves at 35°C. Calculate the tube length for constant heat flux at the wall which is maintained at an average temperature of 50°C.

For liquid metal flowing through a tube, the following correlation may be used

$$Nu_d = 7 + 0.025\,(Pe)^{0.8}$$

where Pe is the Peclet number.

Properties of mercury at 25°C are:

$$\rho = 13582 \text{ kg/m}^3, c_p = 140 \text{ J/kgK}, k = 8.69 \text{ W/mK},$$
$$\nu = 1.5 \times 10^{-7} \text{ m}^2\text{/s}, Pr = 0.0248.$$

**Solution**

$$d = 0.02 \text{ m}, \dot{m} = 1.6 \text{ kg/s}, T_1 = 15°C, T_2 = 35°C.$$

$$Re_d = \frac{\rho v d}{\mu} = \frac{\rho A v d}{A\mu} = \frac{\dot{m} d}{\frac{\pi}{4} d^2 \mu} = \frac{4\dot{m}}{\pi d \mu}$$

$$= \frac{4\dot{m}}{\pi d \rho \nu} = \frac{4 \times 1.6}{\pi \times 0.02 \times 13582 \times 1.5 \times 10^{-7}}$$

$$= 49997$$

$\therefore$ $$Nu_d = 7 + 0.025\,(Re_d \cdot Pr)^{0.8}$$

$$\frac{\bar{h}d}{k} = 7 + 0.025\,(49997 \times 0.0248)^{0.8} = 14.46$$

$\therefore$ $$\bar{h} = 14.46 \times \frac{8.69}{0.02} = 6282.87 \text{ W/m}^2\text{K}$$

$$Q = \bar{h} A\,\Delta T = 6282.87 \times \pi \times 0.02 \times L \times (50 - 25)$$
$$= 9869.11\,L$$

Again, $$Q = \dot{m} c_p (T_2 - T_1) = 1.6 \times 140 \times (35 - 15)$$
$$= 4480 \text{ W}$$

$\therefore$ $$L = \frac{4480}{9869.11} = 0.454 \text{ m} \quad \textit{Ans.}$$

## Summary

The transfer of heat by convection is intimately related to the mechanics of fluid flow in the vicinity of the transfer surface. The nature of heat transfer and the flow phenomena depend greatly on whether the fluid far away from the surface is in laminar or in turbulent flow. The principles of boundary layer theory are discussed in this context. The conservation equations of mass, momentum and energy for laminar flow are derived. Similarity principles applied to heat transfer are discussed and the relevant dimensionless parameters are derived from the differential equations representing conservation of mass, momentum and energy. Convective heat transfer coefficients by dimensional analysis using both Rayleigh's method and Buckingham $\pi$-theorem are evaluated. Analytical solution for laminar flow over a flat plate is first given and then the approximate integral boundary layer analysis is discussed. Afterwards the turbulent flow over a flat plate, the mixing length theory of Prandtl, and the analogy between momentum and heat transfer are explained. Forced

convection inside the tubes and ducts is then taken up and heat transfer coefficients for laminar flow under conditions of constant heat flux and constant wall temperature are derived. The empirical correlations for turbulent flow inside the tubes and over exterior surfaces obtained from experimental data are discussed.

## Important Formulae and Equations

| *Equation Number* | *Equation* | *Remarks* |
|---|---|---|
| (4.2) | $\tau = \mu \dfrac{du}{dy}$ | Newton's law of viscosity |
| (4.3) | $\mathrm{Re} = \dfrac{\rho u_m D}{\mu} = \dfrac{u_m D}{\nu}$ | Reynolds number in pipe flow |
| (4.4) | $\mathrm{Re}_{x_c} = \dfrac{u_\infty x_c}{\nu} = 5 \times 10^5$ | Critical Reynolds number for flow over a flat plate |
| (4.7) | $\tau_x = C_{fx} \rho u_\infty^2 / 2$ | Shear stress related to local drag coefficient |
| (4.9) | $\bar{C}_f = \dfrac{1}{L}\int_0^L C_{fx}\, dx$ | Mean drag coefficient over the flat length $L$ |
| (4.10) | $F = bL\bar{C}_f \dfrac{\rho u_\infty^2}{2}$ | Drag force on a plate of length $L$ and width $b$ |
| (4.11) | $\theta(x, y) = \dfrac{(T_w - T)}{(T_w - T_\infty)}$ | Dimensionless temperature |
| (4.17) | $Q = -k_f A \dfrac{T_\infty - T_w}{\delta_t} = hA(T_w - T_\infty)$ | Convective heat transfer through a fluid film |
| (4.41) | $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0$ | Continuity equation for steady and incompressible flow |
| (4.42) | $u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} + w\dfrac{\partial u}{\partial z}$ $= -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} + \nu\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}\right)$ | $x$-momentum equation for steady compressible flow |
| (4.43) | $u\dfrac{\partial T}{\partial x} + v\dfrac{\partial T}{\partial y} + w\dfrac{\partial T}{\partial z}$ $= \alpha\left(\dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 T}{\partial y^2} + \dfrac{\partial^2 T}{\partial z^2}\right)$ | Energy equation |

(*Contd*)

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| *Equation Number* | *Equation* | *Remarks* |
|---|---|---|
| (4.58) | $h = \frac{k}{L} F(\mathrm{Re}, \mathrm{Pr})$ | Forced convection: heat transfer coefficient is a function of Reynolds number and Prandtl number |
| (4.61) | $\mathrm{Nu} = F(\mathrm{Gr}, \mathrm{Pr})$ | Natural convection: Nusselt number is a function of Grashof number and Prandtl number |
| (4.62) | $\mathrm{Fo} = \frac{\alpha t}{L^2}$ | Fourier number, also referred to as dimensionless time |
| (4.72) | $p_1 - p_2 = \frac{fL}{D}\frac{\rho u^2}{2}$ | Pressure drop of fluid in a pipe of length $L$ |
| (4.89) | $\frac{\delta}{x} = \frac{5}{(\mathrm{Re}_x)^{1/2}}$ | Local hydrodynamic boundary layer thickness depends on the local Reynolds number: Exact solution of Pohlhausen |
| (4.90) | $\tau_w = 0.332\mu \frac{u_\infty}{x}(\mathrm{Re}_x)^{1/2}$ | Wall shear stress |
| (4.91) | $C_{fx} = \frac{0.664}{(\mathrm{Re}_x)^{1/2}}$ | Local friction or drag coefficient |
| (4.92) | $\bar{C}_f = \frac{1.328}{(\mathrm{Re}_L)^{1/2}}$ | Average friction coefficient |
| (4.94) | $\frac{\delta}{\delta_t} = \mathrm{Pr}^{1/3}$ | For Pr < 1, $\delta_t > \delta$ and for fluid having Pr > 1, $\delta_t < \delta$ |
| (4.99) | $\mathrm{Nu}_L = 0.332\,\mathrm{Re}_x^{1/2}\,\mathrm{Pr}^{1/3}$ | Local Nusselt number for flow over a flat plate |
| (4.100) | $\mathrm{Nu}_L = 0.664\,\mathrm{Re}_L^{1/2}\,\mathrm{Pr}^{1/3}$ | Average Nusselt number for flow over a flat plate of length $L$ |
| (4.106) | $\overline{x} = \frac{4.64}{(\mathrm{Re}_x)^{1/2}}$ | Local boundary layer thickness by von Kàrmàn integral method |
| (4.117) | $C_{fx} = \frac{0.647}{\mathrm{Re}_x^{1/2}}$ | The value by approximate method is close to that by exact method, Eq. 4.91 |
| (4.120) | $\mathrm{Nu}_x = 0.331\,\mathrm{Re}_x^{1/2}\,\mathrm{Pr}^{1/3}$ | The expression is very close to the exact solution, Eq. (4.99) |
| (4.127) | $\tau = \rho(\nu + \varepsilon_m)\frac{d\bar{u}}{dy}$ | Shear stress in turbulent flow |

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| *Equation Number* | *Equation* | *Remarks* |
|---|---|---|
| (4.132) | $\frac{Q_t}{A} = \rho c_p(\alpha + \varepsilon_H)\frac{\mathrm{d}\bar{T}}{\mathrm{d}y}$ | Total heat transfer per unit area in trubulent flow |
| (4.133) | $\mathrm{Pr}_t = \frac{\varepsilon_M}{\varepsilon_H} = \frac{\text{Eddy viscosity}}{\text{Eddy diffusivity}}$ | Turbulent Prandtl number |
| (4.138) | $\frac{h_{cx}}{\rho c_p u_\infty} = \frac{C_{fx}}{2} = \mathrm{St}_x$ | Reynolds analogy (for fluids having Pr = 1) |
| (4.139) | $\mathrm{St}_x\,\mathrm{Pr}^{2/3} = \frac{C_{fx}}{2}$ | Reynolds-Colburn analogy |
| (4.140) | $C_{fx} = \frac{0.0576}{(\mathrm{Re}_x)^{0.2}}$ | Local friction coefficient for turbulent flow over a flat plate |
| (4.141) | $\bar{C}_f = \frac{0.072}{(\mathrm{Re}_L)^{0.2}}$ | Average friction coefficient for turbulent flow over a plate of length *L*. |
| (4.142) | $\mathrm{Nu} = 0.0288\,\mathrm{Re}_x^{0.8}\,\mathrm{Pr}^{1/3}$ | Local Nusselt number for turbulent flow for $x > x_c$ |
| (4.143) | $\mathrm{Nu}_L = 0.036\,\mathrm{Re}_L^{0.8}\,\mathrm{Pr}^{1/3}$ | For turbulent flow over a flat surface for $L > x_c$ |
| (4.144) | $\mathrm{Nu}_L = (0.036\,\mathrm{Re}_L^{0.8} - 835)\,\mathrm{Pr}^{1/3}$ | Total heat transfer from a plate where flow in laminar upto $x = x_c$ and turbulent over length $(L - x_c)$ |
| (4.146) | $\bar{C}_f = \frac{0.072}{\mathrm{Re}_L^{0.2}} - \frac{1670}{\mathrm{Re}_L}$ | Average friction coefficient in a flat plate, partly laminar and the rest turbulent flow |
| (4.147) | $\mathrm{Nu}_x = 0.453\,\mathrm{Re}_x^{1/2}\,\mathrm{Pr}^{1/3}$ | Local Nusselt number for constant heat flux |
| (4.153) | $\frac{\delta}{x} = \frac{0.376}{(\mathrm{Re}_x)^{1/5}}$ | Boundary layer thickness (local) in turbulent flow |
| (4.154) | $\delta^* = \frac{\delta}{8}$ | Displacement thickness for turbulent flow over a flat plate |
| (4.159) | $\tau_w = \frac{0.0296\rho u_\infty^2}{(\mathrm{Re}_x)^{0.2}}$ | Local shear stress for turbulent flow over a plate |
| (4.164) | $\mathrm{Re}_d = \frac{u_m D_H}{v}, \mathrm{Nu}_d = \frac{\bar{h}_c D_H}{k}$ where $D_H = \frac{4A}{P}$ | For flow and heat transfer in a non-circular duct, where $D_H$ is the hydraulic diameter |

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| *Equation Number* | *Equation* | *Remarks* |
|---|---|---|
| (4.168) | $\dfrac{u}{u_{max}}=1-\left(\dfrac{r}{R}\right)^2=\dfrac{u}{u_0}$ | Velocity distribution for laminar flow in a tube of radius $R$ |
| (4.170) | $u_0 = 2u_m$ | Centre-line velocity is twice the mean velocity in laminar flow |
| (4.171) | $\Delta p=\dfrac{fL}{D}\dfrac{\rho u_m^2}{2}$ | Pressure drop for flow in a tube where $f$ is the Darey friction factor |
| (4.172) | $C_f=\dfrac{f}{4}$ | Fanning friction coefficient. |
| (4.173) | $\Delta p=\dfrac{32\mu L u_m}{D^2}$ | Hagen–Poiseuille equation for laminar flow in a tube. |
| (4.174) | $f=\dfrac{64}{\mathrm{Re}_d}$ | Darcy friction factor $f$ related to Reynolds number for laminar flow |
| (4.175) | $P=\dot{V}.\Delta p/\eta_p$ | Pumping power $P$ where $\dot{V}$ is the volumetric flow rate ($m^3$/s) and $\eta_p$ is the pumping efficiency |
| (4.184) | $\mathrm{Nu}_d=\dfrac{48}{11}=4.364$ | Nusselt number for laminar flow through a tube under constant heat flux |
| (4.185) | $\mathrm{Nu}_d = 3.66$ | Nusselt number for laminar flow through a tube with constant wall temperature |
| (4.202) | $\mathrm{St}_d=\dfrac{f}{8}$ | Reynolds analogy for momentum and heat transfer for turbulent flow |
| (4.204) | $\mathrm{St}_d=\dfrac{f}{8}\cdot\dfrac{1}{1+5[f/8]^{1/2}(\mathrm{Pr}-1)}$ | Prandtl analogy which reduces to Eq. (202) for Pr = 1 for fully developed turbulent flow |
| (4.205) | $\mathrm{St}_d=\dfrac{f}{8}\dfrac{1}{1+5\left(\dfrac{f}{8}\right)^{1/2}(\mathrm{Pr}-1)+\ln\dfrac{5\,\mathrm{Pr}+1}{6}}$ | Kàrmàn analogy for fully developed turbulent flow |
| (4.210) | $\mathrm{Nu}_d=\dfrac{\bar{h}_c D}{k}=0.023(\mathrm{Re}_d)^{0.8}(\mathrm{Pr})^n$ | Dittus Boelter equation for turbulent flow where $n$ = 0.4 for heating and $n$ = 0.3 for cooling |
| (4.211) | $\mathrm{Nu}_d=0.027\,\mathrm{Re}_d^{0.8}\,\mathrm{Pr}^{1/3}\left(\dfrac{\mu_b}{\mu_w}\right)^{0.14}$ | Sieder-Tate equation for turbulent flow through tube, where all properties except $\mu_w$ are evaluated at the bulk temperature and $\mu_w$ at the wall temperature |
| (4.215) | $\mathrm{Nu}_d=0.625(\mathrm{Re}_d.\mathrm{Pr})^{0.4}$ | For fully developed turbulent flow of liquid metals under uniform heat flux condition |

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## Objective Type Questions

4.1 If we imagine a curve within a fluid, the tangent at every point of which indicates the direction of the velocity of the fluid particle, then the curve is known as a
(a) boundary layer (b) stream line
(c) streak line (d) laminar curve

4.2 A fluid is flowing over a flat plate. To make another plate above the flat plate at a distance $y$ with a velocity $u_0$, a force is necessary, which is proportional to
(a) the velocity $u_0$
(b) the inverse of $y$
(c) the dynamic viscosity of the fluid
(d) all of the above

4.3 Viscous effect of a fluid
(a) is found in the turbulent core
(b) is confined outside the boundary layer
(c) is confined in the boundary layer
(d) is confined in the potential flow

4.4 The zone of transition between the laminar sublayer and the turbulent core is called
(a) transition zone
(b) buffer layer
(c) boundary layer
(d) none of the above

4.5 For flow over a flat plate, the critical value of Reynolds number characterizes
(a) the flow when eddies start forming in the fluid
(b) the end of the laminar boundary layer
(c) the beginning of turbulent boundary layer
(d) all of the above

4.6 For flow over a cylinder, the flow separation occurs at th location $s$ for which
(a) $\left(\frac{\partial u}{\partial y}\right)_s = 0$
(b) $\left(\frac{\partial u}{\partial y}\right)_s > 0$
(c) $\left(\frac{\partial u}{\partial y}\right)_s < 0$
(d) when Reynolds number exceeds 2300

4.7 A fluid is flowing over a flat plate with a free stream velocity $u_\infty$. The distance by which an equivalent uniform stream would have to be displaced from the surface to give the same volume flow of the fluid is called the
(a) boundary layer thickness
(b) displacement thickness
(c) momentum thickness
(d) energy thickness

4.8 A fluid is flowing at a uniform temperature $T_\infty$ along a flat plate maintained at a constant temperature $T_w$. If we define the dimensionless temperature $\theta(x, y)$ as

$$\theta(x, y) = \frac{T_w - T}{T_w - T_\infty}$$

where $T(x, y)$ is the local temperature in the fluid, the thermal boundary layer thickness $\delta_t(x)$ is the locus of points along the plate where $\theta$ is equal to
(a) 1.0 (b) 0
(c) 0.99 (d) 0.95

4.9 A fluid is flowing along a plate having a high uniform wall temperature. The heat transfer coefficient along the length
(a) decreases
(b) increases
(c) remains constant
(d) first decreases and then increases

4.10 For heat transfer from an isothermal plate to a fluid flowing along it, identify the incorrect statement. Higher the value of Reynolds number,
(a) higher will be the rate of mixing
(b) higher the value of $\delta_t$
(c) higher the value of $h$
(d) higher will be the rate of heat transfer

4.11 The conditions which are important to study the analysis of a heat transfer process under steady state are
(a) geometric conditions
(b) physical conditions
(c) boundary conditions
(d) all of the above

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4.12 The momentum and energy equations become identical for fluids having
(a) $\nu = \alpha$ (b) $\nu > \alpha$
(c) $\nu < \alpha$ (d) $\mathrm{Re} = \infty$

4.13 The ratio of kinetic energy of flow relative to boundary layer enthalpy difference is called
(a) Biot number
(b) Eckert number
(c) Grashhoff number
(d) Stanton number

4.14 The ratio of surface shear stress to free stream kinetic energy is called
(a) friction factor (b) Weber number
(c) drag coefficient (d) drag force

4.15 Choose the correct statement:
For a given value of Nusselt number, the convective heat transfer coefficient $h$
(a) decreases with increasing thermal conductivity of the fluid.
(b) increases with distance $x$ from the leading edge.
(c) increases with the thermal conductivity of fluid.
(d) is independent of the distance $x$.

4.16 In laminar boundary layers over a flat plate, the ratio of $\delta_t/\delta$ is equal to
(a) $\mathrm{Pr}^{1/2}$ (b) $\mathrm{Pr}^{1/3}$
(c) $\mathrm{Pr}^{-1/3}$ (d) $\mathrm{Pr}^{1/4}$

4.17 In a fully developed laminar flow through a tube,
(i) $\dfrac{\partial u}{\partial x} = 0$
(ii) $\dfrac{\partial T}{\partial x}$ at any radius $r$ is not zero.
(iii) the temperature profile $T(r)$ continuously changes with $x$
(iv) for constant tube wall temperature, surface heat flux is constant.
Out of the above statements:
(a) only (i) and (ii) are correct
(b) (i), (ii) and (iii) are correct
(c) all four are correct
(d) only (i) and (iv) are correct

4.18 It is a property of a fluid:
(a) Eddy viscosity
(b) Eddy diffusivity
(c) Kinematic viscosity
(d) Turbulent Prandtl number

4.19 **Assertion (A):** In flow over a flat plate, the turbulent heat transfer coefficient is much larger than the laminar heat transfer coefficient at a given value of the Reynolds number.
**Reasoning (R):** Because in turbulent flow the heat transfer is aided by innumerable eddies which carry lumps of fluid across streamlines.
**Codes:**
(a) A is false R is true
(b) A is true, R is false
(c) Both A and R are false
(d) Both A and R are true

4.20 The convective heat transfer coefficient for laminar and turbulent flows over a flat plate varies respectively as
(a) $x^{1/2}$ and $x^{0.2}$ (b) $x^{1/2}$ and $x^{-0.2}$
(c) $x^{-1/2}$ and $x^{-0.2}$ (d) $x^{1/3}$ and $x^{0.2}$

4.21 In liquid metal heat transfer, $\delta_t/\delta$ is
(a) very small
(b) very large
(c) about 1
(d) dependent on thermal conductivity of the film

4.22 In liquid metal heat transfer, Nusselt number is a function of
(a) Prandtl number only
(b) Reynolds number only
(c) Peclet number only
(d) Peclet and Reynolds number

4.23 The average turbulent shear stress is given by
(a) $-\rho\overline{u'v'}$ (b) $\rho\overline{u'v'}$
(c) $-\rho\bar{u}'v'$ (d) $-\overline{\rho u'v'}$

4.24 The flow and heat transfer in coiled tubes is governed by
(a) Reynolds number
(b) Dean number
(c) Nusselt number
(d) Prandtl number

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4.25 By Reynolds analogy between heat and momentum transfer in turbulent flow the Stanton number is given by

(a) $\frac{f}{4}$ (b) $\frac{f}{8}$

(c) $\frac{f}{2}$ (d) $\frac{f}{16}$

where $f$ is the Darcy friction factor.

4.26 Colburn's $j$-factor for turbulent flow over a flat plate is given by

(a) $\text{St}_x\, \text{Pr}^{1/3}$ (b) $\text{St}_x\, \text{Pr}^{1/2}$

(c) $\text{St}_x\, \text{Pr}^{2/3}$ (d) $\text{St}_x^{1/2}\, \text{Pr}^{2/3}$

4.27 The Nusselt number for fully developed laminar flow in a tube under constant heat flux

(a) depends on Reynolds number
(b) depends on Prandtl number
(c) is constant
(d) depends on Graetz number

4.28 This equation can be used to determine experimentally the dynamic viscosity of a fluid

(a) Hagen–Poiseuille equation
(b) Dittus–Boelter equation
(c) Sieder–Tate equation
(d) Hausen's equation

4.29 When a hot fluid is flowing over a cold flat plate, the temperature gradient

(a) is zero at the surface
(b) is negative at the surface
(c) is zero at the edge of the thermal boundary layer
(d) is positive at the edge of the thermal boundary layer

4.30 The boundary layer thickness at a distance $x$ from the leading edge in a laminar boundary layer (Blasius velocity profile is

(a) $5.0\,\text{Re}_x^{-1/2}$ (b) $4.64\,\text{Re}_x^{-1/2}$

(c) $5.0(x^{1/2}\nu^{1/2}u_\infty^{-1/2})$ (d) $5.0\,\text{Re}_x^{1/2}$

4.31 If the local heat transfer coefficient is $h_x$ at a distance $x$ from the leading edge, the average heat transfer coefficient is

(a) $2h_x$ (b) $\sqrt{h_x}$

(c) $3h_x$ (d) $4h_x$

## Answers

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 4.1 | (b) | 4.2 | (d) | 4.3 | (c) | 4.4 | (b) | 4.5 | (d) |
| 4.6 | (a) | 4.7 | (b) | 4.8 | (c) | 4.9 | (a) | 4.10 | (b) |
| 4.11 | (d) | 4.12 | (a) | 4.13 | (b) | 4.14 | (c) | 4.15 | (c) |
| 4.16 | (c) | 4.17 | (b) | 4.18 | (c) | 4.19 | (d) | 4.20 | (c) |
| 4.21 | (b) | 4.22 | (c) | 4.23 | (a) | 4.24 | (b) | 4.25 | (b) |
| 4.26 | (c) | 4.27 | (c) | 4.28 | (a) | 4.29 | (c) | 4.30 | (c) |
| 4.31 | (a) | | | | | | | | |

## Open Book Problems

4.1 Calculate the pressure drop in a 20 mm × 25 mm smooth rectangular duct of 100 m length when water at 40ºC flows through it with a velocity of 1 m/s. Given, $\nu = 0.66 \times 10^{-6}$ m²/s, $\rho = L$ 995 kg/m³.

*Hints:* Find

$D_H = \frac{4A}{p}$ and $\text{Re}_d = \frac{u_m D_H}{\nu}$. For $\text{Re}_d \le 2 \times 10^4$, take $f = 0.316\,\text{Re}_d^{-0.25}$ and for $\text{Re}_d \ge 2 \times 10^4$, take $f = 0.184\,\text{Re}_d^{-0.2}$ [Eq. 4.206 and 4.206a].

Then calculate $\Delta p = \frac{fL}{D_H}\frac{\rho u_m^2}{2}$ where $L$ = 100 m, $u_m$ = 1 m/s and $\rho$ = 995 kg/m³.

4.2 If velocity distribution in laminar boundary layer of a flat plate is assumed to be given by second order polynomial $u = a + by +$

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$cy^2$, determine the form using the necessary boundary conditions.

*Hints:* At $y = 0$, $u = 0$ $\therefore a = 0$

At $y = \delta$, $u = u_\infty$ and $\frac{du}{dy} = 0$, which gives

$u_\infty = b\delta + c\delta^2$

$\left(\frac{du}{dy}\right)_{y=\delta} = b + 2c\delta = 0$. On simplification obtain the relation $\frac{u}{u_\infty} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$

4.3 A 1 kW heater is constructed of a glass plate with an electrically conducting film which produces a constant heat flux. The plate is 60 cm × 60 cm and placed in an air stream at 27ºC, 1 atm with $u_\infty$ = 5 m/s. Calculate the average temperature difference along the plate and the temperature difference in the trailing edge.

*Hints:* Properties are evaluated at the film temperature $\frac{T_w + T_\infty}{2}$. Since $T_w$ is not known, first calculate the properties at $T_\infty$ for $v$, Pr and $k$. Find $\text{Re}_L$ and use Eq. (4.149) to find $\overline{T_w - T_\infty}$ substituting $q_w = \frac{1000}{(0.6)^2}\,\text{W/m}^2$ $L$ = 0.6 m, K, $\text{Re}_L$ and Pr. $T_w = \Delta T + T_\infty$ and $T_f = \frac{T_w + T_\infty}{2}$. At this $T_c$, find properties and use Eq. (4.149) to find $\overline{T_w - T_\infty}$.

4.4 Engine oil at 20ºC is forced over 0.2 m square plate at a velocity of 1.2 m/s. The plate is heated to a uniform temperature of 60ºC. Calculate the heat loss by the plate.

*Hints:* This is the case of a flat plate with constant heat flux. At the film temperature $T_f = \frac{20 + 60}{2} = 40^\circ\text{C}$, from property tables in the appendix, find $\rho$, $k$, $v$, and Pr. Check if $\text{Re}_x < 2000$. Use Eq. (4.150) to find $\text{Nu}_x$, $h_x$, $\bar{h} = 2h_x$ and $Q = \bar{h}A\,(T_w - T_\infty)$ W.

4.5 Air at 20ºC and at a pressure of 1 bar is flowing over a flat plate at a velocity of 3 m/s. If the plate is 280 mm wide and at 56ºC, calculate the following quantities at $x$ = 280 mm, given that properties of air at the bulk mean temperature $\frac{20 + 56}{2}$ = 38ºC are: $\rho$ = 1.1374 kg/m$^3$, $k$ = 0.02732 W/mK, $c_p$ = 1.005 kJ/kgK, $v = 16.768 \times 10^{-6}$ m$^2$/s and Pr = 0.7: (a) the boundary layer thickness, (b) local friction coefficient, (c) the average friction coefficient, (d) shearing stress due to friction, (e) thermal boundary layer thickness, (f) local convective heat transfer coefficient, (g) average friction coefficient, (h) the rate of heat transfer by convection, (i) total drag force on the plate and (j) total mass flow rate through the boundary.

*Hints:* First find $\text{Re}_x = \frac{u_\infty x}{v}$ and establish that the flow is laminar. Then find

(a) $\delta = \frac{5x}{\sqrt{\text{Re}_x}}$

(b) $C_{fx} = \frac{0.664}{\sqrt{\text{Re}_x}}$

(c) $\bar{C}_f = \frac{1.328}{\sqrt{\text{Re}_L}}$

(d) $\tau_w = C_{fx}\,\rho u_\infty^2/2$

(e) $\delta_t = \frac{\delta}{(\text{Pr})^{1/3}}$

(f) $h_x = 0.332\frac{k}{x}\text{Re}_x^{1/2}\,\text{Pr}^{1/3}$

(g) $\bar{h} = 2h_x$

(h) $Q = \bar{h}A(T_w - T_\infty)$

(i) $F_D = \tau_w \times 0.28 \times 0.28$

(j) $\dot{m} = \frac{5}{8}\rho u_\infty(\delta_2 - \delta_1) = \frac{5}{8}\rho u_\infty \delta$

4.6 Air at 20ºC and 1 atm flows over a flat plate at 35 m/s. The plate is 75 cm long and is maintained at 60ºC. Assuming unit width, calculate the heat transfer from the plate.

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*Hints:* At film temperature $T_f = \dfrac{20+60}{2} = 40$ºC $= 313$ K, find the properties of air from the appendix: $\rho$, $\mu$, $c_p$, $k$ and Pr. Check to find $\mathrm{Re}_L = \dfrac{\rho u_m L}{\mu}$ greater than $5 \times 10^5$. It will be a case of mixed boundary layer with laminar flow form $x = 0$ to $x = x_c$ and turbulent from $x_c$ to $L$. Use Eq. 4.145, $\mathrm{Nu}_L = 0.036\,(\mathrm{Re}_L^{0.8} - 835)\,\mathrm{Pr}^{1/3}$ to find $\bar{h}$ and $Q = \bar{h}A\,(T_w - T_\infty)$.

4.7 Water at 40ºC with a mass flow of 0.5 kg/s enters a 2.5 cm ID tube whose wall is maintained at a uniform temperature of 90ºC. Calculate the tube length required for heating the water to 60ºC and also the resulting pressure drop.

*Hints:* At $T_f = \dfrac{40+60}{2} = 50$ºC, find the properties of water from the appendix: $v$, Pr, $\mu$, $\rho$, $c_p$, $k$. From $\dot{m} = \rho \dfrac{\pi}{4} d^2 u_m$, find $u_m = \dfrac{4\dot{m}}{\pi d^2 \rho}$ and $\mathrm{Re}_d$. If the flow is turbulent, you may use Sieder–Tait Eq. (4.211). At $T_w = 90$ºC, find $\mu_w$. Then, $\mathrm{Nu}_d = 0.027\ \mathrm{Re}_d^{0.8}\,\mathrm{Pr}^{1/3}\left(\dfrac{\mu}{\mu_w}\right)^{0.14}$ is used to find $\bar{h}$. With $T_w$ = constant, use Eq. (4.188), to find $Q_c = \bar{h}A(\Delta T)_{\text{l.m.}}$

$$\Delta T_{\text{l.m.}} = \frac{(T_w - T_i) - (T_w - T_e)}{\ln \dfrac{T_w - T_i}{T_w - T_o}},$$

$A = \pi dL$ and $L$. Find friction factor for turbulent flow through a pipe, Eq. (4.206), $f = 0.184\,\mathrm{Re}_d^{-0.2}$ and then $\Delta p = \dfrac{fL}{d}\dfrac{\rho u_m^2}{2}$ can be found.

You could also use Dittus–Boelter equation to find $\bar{h}$.

4.8 In a straight tube of 60 mm diameter, water is flowing at a velocity of 12 mps. The tube surface temperature is maintained at 70ºC and the flowing water is heated from 15ºC to 45ºC. Taking the physical properties of water at its mean bulk temperature, calculate the following: (i) The heat transfer coefficient from the tube surface to the water, (ii) The heat transferred and (iii) The length of the tube.

*Hints:* First find properties of water at the film temperature of 30ºC: $\rho$, $c_p$, $k$, $v$ and Pr. Establish $\mathrm{R_{ed}} > 2300$ and use Dithers–Boelter equation $\mathrm{Nu_d} = 0.023\ \mathrm{Re_d^{0.8}}\ \mathrm{Pr}^{0.333}$ to find $\bar{h}$, $\dot{Q} = \dot{m}q\ (T_2 - T_1)$. Also, $Q = \bar{h}\ \pi dL\ (T_w - T_m)$ from which calculate $L$.

4.9 The main duct of an air conditioning system is rectangular in cross-section (400 mm × 800 mm) and has air at atmospheric pressure and at 20°C flowing with a velocity of 7 m/s. Estimate the heat leakage per metre length per unit temperature difference.

*Hints:* Find properties of air at 20°C from the Appendix: $\rho$, $c_p$, $k$, $\alpha$, $v$ and Pr. The hydraulic diameter $D_h = \dfrac{4A}{P}$ and $\mathrm{Re}_d = \dfrac{u_m D_h}{v}$. From Dittus–Boelter equation $\mathrm{Nu}_d = 0.023\ \mathrm{Re}_d^{0.8}\ \mathrm{Pr}^{0.4}$, find $h$ and $Q/L = h.P(T_w - T_b)$.

4.10 Atomspheric air at $T_w = 300$ K and a bulk stream velocity of $u_\infty = 10$ m/s flows through a tube with $D = 2.5$ cm inside diameter. Calculate the pressure drop per 100 m length of the tube for a smooth tube ($f = 0.028$) and a commerical steel tube ($f = 0.0315$).

*Hints:* Find $\rho$ and $v$ for air from Appendix: Then find $\mathrm{Re} = \dfrac{u_m D}{v}$ to show that the flow is turbulent. Then use $\Delta p = \dfrac{fL}{D}\dfrac{\rho u_\infty^2}{2}$

## Review Questions

4.1 What are Newtonian and non-Newtonian fluids? Give examples.

4.2 What is the difference between dynamic viscosity and kinematic viscosity? What

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are their units? How do the viscosities of a liquid and gas vary with temperature?

4.3 Define laminar and turbulent flows. What is Reynolds number?

4.4 What is boundary layer thickness? What do you mean by laminar and turbulent boundary layers? What is laminar sublayer?

4.5 What is critical Reynolds number for flow over a flat plate? On what does it depend?

4.6 What is displacement thickness?

4.7 Define drag coefficient and drag force.

4.8 What do you mean by thermal boundary layer? How does the ratio $\delta/\delta_t$ vary with Prandtl number?

4.9 Define local and mean heat transfer coefficients. On what factors does the value of $h$ depend?

4.10 Show that the Reynolds number for flow in a circular tube of diameter $D$ can be expressed as $Re = 4\dot{m}/\pi D\mu$.

4.11 Which fluid will require a larger pump to move at a specified velocity in a specified tube: Water or engine oil? Why?

4.12 What are the generally accepted values of the critical Reynolds numbers for (a) flow over a flat plate, (b) flow over a circular tube and (c) flow in a tube?

4.13 In the fully developed region of flow in a circular tube, will the velocity profile change in the flow direction? How does the temperature profile vary?

4.14 Consider the flow of oil in a tube. How will the hydrodynamic and thermal entry lengths compare if the flow is laminar? How would they compare if the flow were turbulent?

4.15 How is the friction factor for flow in a tube related to the pressure drop? How is the pressure drop related to the pumping power for a given mass flow rate?

4.16 Consider laminar forced convection in a circular tube. Will the heat flux be higher near the inlet or near the exit of the tube? What would have happened for turbulent forced convection?

4.17 How does surface roughness affect the pressure drop and the heat transfer in a tube?

4.18 Write the expression of the general continuity equation. What does it reduce to for a steady incompressible flow?

4.19 What do you mean by local and convective differentials?

4.20 Write the Navier-Stokes equations for incompressible viscous liquids and explain the terms in these equations.

4.21 Write the energy equation for a fluid element. What is dissipation function? When is its effect significant? When can it be neglected?

4.22 Express the similarity of momentum and energy equations for flow over a flat plate. How are their solutions identical for fluids having $\nu = \alpha$.

4.23 Explain the principle of similarity applied to heat transfer. How are the relevant dimensionless parameters derived from the appropriate differential equations?

4.24 When is the forced convection heat transfer in two systems physically similar?

4.25 How is the buoyancy force per unit volume derived?

4.26 Show by similarity principle that the three dimensionless parameters relevant to free convection heat transfer are Nusselt number, Grashof number and Prandtl number.

4.27 State the five methods which are available for evaluation of convection heat transfer coefficients.

4.28 State the scope and application of dimensional analysis in heat transfer processes. What are the two methods of determining dimensionless groups to correlate experimental data?

4.29 What are fundamental dimensions? Express thermal resistance and heat transfer coefficient in fundamental dimensions.

4.30 Explain the principle of dimensional homogeneity. How is it utilised in deriving the dimensionless groups?

4.31 Explain Rayleigh's method. How is it applied in deriving the functional relationship of pressure drop of a fluid per unit area of the inside surface of a pipe.

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4.32 Show by Rayleigh's method that in forced convection heat transfer the Nusselt number is a function of Reynolds number and Prandtl number.

4.33 State Buckingham $\pi$-theorem. What are its merits and demerits? What are repeating variables? How are these chosen?

4.34 With the help of Buckingham $\pi$-theorem show that

(a) for forced convection heat transfer

$$\mathrm{Nu}_d = B\,\mathrm{Re}_d^a\,\mathrm{Pr}^b$$

(b) for free convection heat transfer

$$\mathrm{Nu} = B\,\mathrm{Gr}^a\,\mathrm{Pr}^b$$

4.35 How are experimental data for heat transfer by forced convection best correlated?

4.36 Show by order-of-magnitude analysis for flow over a plane surface

$$\frac{\delta}{x} = \frac{1}{(\mathrm{Re}_x)^{1/2}}$$

4.37 Explain the method of deriving the exact solution of Pohlhausen for the drag coefficient and the heat transfer coefficient for flow over a flat plate.

4.38 What is film temperature and its significance?

4.39 What do you mean by von Kármán's integral method? How is it used in deriving the drag force and heat transfer coefficient for flow over a flat plate?

4.40 What do you understand by mean value and the fluctuating component of velocity and of a property in turbulent flow?

4.41 Explain what you mean by Reynolds stress? What is the meaning of the negative sign in the turbulent shear stress $-\overline{\rho u'v'}$?

4.42 What is the difference between laminar shear stress and Reynolds stress?

4.43 Explain Prandtl's mixing length concept to describe turbulent flow over a surface.

4.44 Define eddy viscosity. How is it different from kinematic viscosity?

4.45 Explain eddy diffusivity of heat. How is it related to thermal diffusivity?

4.46 Define turbulent Prandtl number. Why is its value equal to unity?

4.47 Explain Reynolds analogy. Is there any restriction on its use?

4.48 What is Stanton number? What is Reynolds–Colburn analogy? What do you mean by Colburn's $j$-factor?

4.49 By Reynolds–Colburn analogy show that for turbulent flow over a plane surface the local Nusselt number is

$$\mathrm{Nu}_x = 0.0288\,\mathrm{Re}_x^{0.8}\,\mathrm{Pr}^{1/3}$$

and the local drag coefficient is

$$C_{f_x} = \frac{0.0576}{(\mathrm{Re}_x)^{0.2}}$$

4.50 Show that for laminar flow from $x = 0$ to $x_c$ and for turbulent flow from $x = x_c$ to $x = L$ over a flat plate

$$\mathrm{Nu}_L = (0.036\,\mathrm{Re}_L^{0.8} - 835)\,\mathrm{Pr}^{1/3}$$

and

$$C_f = \frac{0.072}{\mathrm{Re}_L^{0.2}} - \frac{1670}{\mathrm{Re}_L}$$

4.51 For constant heat flux boundary condition for laminar flow over a flat plate

$$\mathrm{Nu}_x = 0.453\,\mathrm{Re}_x^{1/2}\,\mathrm{Pr}^{1/3}$$

Show that the average temperature difference along the plate is

$$\overline{T_w - T_\infty} = \frac{q_w\,L/k}{0.6795\,\mathrm{Re}_L^{1/2}\,\mathrm{Pr}^{1/3}}$$

where the $q_w$ is the wall heat flux in W/m$^2$.

4.52 Taking the velocity profile in the turbulent flow over a plane surface as

$$u = u_\infty\left(\frac{y}{\delta}\right)^{1/7}$$

and wall shear stress as

$$\tau_w = 0.0228\rho u_\infty^2\left(\frac{\nu}{u_\infty\delta}\right)^{1/4}$$

show that the boundary layer thickness is given by

$$\frac{\delta}{x} = \frac{0.376}{(\mathrm{Re}_x)^{1/5}}$$

4.53 Hence, show that the displacement thickness is $\delta/8$.

4.54 Show that the thickness of laminar sublayer, where the velocity varies linearly, is

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$$\delta_b/\delta = 191/(\mathrm{Re}_x)^{0.7}$$

and the wall shear stress is

$$\tau_w = \frac{0.0296\,\rho u_\infty^2}{(\mathrm{Re}_x)^{0.2}}$$

4.55 How did von Kármán refine Prandtl's analysis for turbulent flow over a plane surface?

4.56 What do you mean by hydrodynamic and thermal entry lengths?

4.57 Explain the effect of Prandtl number on the temperature gradient in turbulent flow for a given Reynolds number in tubes.

4.58 Show that for laminar flow through a tube the Fanning friction coefficient $C_f$ is equal to

$$C_f = f/4$$

where $f$ is the Darcy friction factor.

4.59 Show that the centre-line velocity of a fluid in steady laminar flow through a tube is twice the mean velocity.

4.60 State and explain the Hagen Poiseiulle flow through a tube and show that for laminar flow, $f = 64/\mathrm{Re}_d$.

4.61 How can Hagen–Poiseuille equation be used to estimate the viscosity of a fluid experimentally?

4.62 Draw and explain the temperature variation of a fluid along the flow direction for (a) constant heat flux and (b) constant wall temperature, boundary conditions.

4.63 What do you mean by mixing-cup temperature of a fluid flowing through a tube?

4.64 What are the values of Nusselt number for laminar flow through a tube when the boundary condition is (a) constant heat flux and (b) constant wall temperature?

4.65 Explain Couette flow for laminar forced convection. Where does the maximum temperature occur?

4.66 What is the physical significance of Eckert number? Where is it meaningful?

4.67 What is the effect of buoyancy force in laminar flow through a tube? Explain the significance of Graetz number.

4.68 Show that by Reynolds analogy for turbulent flow through a tube

$$\mathrm{St}_d = f/8$$

where $f$ is the friction factor.

4.69 Explain the relative magnitudes of $\nu$, $\varepsilon_M$, $\alpha$ and $\varepsilon_H$ in different flow regimes for flow through a tube.

4.70 Explain Prandtl analogy and von Kármán analogy for turbulent flow through a tube.

4.71 How does the friction factor for turbulent flow through a tube depend on the Reynolds number?

4.72 What do you mean by relative roughness? Explain Moody's chart to determine the friction factor.

4.73 Explain the conditions under which Dittus–Boelter equation can be used to determine the heat transfer coefficient.

4.74 How does Sieder–Tate equation improve upon Dittus–Boelter equation in estimating $h$ for turbulent flow through a tube?

4.75 What are the characteristic features of liquid metals as working fluids for heating and cooling?

4.76 Explain heat transfer phenomenon in coiled tubes. What do you mean by Dean number?

4.77 Explain the universal velocity profile. Draw the velocity distribution profiles for the three different layers of flow through a tube.

4.78 Explain the exterior flow of a fluid over a circular cylinder for different values of Reynolds number. When do von Kármán vortex streets appear? When does flow separation occur?

4.79 Explain the heat transfer and pressure drop characteristics for bundles of tubes in cross-flow.

## Problems for Practice

4.1 Air at 25°C and at atmospheric pressure flows over a flat plate at 3 m/s. If the plate is 1 m wide and the wall is maintained at 75°C, calculate the following at locations

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$x$ = 1 m and $x = x_c$ from the leading edge: (a) hydrodynamic and thermal boundary layer thicknesses, (b) local and average friction coefficients, (c) local and average heat transfer coefficients, (d) the total rate of heat transfer and (e) the total drag force due to friction. Properties of air at 50°C are $\rho$ = 1.093 kg/m$^3$, $c_p$ = 1.005 kJ/kg K, $\nu$ = 17.95 × 10$^{-6}$ m$^2$/s and $k$ = 0.0282 W/m K.

(*Ans.* At $x$ = 1 m, $\delta$ = 1.132 cm, $\delta_t$ = 1.272 cm $C_{f_x}$ = 0.00158, $C_f$ = 0.00316, $h_x$ = 3.4 W/m$^2$ K, $\bar{h}$ = 6.8 W/m$^2$ K, $Q$ = 340 W, $F$ = 0.0155 N)

4.2 Air at 20°C flows over a plate 60 cm × 30 cm with a velocity of 20 m/s. The critical Reynolds number is 5 × 10$^5$. Calculate the rate of heat transfer from the plate, assuming the flow to be parallel to the 60 cm side. The plate temperature is maintained at 100°C. Properties of air at 60°C are $\rho$ = 1.06 kg/m$^3$, $c_p$ = 1.005 kJ/kg K, $k$ = 0.0291 W/m K and $\nu$ = 18.97 × 10$^{-6}$ m$^2$/s.

(*Ans.* 457.1 W)

4.3 Air at 27°C and 1 atm pressure flows over a flat plate with a velocity of 2 m/s. Estimate (a) the boundary layer thickness at distances of 20 cm and 40 cm from the leading edge of the plate and (b) the mass flow that enters the boundary layer between $x$ = 20 cm and $x$ = 40 cm. Take m of air at 27°C as 1.85 × 10$^{-5}$ kg/ms. Assume unit depth in $z$-directions. If the plate is heated over its entire length to a temperature of 60°C, calculate the heat transfer in (c) the first 20 cm of the plate and (d) the first 40 cm of the plate. (e) Compute the drag force exerted on the first 40 cm of the plate. Properties of air at 316.5 K are $\nu$ = 17.36 × 10$^{-6}$ m$^2$/s, $k$ = 0.02749 W/m K, Pr = 0.7 and $c_p$ = 1.006 kJ/kg K.

(*Ans.* (a) 5.59 mm, 7.9 mm, (b) 3.4 × 10$^{-3}$ kg/s (c) 81.2 W, (d) 114.8 W, (e) 5.44 mN)

4.4 Water at 60°C enters a tube of 2.54 cm diameter at a mean flow velocity of 2 cm/s. Calculate the exit water temperature if the tube is 3 m long and the wall temperature is constant at 80°C. Properties of water at 60 °C are $\rho$ = 985 kg/m$^3$, $c_p$ = 4.18 kJ/kg K, $\mu$ = 4.71 × 10$^{-4}$ kg/ms, $k$ = 0.651 W/m K and Pr = 3.02. Use Sieder and Tate equation. The value of $\mu_w$ at 80°C is 3.55 × 10$^{-4}$ kg/m s.

(*Ans.* 71.9°C)

4.5 Air at 2 atm, 200°C, is heated as it flows through a tube with a diameter of 2.54 cm at a velocity of 10 m/s. A constant heat flux condition is maintained at the wall and the wall temperature is 20°C above the air temperature all along the length of the tube. Calculate (a) the heat transfer per unit length of tube and (b) the increase in bulk temperature of air over a 3 m length of the tube. Properties of air at 200°C are Pr = 0.681, $\mu$ = 2.57 × 10$^{-5}$ kg/ms, $k$ = 0.0386 W/m K and $c_p$ = 1.025 kJ/kg K.

(*Ans.* (a) 103.5 W/m, (b) $\Delta T_b$ = 40°C)

4.6 Air at atmospheric pressure and 100°C enters a 3 m long tube of 4 cm inner diameter with a velocity of 9 m/s and leaves at 192°C. An electric heater is wound on the outer surface of the tube such that heat absorption rate by air per unit area is uniform throughout the length of the tube. If the mean velocity of air is also 9 m/s, find (a) the mass flow rate of air, (b) the rate of heat absorption by the tube from the heater in kW, and (c) the wall temperature of the tube at outlet. Assume average properties of air at 146°C as $c_p$ = 1.005 kJ/kg K, $k$ = 0.035 W/m K, $\nu$ = 28.8 × 10$^{-6}$ m$^2$/s, Pr = 0.683.

(*Ans.* (a) 0.0107 kg/s, (b) 1 kW, (c) 273.5°C)

4.7 Air at 1 atm and 27°C flows across a 12 mm diameter sphere at a velocity of 4 m/s. A small heater inside the sphere maintains the surface temperature at 77°C. Calculate the rate of heat loss from the sphere.

(*Ans.* 1.553 W)

4.8 Calculate the heat transfer coefficient for water flowing through a 25 mm diameter tube at the rate of 1.5 kg/s, when the mean bulk temperature is 40°C. For turbulent flow of a liquid take

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$$Nu_d = 0.0243\, Re_d^{0.8}\, Pr^{0.4}$$

where all properties are evaluated at the mean bulk temperature.

(*Ans.* 12.53 W/m$^2$ K)

4.9 Air at 20°C flowing at 25 m/s passes over a flat plate, the surface of which is maintained at 270°C. Calculate the rate at which heat is transferred from both sides of the plate per unit width over a distance of 0.25 m from the leading edge. Properties of air at 145°C are Pr = 0.687, $\nu = 2.8 \times 10^{-5}$ m$^2$/s and $k = 3.49 \times 10^{-5}$ kW/m K.

(*Ans.* 4.825 kW)

4.10 The crank case of an automobile is 0.6 m long, 0.2 m wide and 0.1 m deep. Assuming the surface temperature of the crank case is 350 K, estimate the rate of heat transfer from the crank case to atmospheric air at 276 K at a road speed of 30 m/s. Assume that the vibration of the engine and the chassis induce the transition from laminar to turbulent flow so near to the leading edge that, for practical purposes, the boundary layer is turbulent over the entire surface. Neglect radiation and use for the front and rear surface the same average heat transfer coefficient as for the bottom and sides.

(*Ans.* 1898 W)

4.11 For flow over a slightly curved surface, the local shear stress is given by the relation

$$\tau_w(x) = 0.3\left(\frac{\rho\mu}{x}\right)^{0.5} u_\infty^{1.5}$$

Obtain nondimensional relations for the local and average friction coefficients.

$$\left(Ans.\, C_{f_x} = \frac{0.6}{Re_x^{1/2}},\, \bar{C}_f = \frac{1.2}{Re_x^{1/2}}\right)$$

4.12 Used engine oil is required to be recycled in a system in which engine oil flows through a 1 cm inner diameter 0.02 cm wall copper tube at the rate of 0.05 kg/s. The oil enters at 35°C and is to be heated to 45°C by atmospheric pressure steam condensing on the outside. Calculate the length of the tube required. Properties of engine oil at 40°C are $c_p$ = 1964 J/kg K, $\rho$ = 876 kg/m$^3$, $k$ = 0.144 W/m K, $\mu$ = 0.210 N s/m$^2$ and Pr = 2870.

(*Ans.* 9.92 m)

4.13 Calculate the average heat transfer coefficient and the friction factor for flow of $n$-butyl alcohol at a bulk temperature of 20°C through a 0.1 m × 0.1 m square duct, 5 m long with walls at 27°C, if the average velocity is 0.03 m/s.

(*Ans.* $\bar{h}_c$ = 4.97 W/m$^2$ K, $f$ = 0.0691)

4.14 Assume a linear temperature profile

$$T = c + dy$$

for flow of a fluid over a flat plate.

(a) Apply the appropriate boundary conditions and express $T$ in terms of $\delta_t$, $T_w$ and $T_\infty$,

(b) Assume a linear velocity profile $u = a + by$ and obtain an expression for $\delta/\delta t$ as a function of Prandtl number.

(c) Obtain an expression for $Nu_x$.

$$\left(Ans.\,(a)\,\frac{T - T_w}{T_\infty - T_w} = \frac{y}{\delta_t},\,(b)\,\frac{\delta}{\delta_t} = Pr^{1/3},\quad (c)\; Nu = 0.29\, Re^{1/2}\, Pr^{1/3}\right)$$

4.15 A rectangular plate is 120 cm long in the direction of flow and 200 cm wide. The plate is maintained at 80°C when placed in nitrogen that has a velocity of 2.5 m/s and a temperature of 0°C. Determine (a) the average friction coefficient, (b) the viscous drag exerted on the plate, (c) the average heat transfer coefficient and (d) the total heat transfer rate from the plate.

(*Ans.* (a) $3.706 \times 10^{-3}$, (b) $2.55 \times 10^{-2}$ N, (c) 5.63 W/m$^2$ K, (d) 1082 W)

4.16 A liquid metal flows at a mass flow rate of 3 kg/s through a constant heat flux 5 cm inner diameter tube in a nuclear reactor. The fluid at 473 K is to be heated with the tube wall 30 K above the fluid temperature. Using the following relation $\overline{Nu}_d = 0.625\, Pe^{0.4}$, estimate the length of the tube required for a 1 K rise in bulk fluid temperature. Properties of liquid metal are $\rho = 7.7 \times 10^3$ kg/m$^3$, $\nu$ =

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$6 \times 10^{-8}$ m$^2$/s, $c_p$ =130 J/kg K, $k$ = 12 W/m K and Pr = 0.011.

(*Ans.* 0.0307 m)

4.17 Atmospheric air at 20°C flows with a free-stream velocity of 5 m/s over a 2 m diameter spherical tank which is maintained at 80°C. Compute the average heat transfer coefficient and the heat transfer rate from the sphere to the air.

(*Ans.* 63.2 W/m$^2$ K, 29.8 W)

4.18 A fluid at 27°C flows with a velocity of 10 m/s across a 5 cm outer diameter tube whose surface is kept at a uniform temperature of 120°C. Determine the average heat transfer coefficients and the heat transfer rates per metre length of the tube for (a) air at atmospheric pressure, (b) water and (c) ethylene glycol.

(*Ans.* (a) 58.3, (b) 31,800, (c) 10,628 W/m$^2$ K)

4.19 Air at atmospheric pressure and 24°C flows with a velocity of 10 m/s along a flat plate 4 m long, which is maintained at a uniform temperature of 130°C. Assuming Re$_c$ = $2 \times 10^5$, determine (a) the local heat transfer coefficient at 2 m, 3 m and 4 m from the leading edge, (b) the average heat transfer coefficient and (c) the heat transfer rate from the plate.

(*Ans.* (a) 22.6, 21.1, 19.9 W/m$^2$ K, (b) 23.3 W/m$^2$ K, (c) 39.52 kW)

4.20 Atmospheric air at 27°C flows with a free-stream velocity of 10 m/s along a flat plate 4 m long. Calculate the drag coefficient at 2 m and 4 m from the leading edge. Assuming an all turbulent boundary layer, determine the drag force exerted per metre width of the plate.

(*Ans.* $3.56 \times 10^{-3}$, $3.1 \times 10^{-3}$, 0.91 N)

4.21 Air at 27°C flows with a free-stream velocity of 40 m/s along a flat plate 2 m long. Calculate the boundary layer thickness at the end of the plate for air at (a) 1/2 atm, (b) 1 atm and (c) 2 atm.

(*Ans.* (a) 3.18 cm, (b) 3.07 cm, (c) 2.82 cm)

4.22 Determine the hydrodynamic and thermal boundary layer thicknesses at 0.5 m from the leading edge of a flat plate at 74°C for flow at 80°C and atmospheric pressure with a velocity of 3 m/s of air, hydrogen and helium, respectively. Compare the ratio $\delta_t/\delta$.

(*Ans.* $\delta$ = 9.2, 24.1, 25.8 mm; $\delta_t$ = 9.5, 24.8, 26.4 mm)

4.23 Engine oil at 40°C flows with a velocity of 1 m/s over a 2 m long flat plate whose surface is maintained at a uniform temperature of 80°C. Determine the average heat transfer coefficient.

(*Ans.* 74.4 W/m$^2$ K)

4.24 Engine oil ($\rho$ = 868 kg/m$^3$, $\nu$ = $0.75 \times 10^{-4}$ m$^2$/s) flows with a mean velocity of 0.15 m/s inside a circular tube having an inside diameter of 2.5 cm. Calculate the friction factor and the pressure drop over the length 100 m of the tube.

(*Ans.* $f$ = 1.28, $\Delta p$ = 50 kN/m$^2$)

4.25 Atmospheric air at 300 K flows with a velocity of 5 m/s along a flat plate 1 m long. The plate has a width of 0.5 m. The total drag force acting on the plate is $18 \times 10^{-3}$ N. By using Reynolds–Colburn analogy, estimate the corresponding average heat transfer coefficient for flow of air over the plate. Properties of air at 300 K are $\rho$ = 1.177 kg/m$^3$, $c_p$ = 1.006 kJ/kg K and Pr = 0.708.

(*Ans.* $\overline{C}_f$ = $2.447 \times 10^{-3}$, $h_m$ = 9.12 W/m$^2$ K)

4.26 In a heat exchanger water flows through a long 2.2 cm inner diameter copper tube at a bulk velocity of 2 m/s and is heated by steam condensing at 150°C on the outside of the tube. The water enters at 150°C on the outer side of the tube. The water enters at 150°C and leaves at 60°C. Find the heat transfer coefficient for water by (a) Dittus–Boelter equation, (b) Colburn's equation and (c) Sieder–Tate equation.

(*Ans.* (a) 8400, (b) 9700, (c) 8540 W/m$^2$ K)

4.27 (a) In the Couette flow the fluid flows between two infinite parallel plates, one of them stationary, and the other moving with a constant velocity $u_1$. The distance between

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the plates is $2b$. The velocity distribution is given by

$$u/u_1 = \frac{1}{2}\left(\frac{y}{b}+1\right)$$

considering steady, incompressible, laminar fully developed flow without pressure drop. The stationary plate is maintained at temperature $T_o$ and the moving plate at $T_1$. Show that the temperature distribution is given by

$$\theta = \frac{1}{2}(1+\eta) + \frac{\text{Pr Ec}}{8}(1-\eta^2)$$

where $\theta$ and $\eta$ are non-dimensional parameters given by

$$\theta = \frac{T-T_0}{T_1-T_0} \text{ and } \eta = \frac{y}{b}$$

and Ec is Eckert number given by Ec $= \dfrac{u_1^2}{c_p(T_1-T_0)}$.

4.27 (b) In a lubricated bearing, the clearance is 0.05 mm. The surface velocity of the shaft is 30 m/s. The stationary surface is maintained at 50°C. The shaft does not have any provision for cooling. Considering the bearing and journal as two infinite flat parallel plates, determine the steady-state temperature of the shaft and the rate of heat transfer at the bearing surface. Given for oil, $\mu = 0.012$ kg/m s and $k = 1$ W/m K.

(*Ans.* $T_1 = 55.4$°C, $q = 216$ kW/m$^2$)

4.28 An instant water heater consists of 9.1 mm inner diameter tube through which water flows at the rate of 18.4 kg/h entering at 30°C. The tube is externally wound over by a nichrome resistance wire with a heating capacity of 400 W per metre tube length, after which it is insulated. (a) Determine the length of tube required to raise the temperature of water to 75°C. (b) Find the maximum temperature of water at the outlet. (c) What should be done so that there is no boiling in any part of flow. For water at 52.5°C, $\rho = 984$ kg/m$^3$, $\mu = 0.54 \times 10^{-3}$ kg/ms, $k = 0.6445$ W/m K and $c_p = 4187$ J/kg K.

(*Ans.* (a) 2.41 m, (b) 120.2°C, (c) Max. heating capacity limited to 221 W/m)

4.29 Water entering at 10°C is to be heated to 40°C in a tube of 20 mm inner diameter at a mass flow rate of 0.01 kg/s. The outside of the tube is wrapped with an insulated heating element that produces a uniform flux of 5 kW/m$^2$ over the surface. Neglecting any entrance effects, determine (a) the Reynolds number, (b) the heat transfer coefficient, (c) the length of tube needed for 30°C temperature rise, (d) the inner surface temperature, (e) the friction factor, (f) the pressure drop and (g) the pumping power required if the pump is 50% efficient.

(*Ans.* (a) 700, (b) 132.5 W/m$^2$ K, (c) 1.33 m, (d) 153°C, (e) 0.0915, (f) 3.1 N/m$^2$, (g) $6.2 \times 10^{-5}$ W)

4.30 The drag force on a ship is a function of $\rho$, $\mu$, $g$, $L$ and $v$. By making use of Buckingham's p-theorem obtain the nondimensional drag force in terms of dimensionless groups.

4.31 Air at a temperature of 115.6°C enters a smooth pipe of 7.62 cm diameter, the wall of which can be maintained at a constant temperature of 15.6°C. The rate of flow of air is 0.226 m$^3$/s. Estimate the length of pipe necessary if the air is to be cooled to 65.5°C, using the following assumptions: Pr for air = 0.74, $f = 0.00175$ and velocity at the boundary of the sublayer is half the mean velocity in the pipe.

(*Ans.* 12.55 m)

4.32 A flat plate 100 cm wide and 150 cm high is to be maintained at 90°C in air with a free-stream temperature of 10°C. Determine the velocity at which the air must flow over the flat plate along 150 cm side so that the rate of energy dissipation from the plate is 3.75 kW. Properties of air at 50°C are $\rho = 1.09$ kg/m$^3$, $k = 0.028$ W/m K, $c_p = 1.007$ kJ/kg K and $\mu = 2.03 \times 10^{-5}$ kg/ms.

(*Ans.* 15.45 m/s)

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# Heat Transfer by Natural Convection

# 5

In forced convection, fluid motion is imposed externally by a fan, a blower or a pump. Free- or natural-convection flow arises in various ways, for example when a heated object is placed in a fluid at rest, the density of which varies with temperature. Heat is transferred from the surface of the hot object to the fluid layers in its neighbourhood. The density decrease due to a temperature increase causes these layers to rise and create the free-convection flow which now transports heat away from the object. Physically such a flow is described by stating that it is caused by body forces. Here, the body forces are the gravitational forces. Free-convection flows under the influence of gravitational forces have been investigated most extensively because they are encountered frequently in nature as well as in engineering applications. Flows can be caused by other body forces as well. In a rotating system, for instance, centrifugal and Coriolis forces exist as body forces. Flow of cooling air through passages in the rotating blades of gas turbines is an example of such a body force. In the boundary layers which surround missiles flying with supersonic speeds, temperatures may be so high that the air is ionized, which means that the atoms and molecules carry electric charges, as also happens for flow of plasma in a magnetohydrodynamic generator. In this case electric or magnetic forces may arise which as body forces influence the flow. In this chapter only gravitational natural convection will be considered on a number of simple geometries like a vertical flat plate, a horizontal cylinder with constant wall temperature and a fluid enclosed between two plane walls or concentric cylinders.

The flow velocity in free convection is much smaller than that encountered in forced convection. Therefore, heat transfer by free convection is much smaller than that by forced convection. Figure 5.1(a) illustrates the development of velocity field in front of a hot vertical plate owing to the *buoyancy force*. The heated fluid in front of the hot plate rises, entraining fluid from the quiescent outer region. Figure 5.1(b) shows a cold vertical plate in a hot fluid, where the direction of motion is reversed, the fluid in front of the plate being heavier moves vertically down, again entraining fluid from the quiescent outer region. In both cases a velocity boundary layer is developed with a certain peak in it. The velocity is zero both at the plate surface and at the edge of the boundary layer. In the regions near the leading edge, the boundary layer is laminar, then at a certain distance from the leading edge the transition to turbulent layer occurs, and finally a fully developed turbulent layer is established.

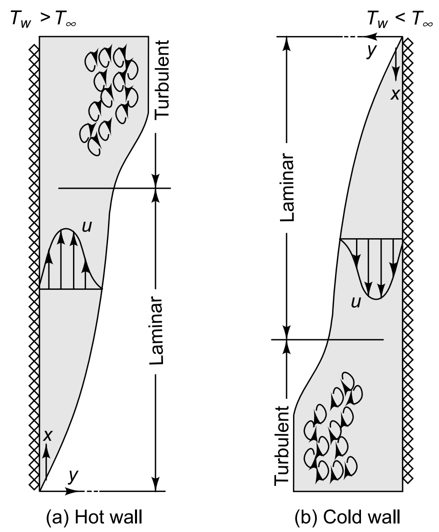


**Fig. 5.1** *Laminar and turbulent velocity boundary layer for natural convection on a vertical plate*

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We now consider a fluid contained in the space between two parallel horizontal plates [Fig. 5.2(a)]. Suppose the lower plate is maintained at a temperature higher than that of the upper plate ($T_1 > T_2$). A temperature gradient will be established in the vertical direction. The layer will be top-heavy, since the density of the cold fluid at the top is higher than that of the hot fluid at the bottom. If the temperature difference is increased beyond a certain critical value, the viscous forces within the fluid can no longer sustain the buoyancy forces, and a convection motion is set up.

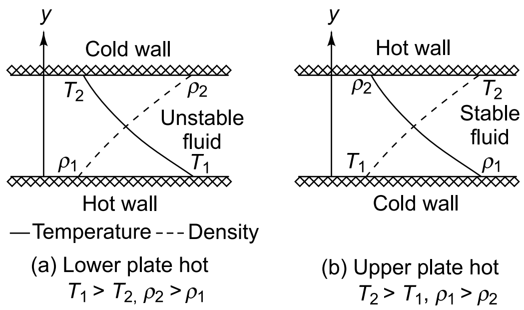


**Fig. 5.2** *Fluid contained between two horizontal plates*

Suppose in Fig. 5.2(b), the lower plate is cold and the upper plate is hot (i.e. $T_1 < T_2$). Here, the density of the top layer is less than that of the bottom layer. The fluid is then always stable, and no natural convection currents are set up.

Figure 5.3 shows the directions of convection currents for horizontal plates, heated or cooled, facing up or down.

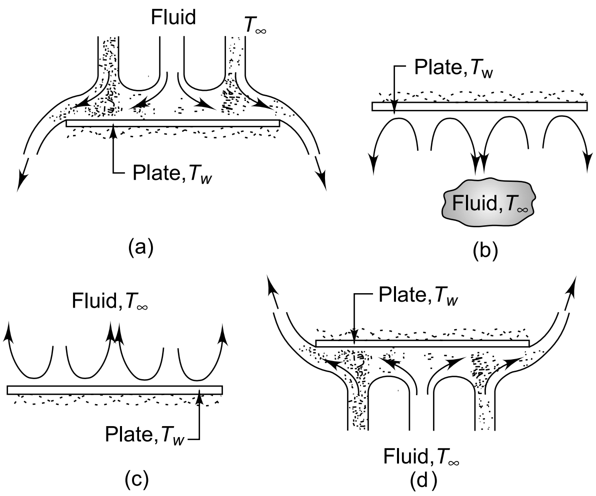


**Fig. 5.3** *Buoyancy-driven flows on horizontal cold ($T_w < T_\infty$) and hot ($T_w > T_\infty$) plates, (a) Top surface of cold plate, (b) bottom surface of cold plate, (c) top surface of hot plate and (d) bottom surface of hot plate*

## 5.1 DIMENSIONLESS PARAMETERS OF NATURAL CONVECTION

To develop the principal dimensionless parameters of natural convection, we consider the natural convection on a vertical plate, as illustrated in Fig. 5.1. For simplicity in the analysis, we assume that the boundary layer flow is steady and laminar. Since small flow velocities are associated with natural convection, the viscous energy dissipation term in the energy equation can be neglected. Then the governing continuity, momentum and energy equations are obtained from the boundary layer equations, as derived in the last chapter, and the appropriate buoyancy term is introduced in the momentum equation:

**Continuity:**
$$\frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \frac{\partial \mathbf{v}}{\partial \mathbf{y}} = \mathbf{0} \tag{5.1}$$

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**Momentum:** $$\rho\left(\mathbf{u}\frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \mathbf{v}\frac{\partial \mathbf{u}}{\partial \mathbf{y}}\right) = -\rho\mathbf{g} - \frac{\partial \mathbf{p}}{\partial \mathbf{x}} + \mu\frac{\partial^2 \mathbf{u}}{\partial \mathbf{y}^2} \tag{5.2}$$

**Energy:** $$\rho c_p\left(\mathbf{u}\frac{\partial \mathbf{T}}{\partial \mathbf{x}} + \mathbf{v}\frac{\partial \mathbf{T}}{\partial \mathbf{y}}\right) = \mathbf{k}\frac{\partial^2 \mathbf{T}}{\partial \mathbf{y}^2} \tag{5.3}$$

Here the term $-\rho g$ on the right hand side of the momentum equation represents the body force exerted on the fluid element in the negative $x$-direction. For small temperature differences, the density $\rho$ in the buoyancy term is considered to vary with temperature, whereas the density appearing elsewhere in these equations is considered constant. This is often referred to as *Boussinesq approximation*.

To determine the pressure gradient term $\partial p/\partial x$, the $x$-momentum equation, Eq. (5.2) is evaluated at the edge of the velocity boundary layer, where $u \to 0$ and $\rho \to \rho_\infty$. We obtain

$$\frac{\partial p}{\partial x} = -\rho g \tag{5.4}$$

where $\rho_\infty$ is the fluid density outside the boundary layer. Then the term $-\rho g - \partial P/\partial x$ appearing in the momentum equation, Eq. (5.2) becomes

$$-\rho g - \frac{\partial p}{\partial x} = (\rho_\infty - \rho)g \tag{5.5}$$

If $\beta$ denotes the volumetric coefficient of thermal expansion of the fluid,

$$\beta = \frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_p = \rho\left(\frac{\partial \frac{1}{\rho}}{\partial T}\right)_p = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p \tag{5.6}$$

$$\Delta\rho = -\beta\rho\Delta T$$

or $$\rho_\infty - \rho = -\beta\rho\,(T_\infty - T) \tag{5.7}$$

Then Eq. (5.5) becomes

$$-\rho g - \frac{\partial p}{\partial x} = -\beta\rho\,(T_\infty - T)\,g \tag{5.8}$$

Substituting Eq. (5.8) into the momentum equation, the resulting equations for natural convection on a vertical plate are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = g\beta\left(T - T_\infty\right) + v\frac{\partial^2 u}{\partial y^2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \tag{5.9}$$

To determine the dimensionless parameters that govern heat transfer in natural convection, we need to nondimensionalise the above governing equations. The following dimensionless parameters are defined:

$$X = \frac{x}{L}, Y = \frac{y}{L}, U = \frac{u}{U_0}, V = \frac{v}{U_0}, \theta = \frac{T - T_\infty}{T_w - T_\infty} \tag{5.10}$$

Here $L$ is the characteristic length, $U_0$ is the reference velocity, $T_w$ is the wall surface temperature and $T_\infty$ is the fluid temperature at a far distance from the hot plate. When these new variables are introduced into Eq. (5.9), the resulting nondimensional equations become

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$$\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}=0$$
$$U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}=\frac{g\beta\left(T_w-T_\infty\right)}{U_0^2}\theta+\frac{1}{\text{Re}}\frac{\partial^2 U}{\partial Y^2} \tag{5.11}$$
$$U\frac{\partial \theta}{\partial X}+V\frac{\partial \theta}{\partial Y}=\frac{1}{\text{Re Pr}}\frac{\partial^2 \theta}{\partial Y^2}$$

Here, the Reynolds and Prandtl numbers are defined as

$$\text{Re}=\frac{U_0 L}{\nu},\ \text{Pr}=\frac{\nu}{\alpha}$$

The dimensionless group in the momentum equation can be rearranged as

$$\frac{g\beta\left(T_w-T_\infty\right)L}{U_0^2}=\frac{g\beta L^3\left(T_w-T_\infty\right)/\nu^2}{\left(U_0 L/\nu\right)^2}$$
$$=\frac{\mathbf{Gr}}{\mathbf{Re^2}} \tag{5.12}$$

where the Grashof number Gr is defined as

$$\mathbf{Gr}=\frac{\mathbf{g\beta L^3\left(T_w-T_\infty\right)}}{\boldsymbol{\nu}^2}$$

The Grashof number represents the ratio of the buoyancy force to the viscous force acting on the fluid. We recall that in forced convection, the Reynolds number represents the ratio of the inertial to viscous forces acting on the fluid. Therefore, the Grashof number in natural convection plays the same role as the Reynolds number in forced convection. In forced convection the transition from laminar to turbulent flow is governed by the critical value of the Reynolds number. Similarly, in natural convection, the transition from laminar to turbulent flow is governed by the critical value of the Grashof number.

Equation (5.11) imply that when the effects of natural and forced convection are of comparable magnitude, the Nusselt number depends on Re, Pr and Gr, or

$$\overline{\text{Nu}}=f(\text{Re, Pr, Gr}) \tag{5.13}$$

The parameter $\text{Gr}/\text{Re}^2$, defined by Eq. (5.12), is a measure of the relative importance of natural convection in relation to forced convection. When $\text{Gr}/\text{Re}^2 \cong 1$, natural and forced convection are of the same order of magnitude; hence both must be considered.

If $(\text{Gr}/\text{Re}^2) << 1$, flow is primarily by forced convection. If $(\text{Gr}/\text{Re}^2) >> 1$, natural convection becomes dominant and the Nusselt number depends on Gr and Pr only:

$$\text{Nu}=f(\text{Gr, Pr}) \tag{5.14}$$

In natural convection, flow velocities are produced by the buoyancy forces only; hence there are no externally induced flow velocities. As a result, the Nusselt number does not depend on the Reynolds number.

Sometimes another dimensionless parameter, called the *Rayleigh number* (Ra), which is defined as

$$\text{Ra}=\text{Gr}\cdot\text{Pr}=\frac{g\beta L^3\left(T_w-T_\infty\right)}{\nu^2}\frac{\nu}{\alpha}$$
$$=\frac{g\beta L^3\left(T_w-T_\infty\right)}{\nu\alpha} \tag{5.15}$$

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is used instead of the Grashof number to correlate heat transfer in natural convection. Then the Nusselt number relation (Eq. (5.14)), becomes

$$\text{Nu} = \phi\,(\text{Ra}) \tag{5.16}$$

Experimental data from various sources for natural convection from horizontal wires and tubes of diameter $D$ are correlated in Fig. 5.4 by plotting the average Nusselt number, $(\bar{h}_c\, D)/k$, against the Rayleigh number, $(c_p\, \rho^2\, g\, \beta \Delta T D^3)\,/\mu k$. The physical properties are evaluated at the film temperatures. We observe that data for fluids as different as air, glycerin and water are correlated over a wide range of Ra from $10^{-5}$ to $10^{9}$ for cylinders ranging from small wires to large pipes.

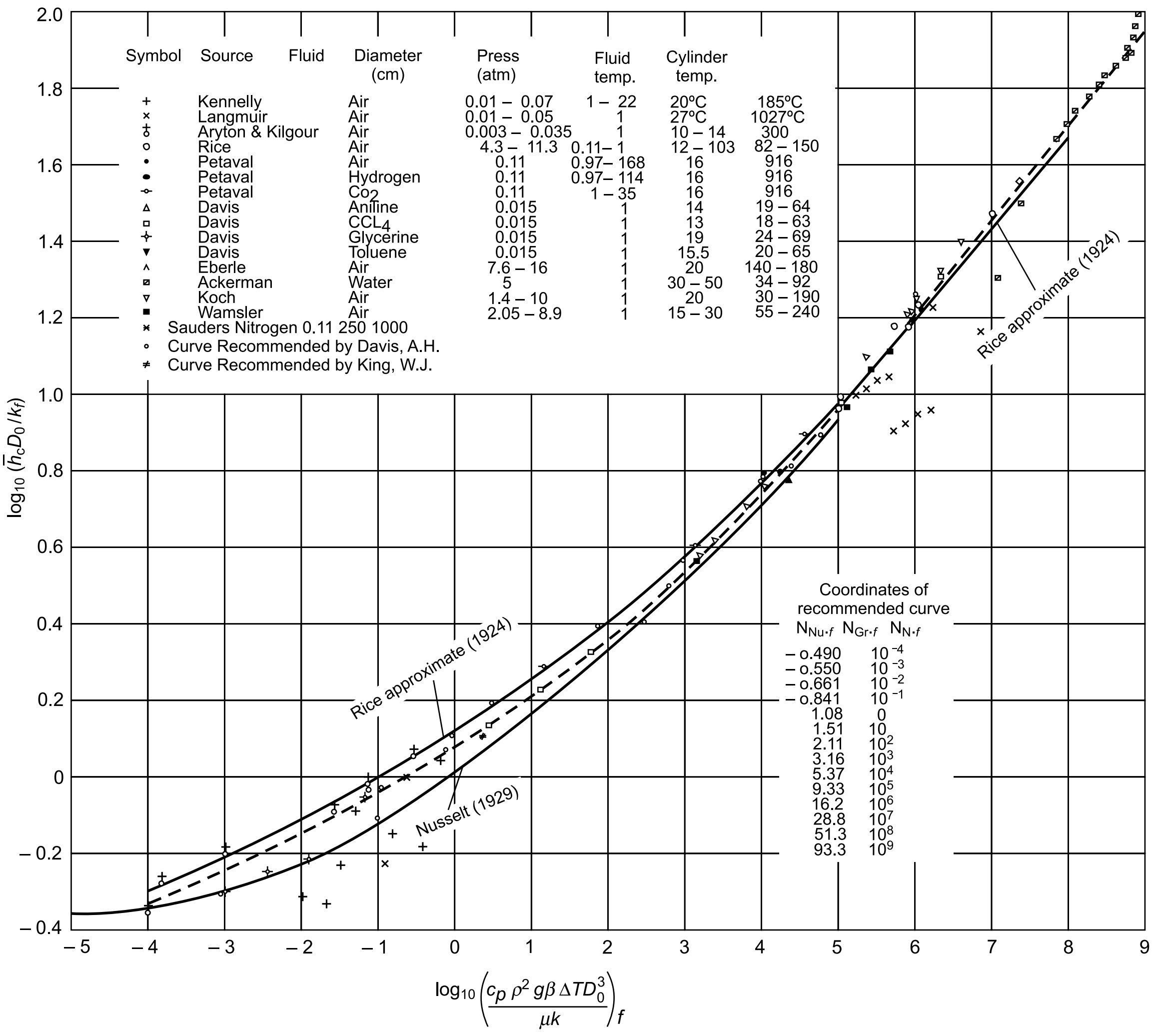


**Fig. 5.4** *Correlation of data for natural convection heat transfer from horizontal cylinders in gases and liquids (McAdams, W.H., Heat Transmission, 3rd ed., McGraw-Hill, 1954)*

For three-dimensional shapes such as short cylinders and blocks the characteristic length $L$ may be determined from

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$$\frac{1}{L} = \frac{1}{L_{hor}} + \frac{1}{L_{vert}}$$

where $L_{vert}$ is the height and $L_{hor}$ the average horizontal dimension of the body.

A correlation for natural convection from vertical plates and vertical cylinders is shown in Fig. 5.5. It is seen that the flow is laminar for Ra ≤ $10^8$, passes through a transition in the range $10^8 < \text{Ra} < 10^{10}$, and becomes fully turbulent for Ra > $10^{10}$.

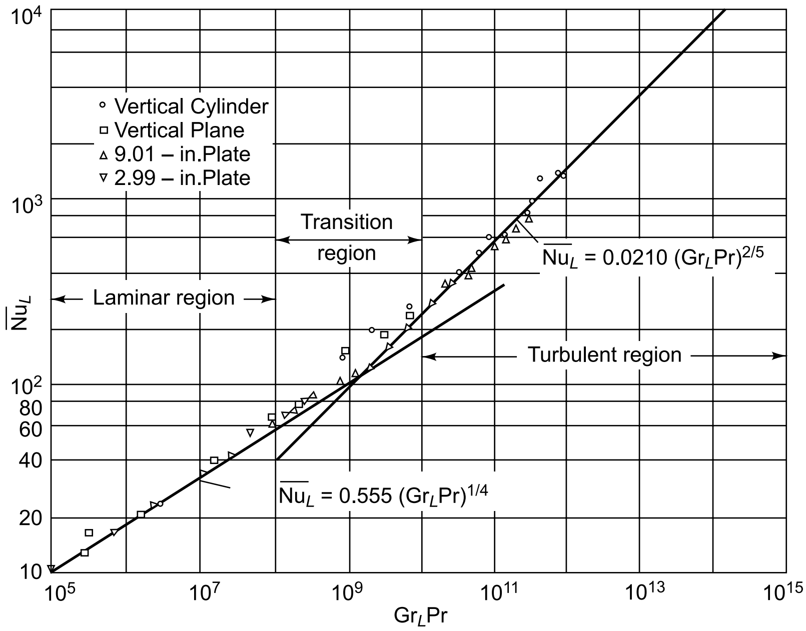


**Fig. 5.5** *Correlation of data for natural convection heat transfer from vertical plates and cylinders*

## 5.2 AN APPROXIMATE ANALYSIS OF LAMINAR NATURAL CONVECTION ON A VERTICAL PLATE

Heat transfer by natural convection on a vertical or an inclined plate has been the subject of many investigations [1–10]. To provide better insight into heat transfer by natural convection, we consider here the simplest situation involving a vertical plate under isothermal conditions. Let $T_w$ and $T_\infty$ be, respectively, the temperature of the wall surface and the bulk temperature of the fluid (Fig. 5.6). The fluid moves upward along the plate for $T_w > T_\infty$ and flows downward for $T_w < T_\infty$, as illustrated in Fig. 5.1. Within the boundary layer the temperature decreases from $T_w$ to $T_\infty$ of the undisturbed or quiescent fluid outside the heated region.

Let $\theta = T - T_\infty$. When $y = 0$, $\theta = \theta_w = T_w - T_\infty$, and when $y = \delta$, $\theta = \theta_\infty = 0$. If $y = 0$, $u = 0$, and if $y = \delta$, $u = 0$.

The velocity and temperature profiles in the neighbourhood of the plate are shown. The integral boundary layer equations for momentum and energy will be used to calculate the heat transfer in natural convection.

To solve the boundary layer equation, the temperature profile is approximated by a parabolic equation of the form

$$T = C_1 + C_2 y + C_3 y^2 \tag{5.17}$$

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At $\quad y = 0,\ T = T_w = C_1$

At $\quad y = \delta = \delta_t,\ T = T_\infty,\ \left(\dfrac{\partial T}{\partial y}\right)_{y=\delta} = 0$

or $\quad C_2 + 2C_3\,\delta = 0, \quad \therefore\ C_2 = -2C_3\delta$

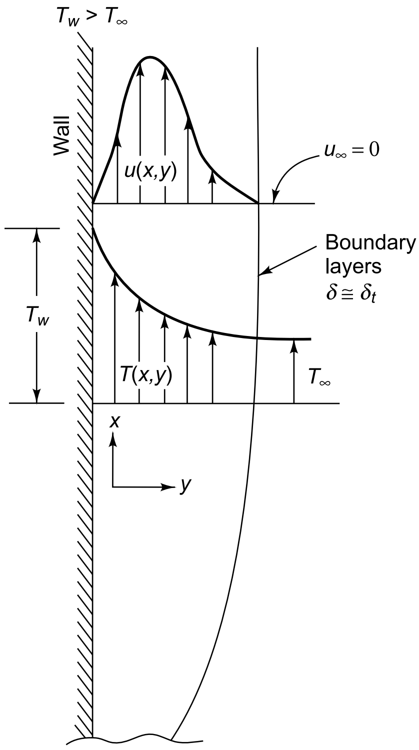


**Fig. 5.6** *Temperature and velocity profiles for natural convection on a hot vertical plate*

For simplicity we assume $\delta = \delta_t$, i.e., equal velocity and thermal boundary layer thicknesses. Substituting in Eq. (5.17),

$$T_\infty = T_w + C_2\delta + C_3\delta^2 = T_w - 3C_3\delta^2 + C_3\delta^2$$

Therefore, $\quad C_3 = \dfrac{T_w - T_\infty}{\delta^2}$ and $C_2 = -2\,\dfrac{T_w - T_\infty}{\delta^2}\cdot\delta$

$$= \frac{-2\,(T_w - T_\infty)}{\delta}$$

$$T = T_w - \frac{2\,(T_w - T_\infty)}{\delta}\,y + \frac{(T_w - T_\infty)}{\delta^2}\,y^2$$

or $\quad T - T_\infty = (T_w - T_\infty) - 2\,\dfrac{y}{\delta}\,(T_w - T_\infty) + (T_w - T_\infty)\,\dfrac{y^2}{\delta^2}$

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or $$\theta = \theta_w \left(1 - \frac{y}{\delta}\right)^2 \tag{5.18}$$

The velocity profile may be assumed to be a cubical parabola given by

$$u = u_1 (a_1 + a_2 y + a_3 y^2 + a_4 y^3) \tag{5.19}$$

where $u_1$ is a reference velocity and is a function of $x$.

1. At $y = 0$, $u = 0$.
2. At $y = \delta$, $u = 0$, $\dfrac{\partial u}{\partial y} = 0$.

Using the first boundary condition

$$u = 0 = u_1 a_1$$

Since $$u_1 \neq 0, \quad \therefore a_1 = 0$$

Now $$\frac{\partial u}{\partial y} = u_1 (a_2 + 2a_3 y + 3a_4 y^2)$$

$$\frac{\partial^2 u}{\partial y^2} = u_1 (2a_3 + 6a_4 y)$$

When, $$y = \delta, \; u = 0, \; \frac{\partial u}{\partial y} = 0$$

$$0 = u_1 (a_2 + a_3 \delta^2 + a_4 \delta^3) \tag{5.20}$$

$$0 = u_1 (a_2 + 2a_3 \delta + 3a_4 \delta^2) \tag{5.21}$$

From momentum equation,

$$\frac{Du}{Dt} = g\beta\rho\theta - \frac{\partial p}{\partial x} + \mu \Delta^2 u$$

Therefore, $$0 = g\beta\theta_w \rho + \mu \frac{\partial^2 u}{\partial y^2}$$

or $$\frac{\partial^2 u}{\partial y^2} = -\frac{g\beta\theta_w}{v}$$

$$(2a_3 + 6a_4 \times 0) = -\frac{g\beta\theta_w}{v}$$

(At $y = 0$, $\theta = \theta_w$)

$\therefore$ $$a_3 = -\frac{g\beta\theta_w}{2v} \tag{5.22}$$

From Eqs (5.20) – (5.22),

$$a_2 = \frac{1}{4} \frac{g\beta\theta_w \delta}{v} \text{ and } a_4 = \frac{g\beta\theta_w}{4v\delta}$$

Substituting in Eq. (5.19),

$$u = u_1 \left( \frac{1}{4} \frac{g\beta\theta_w \delta}{v} y - \frac{g\beta\theta_w}{2v} y^2 + \frac{g\beta\theta_w}{4v\delta} y^3 \right)$$

$$= u_1 \frac{g\beta\theta_w \delta^2}{4v} \frac{y}{\delta} \left( 1 - \frac{2y}{\delta} + \frac{y^2}{\delta^2} \right)$$

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$$= u_1 \frac{g\beta\theta_w\delta^2}{4\nu}\frac{y}{\delta}\left(1-\frac{y}{\delta}\right)^2 = u_0\frac{y}{\delta}\left(1-\frac{y}{\delta}\right)^2 \tag{5.23}$$

where, $$u_0 = u_1\frac{g\beta\theta_w\delta^2}{4\nu} \tag{5.24}$$

The velocity profile is given by

$$u = u_0\frac{y}{\delta}\left(1-\frac{y}{\delta}\right)^2 = u_0\left(\frac{y}{\delta} - 2\frac{y^2}{\delta^2} + \frac{y^3}{\delta^3}\right)$$

There is a certain value of $y$ where $u$ is maximum.

$$\frac{du}{dy} = u_0\left(\frac{1}{\delta} - \frac{4y}{\delta^2} + \frac{3y^2}{\delta^3}\right) = 0$$

or $$3y^2 - 4\delta y + \delta^2 = 0$$

$$y = \frac{4\delta \pm \left(16\delta^2 - 12\delta^2\right)^{1/2}}{6} = \frac{4\delta \pm 2\delta}{6}$$

$$= \delta \text{ or } \frac{1}{3}\delta$$

Since $u = 0$ at $y = \delta$, therefore,

$u$ will be maximum when $y = \frac{1}{3}\delta$. Therefore,

$$\mathbf{u_{max} = u_0 \frac{1}{3}\left(1-\frac{1}{3}\right)^2 = \frac{4}{27}u_0} \tag{5.25}$$

Let us take a control volume of thickness d$x$ at a distance $x$ from the bottom edge within the boundary layer as shown in Fig. 5.7.

Momentum flux across $BC$ is zero

Rate of increase of momentum

$$= \rho\, dx \frac{\partial}{\partial x}\int_0^1 u^2\, dy$$

= Forces acting on the element

$$= -\tau_w\, dx + g\beta\rho\, dx\int_0^l \theta\, dy$$

Integration is limited to $\delta$, as before

$$\frac{d}{dx}\int_0^\delta u^2 dy = -\nu\left(\frac{du}{dy}\right)_{y=0} + g\beta\int_0^\delta \theta\, dy \tag{5.26}$$

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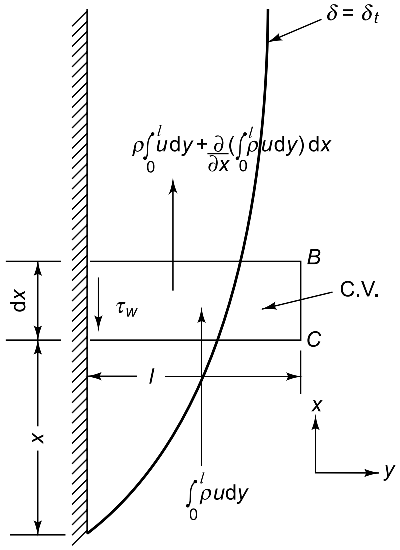


**Fig. 5.7** *Control volume in the boundary layer*

Energy equation for the volume element gives

$$\rho c_p\,\mathrm{d}x \frac{\partial}{\partial x}\int_0^l \theta u\,\mathrm{d}y = -k\left(\frac{\partial \theta}{\partial y}\right)_{y=0}\mathrm{d}x$$

Limiting the integration to $\delta = \delta_t$,

$$\frac{d}{\mathrm{d}x}\int_0^\delta u\theta\,\mathrm{d}y = -\alpha\left(\frac{\mathrm{d}\theta}{\mathrm{d}y}\right)_{y=0} \tag{5.27}$$

Now,

$$\int_0^\delta u^2\mathrm{d}y = \int_0^\delta u_0^2\,\frac{y^2}{\delta^2}\left(1-\frac{y}{\delta}\right)^4\mathrm{d}y$$

$$= \frac{u_0^2}{\delta^2}\int_0^\delta y^2\left(1-\frac{2y}{\delta}+\frac{y^2}{\delta^2}\right)^2\mathrm{d}y$$

$$= \frac{u_0^2\,\delta}{105}$$

$$\int_0^\delta \theta\,\mathrm{d}y = \int_0^\delta \theta_w\left(1-\frac{2y}{\delta}+\frac{y^2}{\delta^2}\right)\mathrm{d}y = \theta_w\,\frac{\delta}{3}$$

$$\int_0^\delta u\theta\,\mathrm{d}y = \int_0^\delta u_0\,\frac{y}{\delta}\left(1-\frac{y}{\delta}\right)^2\theta_w\left(1-\frac{y}{\delta}\right)^2\mathrm{d}y$$

$$= \frac{u_0\,\theta_w\,\delta}{30}$$

Substituting in the momentum equation, Eq. (5.26),

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u_0^2\,\delta}{105}\right) = -\nu\frac{u_0}{\delta} + g\beta\,\frac{\theta_w\,\delta}{3} \tag{5.28}$$

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and in the energy equation, Eq. (5.27),

$$\frac{d}{dx}\left(\frac{u_0\,\theta_w\,\delta}{30}\right) = 2\alpha\frac{\theta_w}{\delta} \tag{5.29}$$

Both $u_0$ and $\delta$ are functions of $x$. Let

$$u_0 = C_1 x^m \text{ and } \delta = C_2 x^n$$

Therefore,

$$\frac{1}{105}\frac{d}{dx}\left(C_1^2 \times x^{2m}\cdot C_2\, x^n\right) = \frac{1}{3}g\beta\theta_w C_2 x^n - \nu\frac{C_1 x^m}{C_2 x^n}$$

$$\frac{C_1^2\,C_2}{105}\frac{d}{dx}\left(x^{2m+n}\right) = \frac{1}{3}g\beta\theta_w C_2 x^n - \nu\frac{C_1}{C_2}x^{m-n}$$

$$\frac{C_1^2\,C_2}{105}(2m+n)x^{2m+n-1} = \frac{1}{3}g\beta\theta_w C_2 x^n - \nu\frac{C_1}{C_2}x^{m-n} \tag{5.30}$$

Again, $$\frac{1}{30}\theta_w\frac{d}{dx}\left(C_1 x^m C_2 x^n\right) = 2\alpha\theta_w\frac{1}{C_2 x^n}$$

or $$\frac{C_1\,C_2}{30}(m+n)x^{m+n-1} = \frac{2\alpha}{C_2}x_2^{-n} \tag{5.31}$$

Equations (5.30) and (5.31) are valid for any value of $x$. Equating the exponents of $x$ in the two equations,

$$2m + n - 1 = n = m - n$$

or $$2m - 1 = 0,\ 2n = m = 1/2$$

or $$m = 1/2,\ n = 1/4$$

Substituting these values of $m$ and $n$ in Eqs (5.30) and (5.31)

$$\frac{5}{4\times 105}C_1^2\,C_2 = \frac{g\beta\theta_w}{3}C_2 - \nu\frac{C_1}{C_2}$$

$$\frac{C_1^2\,C_2}{84} = \frac{g\beta\theta_w\,C_2}{3} - \nu\frac{C_1}{C_2} \tag{5.32}$$

and $$\frac{C_1\,C_2}{30}\times\left(\frac{1}{2}+\frac{1}{4}\right) = \frac{2\alpha}{C_2}$$

$$\frac{C_1\,C_2^2}{40} = 2\alpha \tag{5.33}$$

From Eqs (5.32) and (5.33),

$$C_1 = 5.17\ \nu\left(\frac{20}{21}+\frac{\nu}{\alpha}\right)^{-1/2}\left(\frac{g\beta\theta_w}{\nu^2}\right)^{1/2}$$

$$C_2 = 3.93\left(\frac{20}{21}+\frac{\nu}{\alpha}\right)^{1/4}\left(\frac{g\beta\theta_w}{\nu^2}\right)^{-1/4}\left(\frac{\nu}{\alpha}\right)^{-1/2}$$

$$u_{max} = \frac{4}{27}u_0 = \frac{4}{27}C_1 x^m = \frac{4}{27}C_1 x^{1/2}$$

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$$= \frac{4}{27} \times 5.17\, \nu \left( \frac{20}{21} + \frac{\nu}{\alpha} \right)^{-1/2} \left( \frac{g\beta\theta_w}{\nu^2} \right)^{1/2} x^{1/2}$$

$$= 0.766\ \nu \left( 0.952 + \frac{\nu}{\alpha} \right)^{-1/2} \left( \frac{g\beta\theta_w}{\nu^2} \right)^{1/2} x^{1/2}$$

$$\delta = C_2\, x^n = C_2\, x^{1/4}$$

$$= 3.93 \left( \frac{\nu}{\alpha} \right)^{-1/2} \left( 0.952 + \frac{\nu}{\alpha} \right)^{1/2} \left( \frac{g\beta\theta_w}{\nu^2} \right)^{-1/4} x^{1/4}$$

$$\frac{\delta}{x} = 3.93\, \mathrm{Pr}^{-1/2} (0.952 + \mathrm{Pr})^{1/4} \left( \frac{g\beta\theta_w x^3}{\nu^2} \right)^{-1/4}$$

or,
$$\frac{\boldsymbol{\delta}}{\mathbf{x}} = \mathbf{3.93(0.952 + Pr)^{1/4} \frac{1}{Pr^{1/2}\, (Gr_x)^{1/4}}} \tag{5.34}$$

Heat flux
$$q = -k \left( \frac{\partial \theta}{\partial y} \right)_{y=0} = 2k \frac{\theta_w}{\delta} = h\theta_w$$

$\therefore$
$$h = \frac{2k}{\delta}$$

$$\frac{hx}{k} = \frac{2x}{\delta} = \frac{2}{3.93\, (0.952 + \mathrm{Pr})^{1/4}\ \mathrm{Pr}^{-1/2}\, (\mathrm{Gr}_x)^{-1/4}}$$

$\therefore$
$$\mathbf{Nu_x = 0.508\ Pr^{1/2}\ (0.952 + Pr)^{-1/4}\ (Gr_x)^{1/4}} \tag{5.35}$$

$\therefore$
$$\delta \propto x^{1/4}$$

As $x$ increases, $\delta$ also increases.

$h \propto \dfrac{1}{x^{1/4}}$, as $x$ increases, $h$ decreases,

$$\bar{h} = \frac{1}{L} \int_0^L h\, dx = \frac{1}{L} \int_0^L C x^{-1/4}\, dx$$

$$= (C_1\, x^{3/4})_{x=L}\ \frac{4}{3} = \frac{4}{3} h_L$$

Hence
$$\mathbf{\overline{Nu}_L = \frac{4}{3} Nu_x = 0.677 (Pr)^{1/2} (0.952 + Pr)^{-1/4} (Gr_L)^{1/4}} \tag{5.35a}$$

For air, $\mathrm{Pr} = 0.714$.

Equation (5.35) reduces to

$$\mathrm{Nu}_x = 0.378\ \mathrm{Gr}_x^{1/4} \tag{5.36}$$

Exact solution gives the constant as 0.360

$$\mathrm{Nu}_L = \frac{4}{3}\ (0.378)\ (\mathrm{Gr}_L)^{1/4} = 0.504\ \mathrm{Gr}_L^{1/4} \tag{5.37}$$

Fluid properties are evaluated at the film temperature

$$T^* = (T_w + T_\infty)/2$$

Transit]ion from laminar to turbulent flow occurs at $\mathrm{Ra}_{x,c} = 10^9$.

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Temperature field measurements around a heated body are obtained through the use of the Zehnder-Mach *interferometer*. An interferometer indicates lines of constant density in a fluid flow field. For a gas in natural convection at low pressure these lines of constant density are equivalent to lines of constant temperature. Once the temperature field is obtained, the heat transfer from a surface in natural convection may be calculated by using the temperature gradient at the surface and the thermal conductivity of the gas. Several interferometric studies of natural convection have been carried out (21 – 23), and some typical photographs of the flow fields are shown in Figs 5.8 – 5.11. Figure 5.8 shows the lines of constant temperature around a heated vertical flat plate. It may be noticed that the lines are closest together near the plate surface,

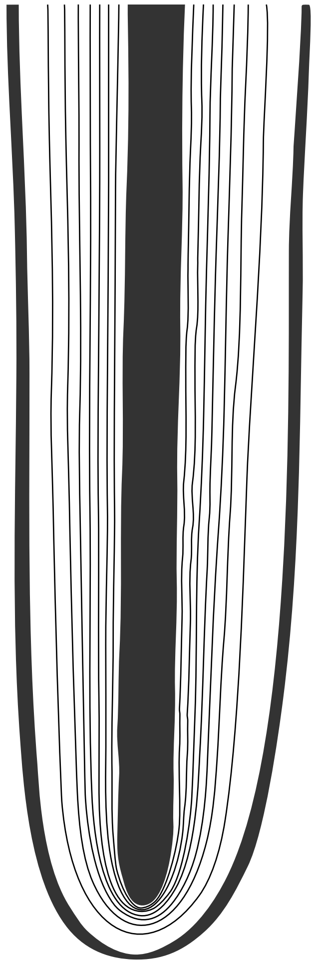

**Fig. 5.8** *Interferometer photograph showing lines of constant temperature around a heated vertical plate in natural convection*

indicating a higher temperature gradient in that region. Figure 5.9 shows the lines of constant temperature around a heated horizontal cylinder in natural convection, and Fig. 5.10 shows the boundary layer interaction between a group of four horizontal cylinders. Interferometric studies help to determine the point at which eddies are formed in the natural convection boundary layers and to predict the start of transition to turbulent flow.

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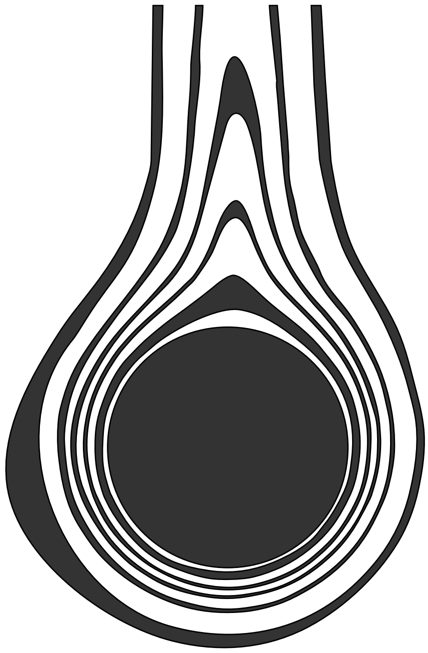

**Fig. 5.9** *Interferometer photograph showing lines of constant temperature around a heated horizontal cylinder in natural convection*

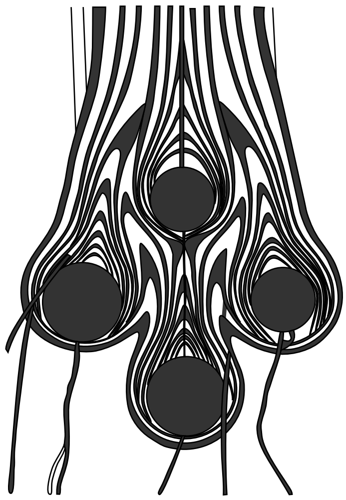

**Fig. 5.10** *Interferometer photograph showing the boundary layer interaction between four heated horizontal cylinders in natural convection*

In natural convection the velocities are so small that they are difficult to measure. A rough visual indication is given in Fig. 5.11, where a natural convection boundary layer wave resulting from a heat pulse near the leading edge of the plate is presented. The maximum points in the isotherms are seen to have a phase lag and a line passing through these maxima has the approximate shape of the velocity profile.

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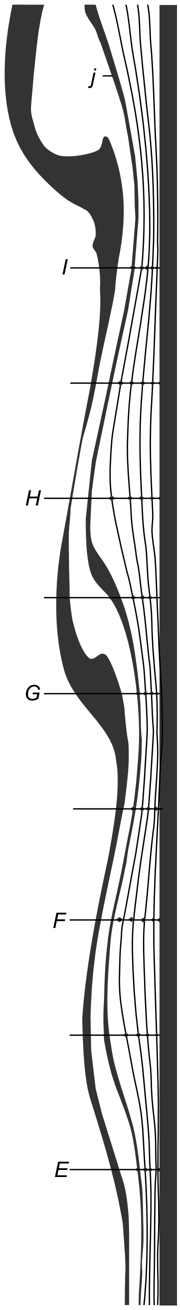


**Fig. 5.11** *Interferometer photograph showing isotherms on a heated vertical flat plate resulting in a periodic disturbance of the boundary layer*

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## 5.3 EMPIRICAL CORRELATIONS FOR VARIOUS SHAPES

Experimental data are correlated by dimensional analysis. It is the general practice to determine an equation for the line that best fits the data. Experimental results are also compared with those obtained by analytic means, if they are available. If the two agree and the analytic method adequately describes the experimental results, one can make out the physical mechanisms that are significant for the problem.

In the following sections we present correlation equations for several important geometries.

### 5.3.1 Vertical Plates and Cylinders

For a flat vertical surface and laminar natural convection, by using the integral boundary layer analysis, the local values of heat transfer coefficient and the boundary layer thickness at a distance $x$ from the leading edge have been derived in the last section in the form of Eqs (5.35) and (5.34) respectively, which can be written as

$$h_x = 0.508\,\mathrm{Pr}^{1/2}\frac{\mathrm{Gr}_x^{1/4}}{(0.952+\mathrm{Pr})^{1/4}}\frac{k}{x} \tag{5.38}$$

and

$$\delta(x) = 3.93\,x\left(\frac{0.952+\mathrm{Pr}}{\mathrm{Pr}^2\,\mathrm{Gr}_x}\right)^{1/4} \tag{5.39}$$

The average value of the heat transfer coefficient for a height $L$ is obtained

$$\bar{h} = \frac{4}{3}h_L = \frac{4}{3}\times 0.508\,\mathrm{Pr}^{1/2}\frac{\mathrm{Gr}_L^{1/4}}{(0.952+\mathrm{Pr})^{1/4}}\frac{k}{L}$$

or

$$\mathbf{\overline{Nu}_L = \frac{\bar{h}L}{k} = 0.68\,Pr^{1/2}\frac{Gr_L^{1/4}}{(0.952+Pr)^{1/4}}} \tag{5.40}$$

For air, $\mathrm{Pr} = 0.714$ and $\mathrm{Nu}_L = 0.505\,\mathrm{Gr}_L^{1/4}$

which is almost the same as Eq. (5.37).

For natural convection over a vertical flat plate or vertical cylinder in the turbulent region ($\mathrm{Gr}_L > 10^9$) McAdams [2] recommends

$$\mathbf{\overline{Nu}_L = 0.13\,(Gr_L\,Pr)^{1/3}} \tag{5.41}$$

For a long vertical plate or a long cylinder tilted at an angle $\theta$ from the vertical with the heated surface facing downward (Fig. 5.12(a)) or cooled surface facing upward [Fig. 5.12(b)], the following equation can be used:

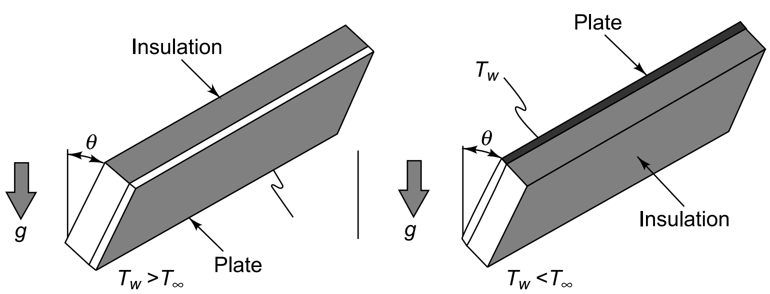


**Fig. 5.12** *(a) Long vertical plate with heated surface facing downward and (b) with cooled surface facing upward*

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$$\overline{\mathrm{Nu}}_L = 0.56\,(\mathrm{Gr}_L\,\mathrm{Pr}\cos\theta)^{1/4} \tag{5.42}$$

which applies in the range

$$10^5 < \mathrm{Gr}\,\mathrm{Pr}\cos\theta < 10^{11} \text{ and } 0 \le \theta \le 89°$$

The exact analysis of natural convection on a vertical plate subject to uniform wall temperature has been performed over a wide range of Prandtl numbers by various investigators. Schliching [6] compiled the mean Nusselt number as presented in Table 5.1 for the expression in the form

$$\overline{\mathrm{Nu}}_m = C(\mathrm{Gr}_L\,\mathrm{Pr})^n \tag{5.43}$$

**Table 5.1** *Exact solutions of the mean Nusselt number for laminar free convection on a vertical plate (For $Gr_L\,Pr < 10^9$)*

| *Pr* | $Nu_m/(Gr\,Pr)^{1/4}$ |
|---|---|
| 0.003 | 0.182 |
| 0.008 | 0.226 |
| 0.01 | 0.242 |
| 0.02 | 0.28 |
| 0.03 | 0.305 |
| 0.72 | 0.516 |
| 0.73 | 0.518 |
| 1 | 0.535 |
| 2 | 0.568 |
| 10 | 0.62 |
| 100 | 0.653 |
| 1000 | 0.665 |
| $\infty$ | 0.670 |

Churchill and Chu [14] proposed two equations for correlating natural convection on a vertical plate under isothermal surface conditions. One expression, which applies to laminar flow only and holds for all values of the Prandtl number, is given by

$$\overline{\mathrm{Nu}} = 0.68 + \frac{0.67\,\mathrm{Ra}_L^{1/4}}{\left[1 + (0.492/\mathrm{Pr})^{9/16}\right]^{4/9}} \tag{5.44}$$

$$10^{-1} < \mathrm{Ra}_L < 10^9$$

The other expression, which applies to both laminar and turbulent flow is given by

$$\overline{\mathrm{Nu}}_L^{1/2} = 0.825 + \frac{0.387\,\mathrm{Ra}_L^{1/6}}{\left[1 + (0.492/\mathrm{Pr})^{9/16}\right]^{8/27}}$$

$$10^{-1} < \mathrm{Ra}_L < 10^{12} \tag{5.45}$$

The physical properties are evaluated at the film temperature $T^* = (T_w + T_\infty)/2$.

Natural convection on a vertical plate subject to uniform heat flux at the wall surface was investigated by Sparrow and Gregg [15], Vliet and Liu [10] and Vliet [16]. On the basis of their experimental data, the following correlations were proposed:

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$$\mathrm{Nu}_x = 0.60\,(\mathrm{Gr}_x^*\,\mathrm{Pr})^{1/5} \qquad 10^5 < \mathrm{Gr}_x^*\,\mathrm{Pr} < 10^{11} \text{ (laminar)} \tag{5.46}$$

$$\mathrm{Nu}_x = 0.568\,(\mathrm{Gr}_x^*\,\mathrm{Pr})^{0.22} \qquad 2\times 10^{13} < \mathrm{Gr}_x^*\,\mathrm{Pr} < 10^{16} \text{ (turbulent)} \tag{5.47}$$

where the modified Grashof number $\mathrm{Gr}_x^*$ is defined as

$$\mathrm{Gr}_x^* = \mathrm{Gr}_x\,\mathrm{Nu}_x = \frac{g\beta\theta_w\,x^3}{\nu^2}\,\frac{q_w x}{k\theta_w} = \frac{g\beta\,q_w x^4}{k\nu^2}$$

$q_w$ being the constant wall heat flux.

### 5.3.2 Horizontal Plates

The average Nusselt number for natural convection on a horizontal plate depends on whether the surface is facing up or down and whether the plate surface is warmer or cooler than the surrounding flux. Again we consider the cases for uniform wall temperature and uniform wall heat flux separately.

#### Uniform Wall Temperature

The mean Nusselt number for natural convection on a horizontal plate as correlated by McAdams [2] is

$$\overline{\mathrm{Nu}}_L = C\,(\mathrm{Gr}_L\,\mathrm{Pr})^n \tag{5.48}$$

For hot surface facing up or cold surface facing down, in the range (laminar) $10^5 < \mathrm{Ra} < 2\times 10^7$, $C = 0.54$, $n = 1/4$, and in the range (turbulent) $2\times 10^7 < \mathrm{Ra} < 3\times 10^{10}$, $C = 0.14$, $n = 1/3$. For hot surface facing down or cold surface facing up, in the range (laminar) $3\times 10^5 < \mathrm{Ra} < 3\times 10^{10}$, $C = 0.27$, $n = 1/4$. The characteristic length $L$ of the plate can be taken as the length of a side for a square, the arithmetic mean of the two sides for a rectangle and $0.9D$ for a circular disk of diameter $D$, as suggested by McAdams, or as $A/P$, where $A$ is the surface area of the plate and $P$ its perimeter.

#### Uniform Heat Flux

For the horizontal plate with heated surface facing upward:

$$\overline{\mathrm{Nu}}_L = 0.13\,(\mathrm{Gr}_L\,\mathrm{Pr})^{1/3} \text{ for } \mathrm{Gr}_L\,\mathrm{Pr} < 2\times 10^8 \tag{5.49}$$

$$\overline{\mathrm{Nu}}_L = 0.16\,(\mathrm{Gr}_L\,\mathrm{Pr})^{1/3} \text{ for } 5\times 10^8 < \mathrm{Gr}_L\,\mathrm{Pr} < 10^{11} \tag{5.50}$$

For the horizontal plate with the heated surface facing downward:

$$\overline{\mathrm{Nu}}_L = 0.58\,(\mathrm{Gr}_L\,\mathrm{Pr})^{1/5} \text{ for } 10^6 < \mathrm{Gr}_L\,\mathrm{Pr} < 10^{11} \tag{5.51}$$

Properties are evaluated at $T^* = T_w - 0.25\,(T_w - T_\infty)$.

### 5.3.3 Spheres and Cylinders

For natural convection on a single isothermal sphere for fluids having $\mathrm{Pr} \simeq 1$,

$$\overline{\mathrm{Nu}}_d = \frac{hD}{k} = 2 + 0.43\,\mathrm{Ra}_d^{1/4} \tag{5.52}$$

for $1 < \mathrm{Ra}_d < 10^5$

For water, $$\overline{\mathrm{Nu}}_d = 2 + 0.50\,\mathrm{Ra}_d^{1/4} \tag{5.53}$$

for $3\times 10^5 < \mathrm{Ra}_d < 8\times 10^8$ and $10 < \overline{\mathrm{Nu}}_d < 90$

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An equation for $h$ from single horizontal wires or pipes in natural convection, based on the experimental data in Fig. 5.4 is

$$\overline{\mathrm{Nu}}_d = 0.53\,(\mathrm{Gr}_d\,\mathrm{Pr})^{1/4} \tag{5.54}$$

which is valid for Pr > 0.5 and $10^3 < \mathrm{Gr} < 10^9$.

### 5.3.4 Simplified Equations for Air

Since air is the most common fluid for natural convection configurations, the foregoing correlations have been simplified for the heat transfer coefficient from different surfaces to air at atmospheric pressure. These simplified equations are given in Table 5.2.

**Table 5.2** *Simplified equations for free convection in air*

| *Geometry* | *Laminar* | *Turbulent* |
|---|---|---|
| 1. Vertical plate or cylinder | $\bar{h} = 1.42\,(\Delta T/L)^{1/4}$<br>$10^4 < \mathrm{Ra} < 10^9$ | $\bar{h} = 1.32\,(\Delta T/L)^{1/3}$<br>$10^9 < \mathrm{Ra} < 10^{12}$ |
| 2. Horizontal cylinder | $\bar{h} = 1.32\,(\Delta T/D)^{1/4}$<br>$10^4 < \mathrm{Ra} < 10^9$ | $\bar{h} = 1.25\,(\Delta T/D)^{1/3}$<br>$10^9 < \mathrm{Ra} < 10^{12}$ |
| 3. Horizontal plate | | |
| (a) Heating surface facing up | $\bar{h} = 1.32\,(\Delta T/L)^{1/4}$<br>$10^5 < \mathrm{Ra} < 2 \times 10^7$ | $\bar{h} = 1.67\,(\Delta T)^{1/3}$<br>$2 \times 10^7 < \mathrm{Ra} < 3 \times 10^{10}$ |
| (b) Heated surface facing down | $\bar{h} = 0.59\,(\Delta T/L)^{1/4}$<br>$3 \times 10^5 < \mathrm{Ra} < 3 \times 10^{10}$ | $\bar{h} = 0.59\,(\Delta T/L)^{1/4}$ |
| 4. Sphere | $\bar{h} = [2 + 0.392\,\mathrm{Gr}_d^{1/4}]$ | $\frac{k}{D}$ for $1 < \mathrm{Gr}_d < 10^5$ |

### 5.3.5 Enclosed Spaces

Engineering applications frequently involve heat transfer between surfaces that are at different temperatures and are separated by an enclosed fluid, like double-glazed windows, flat-plate solar collectors, building walls and so on. The rectangular cavity (Fig. 5.13) consists of two isothermal parallel surfaces at temperatures $T_1$ and $T_2$ spaced a distance $\delta$ apart and of height $L$, and the top and bottom surfaces are insulated. The Grashof number is defined by

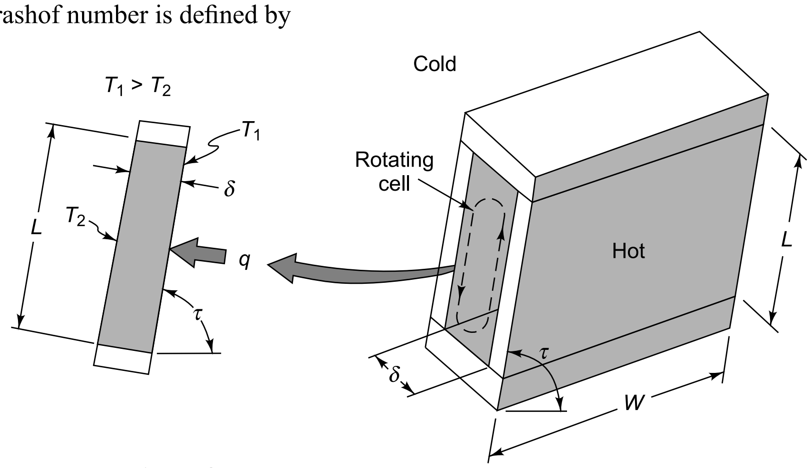


**Fig. 5.13** *Natural convection in inclined enclosed spaces*

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$$\mathrm{Gr}_d = \frac{g\beta(T_1 - T_2)\delta^3}{\nu^2}$$

and the parameter $L/\delta$ is called the *aspect ratio*. A temperature difference produces a flow in the enclosure. In vertical cavities ($\tau$ = 90°) Hollands and Konicek [17] found that for $\mathrm{Gr}_d$ < 8000 the flow consists of one large cell rotating in the enclosure with heat transfer taking place essentially by conduction. As Gr increases, the flow becomes more of a boundary-layer type with fluid rising in a layer near the heated surface, turning the corner at the top, and flowing downward in a layer near the cooled surface. The boundary layer thickness decreases with $\mathrm{Gr}_d^{1/4}$, and the core region is more or less inactive and thermally stratified. For the geometry in Fig. 5.13,

$$\overline{\mathrm{Nu}}_\delta = 0.22\left(\frac{L}{\delta}\right)^{-1/4}\left(\frac{\mathrm{Pr}}{0.2+\mathrm{Pr}}\cdot\mathrm{Ra}_d\right)^{0.28} \tag{5.55}$$

in the range 2 < $L/\delta$ < 10, Pr < 10 and Ra < $10^{10}$ and

$$\overline{\mathrm{Nu}}_\delta = 0.18\left(\frac{\mathrm{Pr}}{0.2+\mathrm{Pr}}\cdot\mathrm{Ra}_\delta\right)^{0.29} \tag{5.55a}$$

in the range 1 < $L/\delta$ < 2, $10^{-3}$ < Pr < $10^5$ and,

$$\frac{\mathrm{Ra}_\delta\cdot\mathrm{Pr}}{0.2+\mathrm{Pr}} > 10^3$$

For large aspect ratios and $\tau$ = 90° the following relations are recommended:

$$\overline{\mathrm{Nu}} = 0.42\,\mathrm{Ra}_\delta^{0.25}\,\mathrm{Pr}^{0.012}/(L/\delta)^{0.3} \tag{5.56}$$

in the range 10 < $L/\delta$ < 40, 1 < Pr < 2 × $10^4$ and $10^4$ < $\mathrm{Ra}_\delta$ < $10^7$ and

$$\overline{\mathrm{Nu}}_\delta = 0.046\,\mathrm{Ra}_\delta^{0.33} \tag{5.56a}$$

in the range 1 < $L/\delta$ < 40 and 1 < Pr < 20.

For natural convection inside spherical cavities of diameter $D$ the relation

$$\frac{\bar{h}_c D}{k} = C\,(\mathrm{Gr}_d\,\mathrm{Pr})^n \tag{5.57}$$

is recommended, with the constants $C$ and $n$ selected from the following table:

| $Gr_d Pr$ | $C$ | $n$ |
|---|---|---|
| $10^4$–$10^9$ | 0.59 | 1/4 |
| $10^4$–$10^{12}$ | 0.13 | 1/3 |

For natural convection heat transfer across the gap between two horizontal concentric cylinders (Fig. 5.14) the following correlation is suggested for heat flow per unit length (W/m)

$$q' = \frac{2\pi k_{\mathrm{eff}}}{\ln(D_o/D_i)}(T_i - T_o) \tag{5.58}$$

where the effective thermal conductivity $k_{\mathrm{eff}}$ is given by

$$\frac{k_{\mathrm{eff}}}{k} = 0.386\left(\frac{\mathrm{Pr}}{0.861+\mathrm{Pr}}\right)^{1/4}\mathrm{Ra}_{\mathrm{cyl}}^{*1/4} \tag{5.59}$$

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$$(\mathrm{Ra}^*_{\mathrm{cyl}})^{1/4} = \frac{\ln D_o / D_i}{L^{3/4}\left(D_i^{-3/5} + D_o^{-3/5}\right)^{5/4}} \cdot \mathrm{Ra}_L^{1/4}$$

$$\mathrm{Ra}_L = \frac{g\beta\,(T_i - T_o)\,L^3}{\nu^2} \cdot \mathrm{Pr}$$

$$L = \frac{D_o - D_i}{2}$$

which is valid in the range $0.7 < \mathrm{Pr} < 6000$ and $10 \le \mathrm{Ra}_{\mathrm{cyl}} \le 10^7$.

For concentric spheres the following correlation is recommended

$$Q = k_{\mathrm{eff}} \frac{\pi D_i D_o}{L}(T_i - T_o)\ \text{watts} \tag{5.60}$$

where, $$\frac{k_{\mathrm{eff}}}{k} = 0.74\left(\frac{\mathrm{Pr}}{0.861 + \mathrm{Pr}}\right)^{1/4} (\mathrm{Ra}^*_{\mathrm{sph}})^{1/4}$$

and $$(\mathrm{Ra}^*_{\mathrm{sph}})^{1/4} = \frac{L^{1/4}\,\mathrm{Ra}_L^{1/4}}{D_i D_o \left(D_i^{7/5} + D_o^{-7/5}\right)^{5/4}}$$

valid in the range of $10^2 < \mathrm{Ra}^*_{\mathrm{sph}} < 10^4$.

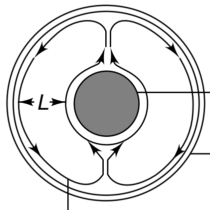


**Fig. 5.14** *Natural convection heat transfer in the annular space between long concentric cylinders or concentric spheres*

## 5.4 ROTATING CYLINDERS, DISKS AND SPHERES

Heat transfer by convection between a rotating body and a surrounding fluid is of importance in the thermal analysis of flywheels, turbine rotors and other rotating components of various machines. With heat transfer, a critical velocity is reached when the circumferential speed of the cylinder surface becomes approximately equal to the upward natural convection velocity at the side of a heated stationary cylinder. Below the critical velocity, simple natural convection, characterised by the conventional Grashof number, $[g\beta(T_w - T_\infty)D^3]\ \nu^2$, controls the rate of heat transfer. At speeds greater than critical ($\mathrm{Re}_w > 8000$ in air) the peripheral-speed Reynolds number $\pi D^2 w/\nu$ becomes the controlling parameter. The combined effects of the Reynolds, Prandtl and Grashof numbers on the average Nusselt number for a horizontal cylinder rotating in air above the critical velocity (Fig. 5.15) can be expressed by the empirical equation [18].

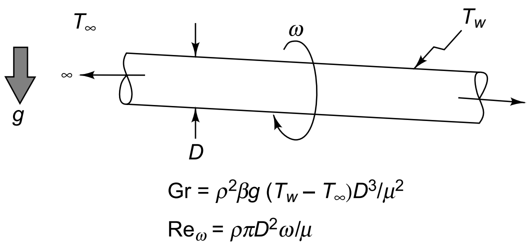


**Fig. 5.15** *Horizontal cylinder rotating in air*

$$\overline{\mathrm{Nu}}_d = \frac{\bar{h}_c D}{k} = 0.11\,(0.5\,\mathrm{Re}_w^2 + \mathrm{Gr}_d\,\mathrm{Pr})^{0.35} \tag{5.61}$$

The boundary layer on a rotating disk is laminar and of uniform thickness at rotational Reynolds numbers $wD^2/\nu$ below about $10^6$. At higher Reynolds numbers the flow becomes turbulent near the outer edge, and as $\mathrm{Re}_w$ is increased, the transition point moves radially inward. The boundary layer thickens with increasing radius (Fig. 5.16). For the laminar regime in air

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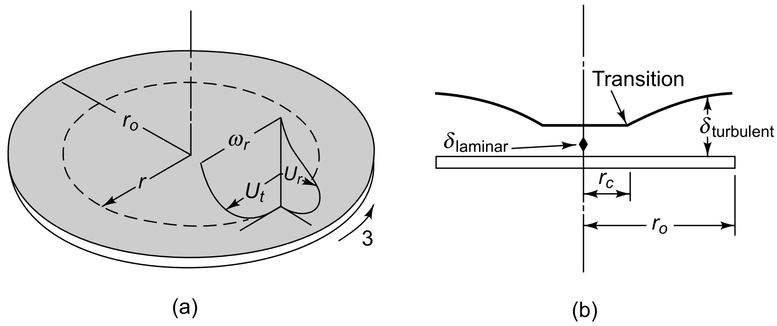


**Fig. 5.16** *Velocity and boundary layer profiles for a disk rotating in an infinite environment*

$$\overline{\text{Nu}}_d = \frac{\bar{h}D}{k} = 0.36\left(\frac{wD^2}{\nu}\right)^{1/2} \tag{5.62}$$

for $wD^2/\nu < 10^6$.

In the turbulent flow regime ($wD^2/\nu > 10^6$) of a disk in air, the local value at a radius $r$ is

$$\text{Nu}_r = \frac{h_c r}{k} = 0.0195\left(\frac{wr^2}{\nu}\right)^{0.8} \tag{5.63}$$

For a sphere of diameter $D$ rotating in an infinite environment with Pr > 0.7 in laminar regime ($\text{Re}_w = wD^2/\nu < 5 \times 10^4$), the average Nusselt number ($\bar{h}_c D/k$) can be obtained from

$$\overline{\text{Nu}}_d = 0.43\ \text{Re}_w^{0.5}\ \text{Pr}^{0.4} \tag{5.64}$$

while in the regime $5 \times 10^4 \le \text{Re}_w \le 7 \times 10^5$ the equation is

$$\overline{\text{Nu}}_d = 0.066\ \text{Re}_w^{0.67}\ \text{Pr}^{0.4} \tag{5.65}$$

## 5.5 COMBINED FORCED AND NATURAL CONVECTION

The relative magnitude of the dimensionless parameter $\text{Gr}/\text{Re}^2$ governs the relative importance of natural convection in relation to forced convection where

$$\frac{\text{Gr}}{\text{Re}^2} = \frac{g\beta\left(T_w - T_\infty\right)L}{U_0^2}$$

which represents the ratio of the buoyancy forces to inertia forces. When this ratio is of the order of unity, i.e. $\text{Gr} \simeq \text{Re}^2$, the natural and forced convection are of comparable magnitude, and hence they should be analysed simultaneously. If

$\dfrac{\text{Gr}}{\text{Re}^2} >> 1$ Natural convection dominates

$\dfrac{\text{Gr}}{\text{Re}^2} \simeq 1$ Natural and forced convection are of comparable magnitude

$\dfrac{\text{Gr}}{\text{Re}^2} << 1$ Forced convection dominates

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Combined natural and forced convection has been studied by various investigators, for flow over vertical [15, 16] and horizontal [17, 18] plates. Figure 5.17 illustrates the role of the parameter $Gr_x/Re_x^2$ on the effects of natural convection on forced convection on a vertical heated (i.e., $T_w > T_\infty$) isothermal flat plate subjected to upward forced flow with a free stream velocity $u_\infty$. The results also apply for a cold plate (i.e., $T_w < T_\infty$) when the forced flow velocity $u_\infty$ is downward. In such cases, the effect of natural convection is to enhance the total heat transfer.

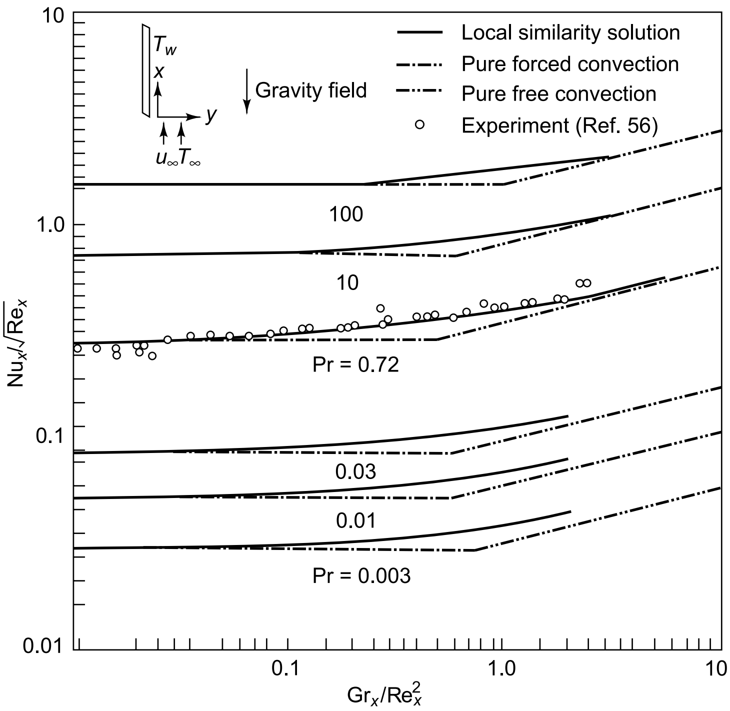


**Fig. 5.17** *Local Nusselt number for combined, forced and natural convection from an isothermal vertical plate*

In Fig. 5.17, the solid lines represent the theoretical predictions of the local Nusselt number when heat transfer by combined natural and forced convection is considered. The horizontal and the slanted chain dotted lines are for the cases of pure forced and natural convection, respectively. For larger values of $Gr_x/Re_x^2$ the solid lines asymptotically join the lines for pure natural convection. For smaller values of $Gr_x/Re_x^2$, the solid lines join the lines for pure forced convection. Included in this figure are the experimental data for Pr = 0.72. The agreement between the experiment and the analysis is very good. The effects of natural convection on forced convection become theoretically pronounced at low Prandtl numbers. The threshold values of $Gr_x/Re_x^2$ for 5% deviation of the Nusselt number resulting from the neglect of natural convection in forced flow along a vertical plate for $T_w > T_\infty$ when $u_\infty$ is upward are as follows:

| | | | | |
|---|---|---|---|---|
| Pr: | 100 | 10 | 0.72 | 0.03–0.003 |
| $Gr_x/Re_x^2$: | 0.24 | 0.13 | 0.08 | 0.056–0.05 |

For combined free and forced convection in the laminar flow regime inside a circular tube, Brown and Gauvin [23] recommend the following correlation for the Nusselt number

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$$\text{Nu}_d = 1.75\,[\text{Gz} + 0.012\,(\text{Gz}\,\text{Gr}_d^{1/3})^{4/3}]^{1/3}\left(\frac{\mu_b}{\mu_w}\right)^{0.14} \tag{5.66}$$

where Gz is the Graetz number, defined as

$$\text{Gz} = \text{Re}_d\,\text{Pr}\left(\frac{D}{L}\right) \tag{5.67}$$

where $\text{Gr}_d$ and $\text{Re}_d$ are based on the tubes inside diameter with $\Delta T = T_w - T_b$, the difference between tube wall and fluid bulk temperature.

Figure 5.18 shows the regimes of pure natural convection in boundary layer flow, mixed convection and pure forced convection.

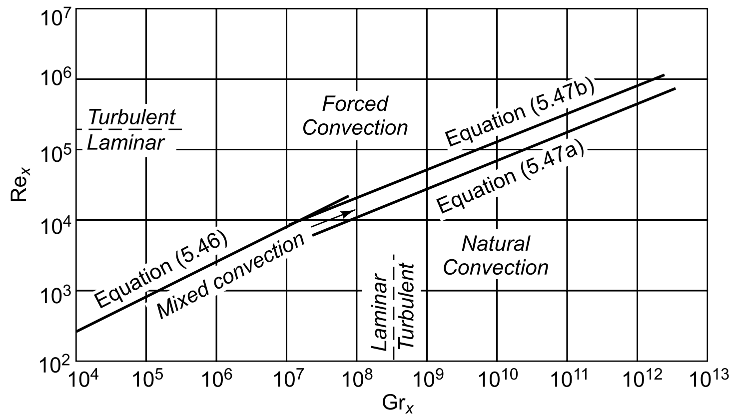


**Fig. 5.18** *Regimes of convection for flow and buoyancy effects parallel boundary layer processes (B. Gebhart [8]).*

## Solved Examples

**Example 5.1** A metal plate 0.609 m in height forms the vertical wall of an oven and is at a temperature of 171 °C. Within the oven is air at a temperature of 93.4 °C and atmospheric pressure. Assuming that natural convection conditions hold near the plate, and that for this case

$$\text{Nu} = 0.548\,(\text{Gr Pr})^{1/4}$$

find the mean heat transfer coefficient and the heat taken up by air per second per metre width. For air at 132.2°C, take $k$ = 33.2 × $10^{-6}$ kW/m K, $\mu$ = 0.232 × $10^{-4}$ kg/ms, $c_p$ = 1.005 kJ/kg K. Assume air as an ideal gas and $R$ = 0.287 kJ/kg K.

**Solution** Prandtl number $\text{Pr} = \dfrac{\mu c_p}{k} = \dfrac{0.232\times10^{-4}\times1.005\times10^{3}}{32.2\times10^{-6}\times10^{3}}$

$= 0.7241$

$$T_{\text{mean}} = \frac{171+93.4}{2} = 132.2°\text{C} = 405.2\,\text{K}$$

$$\beta = \frac{1}{T_{\text{mean}}} = 2.47\times10^{-3}\,\text{K}^{-1}$$

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$$\text{Nu}_d = 1.75\,[\text{Gz} + 0.012\,(\,\text{Gz}\,\text{Gr}_d^{1/3}\,)^{4/3}]^{1/3}\left(\frac{\mu_b}{\mu_w}\right)^{0.14} \tag{5.66}$$

where Gz is the Graetz number, defined as

$$\text{Gz} = \text{Re}_d\,\text{Pr}\left(\frac{D}{L}\right) \tag{5.67}$$

where $\text{Gr}_d$ and $\text{Re}_d$ are based on the tubes inside diameter with $\Delta T = T_w - T_b$, the difference between tube wall and fluid bulk temperature.

Figure 5.18 shows the regimes of pure natural convection in boundary layer flow, mixed convection and pure forced convection.

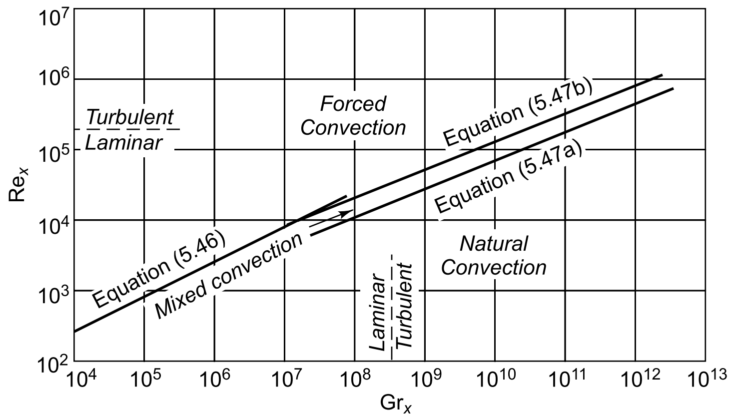


**Fig. 5.18** *Regimes of convection for flow and buoyancy effects parallel boundary layer processes (B. Gebhart [8]).*

## Solved Examples

**Example 5.1** A metal plate 0.609 m in height forms the vertical wall of an oven and is at a temperature of 171 °C. Within the oven is air at a temperature of 93.4 °C and atmospheric pressure. Assuming that natural convection conditions hold near the plate, and that for this case

$$\text{Nu} = 0.548\,(\text{Gr Pr})^{1/4}$$

find the mean heat transfer coefficient and the heat taken up by air per second per metre width. For air at 132.2°C, take $k$ = 33.2 × 10$^{-6}$ kW/m K, $\mu$ = 0.232 × 10$^{-4}$ kg/ms, $c_p$ = 1.005 kJ/kg K. Assume air as an ideal gas and $R$ = 0.287 kJ/kg K.

**Solution** Prandtl number $\text{Pr} = \dfrac{\mu c_p}{k} = \dfrac{0.232 \times 10^{-4} \times 1.005 \times 10^3}{32.2 \times 10^{-6} \times 10^3}$

$$= 0.7241$$

$$T_{\text{mean}} = \frac{171 + 93.4}{2} = 132.2°\text{C} = 405.2\,\text{K}$$

$$\beta = \frac{1}{T_{\text{mean}}} = 2.47 \times 10^{-3}\,\text{K}^{-1}$$

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$$\theta = T_w - T_\infty = 171 - 93.4 = 77.6°C = 77.6 \text{ K}$$

$$\rho = \frac{p}{RT} = \frac{101.325}{0.287 \times 405.2} = 0.8713 \text{ kg/m}^3$$

$$\nu = \frac{\mu}{\rho} = \frac{0.232 \times 10^{-4}}{0.8713} = 2.663 \times 10^{-5} \text{ m}^2\text{/s}$$

$$\text{Gr} = \frac{g\beta\theta L^3}{\nu^2} = \frac{9.81 \times 2.47 \times 10^{-3} \times 77.6 \times (0.609)^3}{(2.663 \times 10^{-5})^2}$$

$$= 5.985 \times 10^8$$

$$\text{Nu} = 0.548\,(5.985 \times 10^8 \times 0.7241)^{1/4} = 79.07$$

$$= (h_m L)/k$$

$$h_m = \frac{79.07 \times 32.2 \times 10^{-6}}{0.609} = 4.181 \times 10^{-3} \text{ kW/m}^2\text{ K}$$

$$= 4.181 \text{ W/m}^2\text{ K} \quad \textit{Ans.}$$

$$Q = h_m\,(Lb)\,\theta$$

$$Q/b = 4.181 \times 0.609 \times 77.6 = 197.57 \text{ W/m} \quad \textit{Ans.}$$

**Example 5.2** The maximum allowable surface temperature of an electrically heated vertical plate 15 cm high and 10 cm wide is 140°C. Estimate the maximum rate of heat dissipation from both sides of the plate in an atmosphere at 20°C. The radiation heat transfer coefficient is 8.72 W/m$^2$ K. For air at 80°C, take $\nu = 21.09 \times 10^{-6}$ m$^2$/s, Pr = 0.692 and $k$ = 0.03 W/m K.

**Solution** Rayleigh number $\text{Ra} = \text{Gr Pr} \dfrac{g\beta\theta L^3}{\nu^2}\text{Pr}$

$$= \frac{9.81 \times (1/353) \times 120 \times (0.15)^3}{\left(21.09 \times 10^{-6}\right)^2} \times 0.692$$

$$= 17{,}510{,}650 \; (< 10^9)$$

$$\text{Nu} = 0.59\,(\text{Ra})^{1/4} = 0.59 \times (17{,}510{,}650)^{1/4}$$

$$= 38.166 = \frac{h_c L}{k}$$

$$h_c = \frac{38.166 \times 0.03}{0.15} = 7.6332 \text{ W/m}^2\text{ K}$$

$$Q_c = 2h_c A(T_w - T_\infty)$$

$$= 2 \times 7.6332 \times 0.15 \times 0.1 \times 120 = 27.48 \text{ W}$$

$$Q_r = 2h_r A\,(T_w - T_\infty) = 2 \times 8.72 \times 0.15 \times 0.1 \times 120$$

$$= 31.392 \text{ W}$$

$$Q_{\text{total}} = Q_c + Q_r = 58.872 \text{ W} \quad \textit{Ans.}$$

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**Example 5.3** *Given:* A 0.15 m o.d. steel pipe lies 2 m vertically and 8 m horizontally in a large room with an ambient temperature of 30°C. The pipe surface is at 250°C and has an emissivity of 0.60.
*To estimate*: The total rate of heat loss from the pipe to the atmosphere.

**Solution** Heat is lost by the pipe to the atmosphere both by natural convection and radiation.

Natural convection $\beta = \dfrac{1}{T_f} = \dfrac{1}{140+273} = \dfrac{1}{413}\,\text{K}^{-1}$

(a) *Vertical Part*: Grashof number, $\text{Gr} = \dfrac{g\beta\theta L^3}{v^2}$

or, $$\text{Gr} = \frac{9.81\times(1/413)\times(250-30)\times 2^3}{(27.8\times10^{-6})^2}$$

$$= 0.0541 \times 10^{12}$$

$$\text{Gr Pr} = 0.0541 \times 10^{12} \times 0.684$$

$$= 3.7 \times 10^{10}$$

Since Gr Pr > $10^9$, the flow is turbulent over the pipe. From Eq. (5.41)

$$\text{Nu} = 0.13\,(\text{Gr.Pr})^{1/3}$$

$$= 0.13\,(3.7 \times 10^{10})^{1/3}$$

$$= 432.7 = \frac{h_v L}{k}$$

$$\therefore \quad h_v = \frac{432.7\times0.035}{2} = 7.572\ \text{W/m}^2\text{K}.$$

(b) *Horizontal part:*

$$\text{Gr}_d = \frac{g\beta\theta D^3}{v^2}$$

$$= \frac{9.81\times\dfrac{1}{413}\times220\times(0.15)^2}{(27.8\times10^{-6})^2}$$

$$= 2.282 \times 10^7$$

$$\text{Gr}_d.\ \text{Pr} = 2.282 \times 10^7 \times 0.684$$

$$= 1.56 \times 10^7$$

Since $\text{Gr}_d$. Pr < $2 \times 10^7$, the flow is laminar. From Eq. (5.54),

$$\text{Nu}_d = 0.53\,(\text{Gr}_d\,.\ \text{Pr})^{1/4}$$

$$= 0.53\,(1.56 \times 10^7)^{1/4}$$

$$= 33.3 = \frac{h_H D}{k}$$

$$h_H = \frac{33.3\times0.035}{0.15} = 7.77\ \text{W/m}^2\text{K}$$

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Total heat loss by natural convection

$$Q_C = Q_H + Q_V$$
$$= (h_H A_H + h_V A_V)(T_\omega - T_\infty)$$
$$= [7.572 \times \pi \times 0.15 \times 2 + 7.77\ \pi \times 0.15 \times 8]\ (250 - 30)$$
$$= 6444 \text{ W} = 6.444 \text{ kW}$$

*Radiation*: Heat lost by radiation

$$Q_r = \sigma A \Sigma\ (T_w^4 - T_\infty^4)$$
$$= 5.67 \times \pi \times 0.15 \times 10 \times 0.6\ (5.23^4 - 3.03^4)$$
$$= 10643 \text{ W} = 10.643 \text{ kW}$$

Total heat loss $= Q_c + Q_r$

$$= 6.444 + 10.643$$
$$= 18.657 \text{ kW } Ans.$$

**Example 5.4** A nuclear reactor with its core constructed of parallel vertical plates 2.2 m high and 1.45 m wide has been designed on free convection heating of liquid bismuth. The maximum temperature of the plate surfaces is limited to 960°C, while the lowest allowable temperature of bismuth is 340°C. Calculate the maximum possible heat dissipation from both sides of each plate. For the convection coefficient the appropriate correlation is

$$\text{Nu} = 0.13\ (\text{Gr} \cdot \text{Pr})^{1/3}$$

where the properties evaluated at the mean film temperature of 650°C for bismuth are: $\rho = 10^4$ kg/m$^3$, $\mu$ = 3.12 kg/m-h, $c_p$ = 150.7 J/kgK, $k$ = 13.02 W/mK.

**Solution**

$$\beta = \frac{1}{T_f} = \frac{1}{650+273} = 1.08 \times 10^{-3}\ \text{K}^{-1}$$

where $T_f = \dfrac{960+340}{2} = 650°\text{C}$

$$\text{Pr} = \frac{\mu c_p}{k} = \frac{\frac{3.12}{3600} \times 150.7}{13.02} = 0.01$$

$$\text{Gr} = \frac{g \beta \theta L^3 \rho^2}{\mu^2} = \frac{9.81 \times 1.08 \times 10^{-3} \times (960-340)(2.2)^3 \times (10^4)^2}{(3.12/3600)^2}$$
$$= 9.312 \times 10^{15}$$

$$\text{Gr} \cdot \text{Pr} = 9.312 \times 10^{15} \times 0.01 = 93.12 \times 10^{12}$$

For the given correlation

$$\text{Nu} = \frac{hL}{k} = 0.13(93.12 \times 10^{12})^{0.333}$$
$$= 0.13\ (93.12)^{0.333} \times 10^4 = 0.5883 \times 10^4$$

$$\therefore \quad h = \frac{13.02}{2.2} \times 0.5883 \times 10^4 = 34820 \text{ W/m}^2\text{ K}$$

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$\therefore$ Heat dissipation from both sides of each plate

$$Q = 2\, hA\, \Delta T$$
$$= 2 \times 34{,}820 \times (2.2 \times 1.45) \times (960 - 340)$$
$$= 137734 \text{ W} = 137.734 \text{ kW} \quad \textit{Ans.}$$

**Example 5.5** *Given*: A 0.3 m glass plate at 77°C is hung vertically in the air at 27°C. A similar plate is placed in a wind tunnel and air is blown over it at a velocity of 4 m/s.
*To determine*: The boundary layer thickness at the trailing edge of the plate and the average heat transfer coefficient in both cases of natural and forced convection.

**Solution** Film temperature of air, $T_f = \dfrac{27+77}{2} = 52°\text{C}$

Properties of air at 52°C taken from the Appendix (Table A 4) are:

$$\beta = 30.7 \times 10^{-3}\ \text{K}^{-1},\ k = 28.15 \times 10^{-3}\ \text{W/mK}$$
$$\nu = 18.41 \times 10^{-6}\ \text{m}^2/\text{s},\ \text{Pr} = 0.7$$

*Natural Convection:* Grashof number, $\text{Gr} = \dfrac{g\beta\theta L^3}{\nu^2} = \dfrac{9.81\times 3.07\times 10^{-3}\times(77-27)\times(0.3)^3}{(18.41\times 10^{-6})^2} = 1.2 \times 10^8$

Rayleigh number, $\text{Ra} = \text{Gr. Pr}$

$$= 1.2 \times 10^8 \times 0.7 = 8.4 \times 10^7$$

$\therefore$ The boundary layer is in laminar flow. By using Eq. (5.34) and $x = 0.3$ m,

$$\frac{\delta}{x} = 3.93\times(0.952+\text{Pr})^{1/4}\frac{1}{\text{Pr}^{1/2}(\text{Gr}_x)^{1/4}}$$

$$\therefore \qquad \delta = 3.93 \times 0.3 \times (0.952 + 0.7)^{1/4}\ \frac{1}{(0.7)^{1/2}(1.2\times 10^8)^{1/4}}$$

$$\delta = 1.179 \times 1.1337 \times \frac{1}{0.8367\times 104.66}$$
$$= 0.0153 \text{ m} = 15.3 \text{ mm} \quad \textit{Ans.}$$

From Eq. (5.35),

$$\text{Nu}_x = 0.508\ \text{Pr}^{1/2}\ (0.952 + \text{Pr})^{1/4}\ (\text{Gr}_x)^{1/4}$$

$$h \propto \frac{1}{x^{1/4}}$$

$$\therefore \qquad \bar{h} = \frac{1}{L}\int_0^L h\text{d}x = \frac{1}{L}\int_0^L Cx^{-1/4}\text{d}x = \frac{4}{3}h_L$$

$$\therefore \qquad \text{Nu}_L = \frac{\bar{h}L}{k} = 0.676\ \text{Pr}^{1/2}(0.952+\text{Pr})^{1/4}(\text{Gr}_L)^{1/4}$$
$$= 0.676\ (0.7)^{1/2}\ (0.952 + 0.7)^{1/4}\ (1.2 \times 108)^{1/4}$$
$$= 52.2$$

$$\therefore \qquad \bar{h} = \frac{52.2\times 28.15\times 10^{-3}}{0.3} = 4.9\ \text{W/m}^2\text{K} \quad \textit{Ans.}$$

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*Forced convection*

$$u_\infty = 4 \text{ m/s}$$

$$\text{Re}_L = \frac{u_\infty L}{\nu} = \frac{4 \times 0.3}{18.41 \times 10^{-6}} = 6.51 \times 10^4$$

The flow is considered laminar

$$\delta_L = \frac{5L}{\sqrt{\text{Re}_L}} = \frac{5 \times 0.3}{(6.51 \times 10^4)^{1/2}}$$

$$= 5.88 \times 10^{-3} \text{ m} = 5.88 \text{ mm} \quad \textit{Ans.}$$

The boundary layer thickness in forced convection (5.88 mm) is less than that in natural convection (15.3 mm).

For forced convection laminar flow over a plate,

$$\text{Nu} = 0.664\,(\text{Re}_L)^{1/2}\,(\text{Pr})^{1/3}$$

$$= 0.664\,(6.51 \times 10^4)^{1/2}\,(0.7)^{1/3}$$

$$= 150.4$$

$$\therefore \qquad \bar{h} = \frac{150.4 \times 28.15 \times 10^{-3}}{0.3}$$

$$= 14.11 \text{ W/m}^2\text{K} \quad \textit{Ans.}$$

Thus, the heat transfer coefficient in forced convection is much larger than in natural convection.

**Example 5.6** For natural convection heat transfer from a horizontal circular cylinder, the following correlation can be used for Rayleigh number in the range of $10^5$ and $10^{12}$.

$$\overline{\text{Nu}}_D = \left[0.60 + \frac{0.387\,\text{Ra}_D^{1/6}}{\left[1 + (0.559/\text{Pr})^{9/16}\right]^{8/27}}\right]^2$$

(a) Determine the rate of heat loss per metre length from a 0.1 m outer diameter steam pipe placed horizontally in ambient air at 30°C. The pipe has an outside wall temperature of 170°C and an emissivity of 0.9.

(b) By using the simplified relation of Table 5.2, what would have been the convective heat loss?

**Solution** The properties of air at the film temperature of $\frac{170 + 30}{2}$ = 100°C are:

$$\nu = 23.13 \times 10^{-6} \text{ m}^2\text{/s}, \ \text{Pr} = 0.688$$

$$k = 32.10 \times 10^{-3} \text{ W/mK}, \ \beta = \frac{1}{373} = 2.68 \times 10^{-3} \text{ K}^{-1}$$

$$\text{Ra}_D = \text{Gr}_D\,\text{Pr} = \frac{g\beta (T_w - T_\infty) D^3}{\nu^2} \cdot \text{Pr}$$

$$= \frac{9.81 \times 2.68 \times 10^{-3} (170 - 30)\,(0.1)^3 \times 0.688}{(23.13 \times 10^{-6})^2}$$

$$= 4.72 \times 10^6$$

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Using the given correlation

$$\overline{\text{Nu}}_D = \left[0.60 + \frac{0.387 \times (4.72 \times 10^6)^{\frac{1}{6}}}{\left[1 + (0.559/0.668)^{9/16}\right]^{8/27}}\right]^2$$

$$= 22.8$$

$$\bar{h} = \frac{22.8 \times 0.0321}{0.1} = 7.32 \text{ W/m}^2 \text{ K}$$

$$Q_{\text{conv.}} = hA(T_w - T_\infty)$$

$$= 7.32 \times \pi \times 0.1 \times 1(170 - 30) = 321.95 \text{ W}$$

$$Q_{\text{rad.}} = \sigma A F_{1-2} (T_w^4 - T_\infty^4)$$

$$= 5.67 \times 10^{-8} \times \pi \times 0.1 \times 1 \times 0.9 (443^4 - 303^4)$$

$$= 482.3 \text{ W}$$

$\therefore$
$$Q_{\text{total}} = Q_{\text{conv.}} + Q_{\text{rad.}}$$

$$= 321.95 + 482.3 = 804.25 \text{ W} \quad \textit{Ans.}$$

(b) From Table 5.2,

$$\bar{h} = 1.32 (\Delta T/D)^{1/4}$$

$$= 1.32 \left(\frac{170 - 30}{0.1}\right)^{1/4} = 8.07 \text{ W/m}^2 \text{ K}$$

$\therefore$
$$Q_{\text{conv.}} = 8.07 \times \pi \times 0.1 \times 1 \times (170 - 30)$$

$$= 355.12 \text{ W} \quad \textit{Ans.} \text{ (b)}$$

which is about 10% higher than in (a).

**Example 5.7** A square plate 0.4 m × 0.4 m maintained at a uniform temperature of $T_w$ = 400 K is suspended vertically in quiescent atmospheric air at 27°C. Determine (a) the boundary layer thickness at the trailing edge of the plate (i.e. at $x$ = 0.4 m), (b) the average heat transfer coefficient over the entire length by using theoretical analysis (Table 5.1) and (c) compare the latter value with that obtained from Eq. (5.44). Properties of air at 350 K are $\nu = 20.75 \times 10^{-6}$ m²/s, Pr = 0.697 and $k$ = 0.03 W/m K.

**Solution**

$$\beta = \frac{1}{T^*} = \frac{1}{350} = 2.86 \times 10^{-3} \text{ K}^{-1}$$

$$\text{Gr}_{L=0.4} = \frac{g\beta\theta L^3}{\nu^2} = \frac{9.81 \times 2.86 \times 10^{-3} (400 - 300)(0.4)^3}{(20.75 \times 10^{-6})^2}$$

$$= 4.16 \times 10^8$$

From Eq. (5.34),

$$\frac{\delta}{x} = 3.93 (0.952 + \text{Pr})^{1/4} \frac{1}{\text{Pr}^{1/2} (\text{Gr}_x)^{1/4}}$$

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$$\delta_{L\,=\,0.4\text{ m}} = 3.93\,(0.952 + 0.697)^{1/4}\,\frac{0.4}{(0.697)^{1/2}\left(4.16\times10^{8}\right)^{1/4}}$$

$$= 1.5 \times 10^{-2}\text{ m} = 1.5\text{ cm} \quad \textit{Ans.}\text{ (a)}$$

(b)
$$\text{Nu}_m = \frac{hL}{k} = 0.518\,(\text{Gr}_L\,\text{Pr})^{1/4}$$

$$h = 0.518\,(4.16 \times 10^{8} \times 0.697)^{1/4}\,\frac{0.03}{0.4}$$

$$= 5.07\text{ W/m}^2\text{ K} \quad \textit{Ans.}\text{ (b)}$$

(c)
$$\text{Nu}_m = \frac{hL}{k} = 0.68 + \frac{0.67\left(4.16\times10^{8}\times0.697\right)^{1/4}}{\left(1+(0.492/0.697)^{9/16}\right)^{4/9}}$$

$$= 72.61$$

$$h = \frac{72.61 \times 0.03}{0.4} = 5.45\text{ W/m}^2\text{ K} \quad \textit{Ans.}\text{ (c)}$$

**Example 5.8** Determine the mean heat transfer coefficient for natural convection from the surface of a cabinet. The cabinet is mounted on a vertical wall. Its surface temperature is 125°C and the ambient temperature is 25°C. What is the rate of heat loss from the surface?

**Solution** Properties of air at 75°C are $\nu = 2.06 \times 10^{-6}$ m²/s, Pr = 0.697, $k = 0.0299$ W/m K and $\beta = 2.87 \times 10^{-3}$ K$^{-1}$.

The characteristic length $\delta$ for the solid

$$\frac{1}{\delta} = \frac{1}{L_{\text{hor}}} + \frac{1}{L_{\text{vert}}} = \frac{1}{(0.16+0.318)/2} + \frac{1}{0.418}$$

$$\delta = 0.152\text{ m}$$

$$\text{Gr}_\delta = \frac{g\beta\theta\,\delta^3}{\nu^2} = \frac{9.81\times2.87\times10^{-3}\times(125-25)(0.152)^3}{\left(2.06\times10^{-6}\right)^2}$$

$$= 2.33 \times 10^{7}$$

$$\text{Ra}_\delta = 2.33 \times 10^{7} \times 0.697 = 1.62 \times 10^{7}$$

$$\overline{\text{Nu}} = 0.55\ \text{Ra}_d^{1/4} = 0.55\,(1.62 \times 10^{7})^{1/4}$$

$$= 34.9 = \frac{h\delta}{k}$$

$$\therefore \quad h = \frac{34.9 \times 0.0299}{0.152} = 6.87\text{ W/m}^2\text{ K} \quad \textit{Ans.}$$

Total heat loss $= 6.87 \times 0.268 \times 100 = 1.84$ W  *Ans.*

**Example 5.9** A wall of a cold storage having an air gap is 6 m high and 11 m wide. The air gap width is 2.5 cm. If the two wall surfaces across the air gap have temperatures of 45°C and 35°C, find the heat gain by natural convection and conduction through the air gap.

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**Solution**

$$T^* = \frac{45+35}{2} = 40°\text{C} = 313 \text{ K}$$

Properties of air at 40°C are $\beta = 3.195 \times 10^{-3}$ K$^{-1}$, $\nu = 17 \times 10^{-5}$ m$^2$/s, $k = 0.02723$ W/m K and Pr = 0.705.

$$\text{Gr}_\delta \text{Pr} = \frac{9.81 \times 3.195 \times 10^{-3} \times 10 \times (0.025)^3 \times 0.705}{(17 \times 10^{-5})^2}$$

$$= 1.1947 \times 10^4$$

Flow is laminar. Using Eq. (5.56a)

$$\text{Nu}_\delta = 0.42 \text{ Ra}_\delta^{0.25} \text{ Pr}^{0.012} \left(\frac{L}{\delta}\right)^{-0.3}$$

$$= 0.42\,(1.1947 \times 10^4)^{1/4}\,(0.705)^{0.012} \left(\frac{6}{0.025}\right)^{-0.3}$$

$$= 0.4783$$

$$k_{\text{eff}} = k\,\text{Nu}_\delta = 0.02723 \times 0.4783 = 0.013 \text{ W/m K}$$

Heat gain by natural convection is

$$Q = k_{\text{eff}} A \frac{\Delta T}{\delta} = 0.013\,(6 \times 10)\,\frac{10}{0.025}$$

$$= 312 \text{ W} \quad \textit{Ans.}$$

Heat gain by conduction

$$Q = kA\frac{\Delta T}{\delta} = 0.02723 \times 60 \times \frac{10}{0.025}$$

$$= 653.52 \text{ W} \quad \textit{Ans.}$$

**Example 5.10** *Given*: A 50 cm long fine wire of 0.02 mm dia is maintained constant at 54°C by an electric current when exposed to air at 0°C.

*To find*: The electric power necessary to maintain the wire at 54°C.

**Solution** Air film temperature $T_f = (54 + 0)/2$

$$= 27°\text{C} = 300 \text{ K}$$

Properties of air at 300 K taken from the Appendix are:

$$\nu = 15.69 \times 10^{-6} \text{ m}^2\text{/s}, \ \beta = \frac{1}{T_f} = \frac{1}{300} = 0.00333 \text{ K}^{-1}$$

$$k = 0.02624 \text{ W/mK and Pr} = 0.708.$$

Now,

$$\text{Gr} \cdot \text{Pr} = \frac{g\beta\theta D^3}{\nu^2} \cdot \text{Pr}$$

$$= \frac{9.81 \times 0.00333 \times 54 \times (0.02 \times 10^{-3})^3}{(15.69 \times 10^{-6})^2} \times 0.78$$

$$= 4.05 \times 10^5$$

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From Holman's book, when Gr · Pr = $4.05 \times 10^5$ in Eq. Nu = $C$(Gr · Pr)$^m$, $C = 0.675$ and $m = 0.058$

$$\therefore \quad \overline{\text{Nu}} = 0.675\,(4.05 \times 10^5)^{0.058} = 0.375$$

$$= \frac{\bar{h}D}{k}$$

$$\therefore \quad \bar{h} = \frac{0.375 \times 0.02624}{0.02 \times 10^{-3}} = 492.6 \text{ W/m}^2\text{K}$$

$$Q = 492.6 \times \pi\,(0.02 \times 10^{-3} \times 0.50) \times 54$$

$$= 0.836 \text{ W} \quad \textit{Ans.}$$

**Example 5.11** Air at atmospheric pressure is confined between two 0.5 m square vertical plates separated by a distance of 15 mm. The temperatures of the plates are 100°C and 40°C, respectively. Calculate (a) the heat transfer across the air space by natural convection and (b) the radiation heat transfer across the air space if both surfaces have $\varepsilon = 0.2$. Use the equation $k_{eff}/k = 0.197\,(\text{Gr Pr})^{1/4}\,(L/\delta)^{-1/9}$.

**Solution** At 70°C, the properties of air are $\rho = 1.029$ kg/m$^3$, $\beta = 2.915 \times 10^{-3}$ K$^{-1}$, $\mu = 2.043 \times 10^{-5}$ kg/ms, $k = 0.0295$ W/m K and Pr = 0.7.

$$\text{Gr Pr} = \frac{9.81 \times (1.029)^2 \times 2.915 \times 10^{-3} \times (100 - 40) \times (0.015)^3}{(2.043 \times 10^{-5})^2} \times 0.7$$

$$= 1.027 \times 10^4$$

$$\frac{k_{\text{eff}}}{k} = 0.197 \times (1.027 \times 10^4)^{1/4} \left(\frac{0.5}{0.015}\right)^{-1/9}$$

$$= 1.343$$

$$Q_c = k_{\text{eff}} A \frac{T_1 - T_2}{\delta} = \frac{1.343 \times 0.0295 \times (0.5 \times 0.5)(100 - 40)}{0.015}$$

$$= 39.62 \text{ W} \quad \textit{Ans.}\text{ (a)}$$

$$\left(\frac{Q}{A}\right)_r = \sigma \frac{1}{(1/\varepsilon_1) + (1/\varepsilon_2) - 1}\left(T_1^4 - T_2^4\right)$$

$$= (5.67 \times 10^{-8})\,\frac{1}{(1/0.2) + (1/0.2) - 1}\left(373^4 - 313^4\right)$$

$$= 61.47 \text{ W/m}^2$$

$$Q_r = 61.47 \times 0.5 \times 0.5 = 15.37 \text{ W}$$

**Example 5.12** Two horizontal plates 30 cm on a side are separated by a gap of 1 cm with air at 1 atm in the space. The temperatures are 100°C for the lower and 40°C for the upper plate. Calculate the heat transfer across the air space. Use $k_{eff}/k = C(\text{Gr Pr})^n\,(L/\delta)^m$, where $C = 0.059$, $n = 0.4$ and $m = 0$.

**Solution** Properties of air at 70°C are $\rho = 1.029$ kg/m$^3$, $\beta = 2.915 \times 10^{-3}$ K$^{-1}$, $\mu = 2.043 \times 10^{-5}$ kg/ms, $k = 0.0295$ W/m K and Pr = 0.7.

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$$\text{Gr Pr} = \frac{9.81\times(1.029)^2\times 2.915\times10^{-3}\times(100-40)\times(0.01)^3}{\left(2.043\times10^{-5}\right)^2}\times 0.7$$

$$= 3043$$

$$\frac{k_{\text{eff}}}{k} = 0.059\ (3043)^{0.4}\left(\frac{0.2}{0.01}\right)^{\circ} = 1.46$$

$$Q = k_{\text{eff}}\frac{A\,(T_1-T_2)}{\delta} = \frac{1.46\times 0.0295\times(0.3)^2\times(100-40)}{0.1}$$

$$= 23.25 \text{ W} \quad \textit{Ans.}$$

**Example 5.13** A 15 cm diameter steel shaft is heated to 350°C for heat treatment. The shaft is then allowed to cool in air (at 20°C) while rotating about its own horizontal axis at 4 rpm. Compute the rate of convection heat transfer from the shaft when it has cooled to 100°C.

**Solution**

$$w = \frac{2\pi N}{60} = \frac{2\pi\times 4}{60} = 0.419 \text{ rad/s}$$

At (100 + 20)/2, i.e., 60°C or 333 K, properties of air are $\nu = 1.94\times10^{-5}$ m²/s, $\beta = (333\text{ K})^{-1}$, Pr = 0.71 and $k = 0.0279$ W/m K.

$$\text{Re}_w = \frac{\pi D^2 w}{\nu} = \frac{\pi\,(0.15)^2\times 0.419}{1.94\times10^{-5}} = 1527$$

$$\text{Ra} = \frac{g\beta\theta\,D^3}{\nu^2}\,\text{Pr} = \frac{9.81\,(1/333)\times(100-20)\,(0.15)^3}{\left(1.94\times10^{-5}\right)^2}\times 0.71$$

$$= 2.91\times10^7$$

From Eq. (5.61),

$$\overline{\text{Nu}}_d = 0.11\ (0.5\ \text{Re}_w^2 + \text{Gr}_d\ \text{Pr})^{0.35}$$

$$= 0.11\ [0.5\times(1527)^2 + 2.91\times10^7]^{0.35}$$

$$= 45.68$$

$$h_c = \frac{45.68\times 0.0279}{0.15} = 8.496 \text{ W/m}^2\text{ K}$$

$$Q = h_c\times\pi DL\times(T_w - T_\infty)$$

$$= 8.496\times\pi\times 0.15\times 1\times 80 = 320.3 \text{ W/m} \quad \textit{Ans.}$$

**Example 5.14** Air at 1 atm and 30°C is forced through a horizontal 30 mm diameter 0.5 m long tube at an average velocity of 0.25 m/s. The tube wall is maintained at 137°C. Calculate (a) the heat transfer coefficient and (b) percentage error if the calculation is made strictly on the basis of laminar forced convection.

**Solution**

$$T^* = \frac{30+137}{2} = 83.5°\text{C} = 356.5 \text{ K}$$

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at which for air $\rho = 0.99$ kg/m$^3$, $\beta = 2.805 \times 10^{-3}$ K$^{-1}$, Pr = 0.695, $\mu_w$ (at 137°C) = $2.837 \times 10^{-5}$ kg/ms, $\mu_b$ (at 30°C) = $1.8462 \times 10^{-5}$ kg/ms, $\mu = 2.102 \times 10^{-5}$ (at 83°C) kg/ms and $k = 0.0305$ W/m K.

$$\text{Re} = \frac{\rho u D}{\mu} = \frac{0.99 \times 0.25 \times 0.03}{2.102 \times 10^{-5}} = 353$$

$$\text{Gr} = \frac{g\beta\theta_w \rho^2 d^3}{\mu^2} = \frac{9.81 \times 2.805 \times 10^{-3} \times (137 - 30)(0.03)^3}{(2.102 \times 10^{-5})^2}$$

$$= 3.78 \times 10^5$$

$$\text{Gr Pr}\frac{d}{L} = 3.78 \times 10^5 \times 0.695 \times \frac{0.03}{0.5} = 15{,}763$$

$$\text{Gz} = \text{Re Pr}\frac{d}{L} = 353 \times 0.695 \times \frac{0.03}{0.5} = 14.72$$

From Eq. (5.66),

$$\text{Nu}_d = 1.75\,[\text{Gz} + 0.012\,(\text{Gz Gr}_d^{1/3})^{4/3}]^{1/3}\left(\frac{\mu_b}{\mu_a}\right)^{0.14}$$

$$= 1.75\left(\frac{1.8462}{2.337}\right)^{0.14}$$

$$\{14.72 + 0.012 \times [14.72 \times (3.78 \times 10^5)^{1/3}]^{4/3}\}^{1/3}$$

$$= 1.693\,(5.2337) = 8.86$$

$$\bar{h} = \frac{8.86 \times 0.0305}{0.03} = 9.0 \text{ W/m}^2\text{ K}$$

We may compare this value with that which would be obtained for strictly laminar forced convection. The Sieder–Tate equation may be used

$$\text{Nu}_d = 1.86\,(\text{Re}_d\,\text{Pr})^{1/3}\left(\frac{d}{L}\right)^{1/3}\left(\frac{\mu}{\mu_w}\right)^{0.14}$$

$$= 1.86\,(\text{Gz})^{1/3}\left(\frac{\mu}{\mu_w}\right)^{0.14}$$

$$= 1.86\,(14.72)^{1/3}\left(\frac{2.102}{2.337}\right)^{0.14} = 4.48$$

$$h = \frac{4.48 \times 0.0305}{0.03} = 4.55 \text{ W/m}^2\text{ K}$$

Percentage error would have been $(9.0 - 4.55)/9.0 \times 100$ or 49.4% *Ans.*

**Example 5.15** A cylindrical body of 300 mm diameter and 1.6 m height is maintained at a constant temperature of 36.5°C. The surrounding air temperature is 13.5°C. Find out the amount of heat to be generated by the body per hour if $\rho = 1.025$ kg/m$^3$, $c_p = 0.96$ kJ/kgK, $\nu = 15.06 \times 10^{-6}$ m$^2$/s, $k = 0.0892$ kJ/mhK and $\beta = 1/298$ K$^{-1}$. Assume $Nu = 0.12\,(\text{Gr. Pr})^{1/3}$, the symbols having their usual meanings.

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*Given*: $D$ = 300 mm = 0.03 m, $L$ = 1.6 m, $T_w$ = 36.5°C, $T_\infty$ = 13.5°C, $\rho$ = 1.025 kg/m³, $c_p$ = 0.96 kJ/kgK, $k$ = 0.0892 kJ/mhK, $\beta = \frac{1}{298}$ K$^{-1}$, Nu = 0.12 (Gr. Pr)$^{1/3}$.

*To find*: The amount of heat to be generated per hour.

**Solution** Grashof number, $\text{Gr} = \dfrac{g\beta\theta L^3}{\nu^2}$

or,
$$\text{Gr} = \frac{9.81 \times \frac{1}{298} \times (36.5 - 13.5)(1.6)^3}{(15.06 \times 10^{-6})^2} = 1.3674 \times 10^{10}$$

Prandtl number,
$$\text{Pr} = \frac{\mu c_p}{k} = \frac{1.025 \times 15.06 \times 10^{-6} \times 3600 \times 0.96}{0.0892}$$

Nusselt number,
$$\text{Nu} = 0.12\,(\text{Gr} \cdot \text{Pr})^{1/3}$$
$$= 0.12\,(1.3674 \times 10^{10} \times 0.598)^{1/3}$$
$$= 241.75 = \frac{hL}{k}$$

∴
$$h = 241.75 \times 0.0892/1.6$$
$$= 13.478 \text{ kJ/hm}^2\text{K}$$

Heat loss from the surface by natural convection, $Q = hA(T_w - T_\infty)$
$$= 13.478 \times \pi \times 0.3 \times 1.6 \times (36.5 - 13.5)$$
$$= 467.5 \text{ kJ/h} \quad \textit{Ans.}$$

This is the amount of heat to be generated per hour to maintain the cylinder surface at 36.5°C while the surrounding air is at 13.5°C.

## Summary

Convective flows that originate in part or exclusively from buoyancy forces and the associated heat transfer rates are considered here introducing the dimensionless parameters needed to characterise such flows. An approximate analysis of laminar natural convection heat transfer from an isothermal vertical plate has been made. Empirical correlations for heat transfer from various shapes and geometries are considered. Natural convection heat transfer in enclosed spaces and from rotating cylinders, disks and spheres are discussed. The mixed convection processes with relative magnitudes and dominance of free and forced convection are finally brought into consideration for a number of physical situations.

## Important Formulae and Equations

| *Equation Number* | *Equation* | *Remark* |
|---|---|---|
| (5.12) | $\text{Gr} = \dfrac{g\beta(T_w - T_\infty)L^3}{\nu^2}$ | Grashof number |

*(Contd)*

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| *Equation Number* | *Equation* | *Remark* |
|---|---|---|
| (5.15) | $\text{Ra} = \dfrac{g\beta L^3 (T_w - T_\infty)}{\nu\alpha} = \text{Gr.Pr}$ | Rayleigh number |
| (5.34) | $\dfrac{\delta}{x} = 3.93(0.952 + \text{Pr})^{1/4} \dfrac{1}{\text{Pr}^{1/2}(\text{Gr}_x)^{1/4}}$ | Boundary layer thickness in a natural convection film on a vertical plate |
| (5.35) | $\text{Nu}_x = 0.508\,\text{Pr}^{1/2}(0.952 + \text{Pr})^{-1/4}(\text{Gr}_x)^{1/4}$ | Local Nusselt number for natural convection on a vertical plate |
| (5.36) | $\text{Nu}_x = 0.378(\text{Gr}_x)^{1/4}$ | Nusselt number for air (Pr = 0.714) |
| (5.37) | $\text{Nu}_L = 0.504(\text{Gr}_L)^{1/4}$ | Average Nusselt number for natural convection on a vertical plate of length $L$ |
| (5.38) | $h_x = 0.508\,\text{Pr}^{1/2} \dfrac{\text{Gr}_x^{1/4}}{(0.952 + \text{Pr})^{1/4}} \dfrac{k}{x}$ | Local heat transfer coefficient at a distance $x$ from the leading edge |
| (5.40) | $\overline{\text{Nu}}_L = 0.68\,\text{Pr}^{1/2} \dfrac{\text{Gr}_L^{1/4}}{(0.952 + \text{Pr})^{1/4}}$ | Average value of Nusselt number for a height $L$ |
| (5.41) | $\overline{\text{Nu}}_L = 0.13(\text{Gr}_L\,\text{Pr})^{1/3}$ | For natural convection over a vertical plate or cylinder in the turbulent region ($\text{Gr}_L > 10^9$). |
| (5.45) | $\overline{\text{Nu}}_L = 0.825 + \dfrac{0.387\text{Ra}_L^{1/6}}{[1 + (0.492/\text{Pr})^{9/16}]^{8/27}}$ for $10^{-1} < \text{Ra} < 10^{12}$ | Natural convection on a vertical plate for both laminar and turbulent flows |
| (5.48) | $\overline{\text{Nu}}_L = C(\text{Gr}_L\,\text{Pr})^n$<br>where $C = 0.54$ and $n = 1/4$<br>for $10^5 < \text{Ra} < 10^7$<br>and $C = 0.14$, $n = 1/3$<br>for $2 \times 10^7 < \text{Ra} < 3 \times 10^{10}$ | Nusselt number for natural convection on a horizontal plate for hot surface facing up or cold surface facing down. For hot surface facing down or cold surface facing up, $C$ = 0.27, $n = 1/4$, for $3 \times 10^5 < \text{Ra} < 3 \times 10^{10}$ |
| (5.54) | $\overline{\text{Nu}}_d = 0.53(\text{Gr}_d.\text{Pr})^{1/4}$<br>for $\text{Pr} > 0.5$, $10^3 < \text{Gr} < 10^9$ | Horizontal pipes or wires in natural convection |

## Review Questions

5.1 Why are heat transfer coefficients for natural convection much less than those in forced convection?

5.2 How is the velocity field developed in front of a vertical plate which is maintained at a temperature (a) higher and (b) lower, than the surrounding fluid?

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5.3 Explain the velocity and temperature profiles for air filling the gap between two horizontal plates when (a) the bottom plate is maintained at a temperature higher than the top plate and (b) the bottom plate is maintained at a temperature lower than the top plate.

5.4 How are the principal dimensionless parameters of natural convection determined from the boundary layer equations concerning continuity, momentum and energy?

5.5 What do you mean by Boussinesq approximation?

5.6 What is the physical significance of Grashof number with reference to heat transfer by natural convection? What is Rayleigh number?

5.7 What do you mean by critical value of Rayleigh number?

5.8 Find the location and magnitude of maximum velocity in the boundary layer formed on a heated or cooled vertical plate.

5.9 Show that for laminar flow of air (Pr = 0.714), the local and average values of Nusselt number for natural convection heat transfer from or to a vertical plate are given by

$$\mathrm{Nu}_x = 0.378\ \mathrm{Gr}_x^{1/4} \text{ or } \overline{\mathrm{Nu}}_L = 0.504\ \mathrm{Gr}_L^{1/4}$$

5.10 What is the recommended correlation for natural convection over a vertical plate or cylinder in the turbulent flow region?

5.11 What is modified Grashof number? Where does it appear?

5.12 Explain how do the average values of Nusselt number for natural convection depend on (a) whether the hot surface is facing up or down, (b) whether the plate surface is warmer or cooler than the surrounding fluid and (c) whether the plate is subjected to uniform wall heat flux or uniform wall temperature.

5.13 Explain how does heat transfer occur in a rectangular vertical cavity consisting of two isothermal parallel plates spaced a distance $\delta$ apart. When does heat transfer essentially occur by conduction?

5.14 What is the mechanism of heat transfer by natural convection across a gap between two horizontal concentric cylinders?

5.15 Explain the significance of heat transfer by convection between a rotating body and a surrounding fluid. How is the peripheral speed Reynolds number defined? What is its critical value?

5.16 What is rotational Reynolds number for a disk? What is its critical value?

5.17 Explain the significance of combined forced and natural convection. What is the role of the parameter Gr/Re$^2$ in this regard?

5.18 When does the effect of buoyancy force become important in forced convection heat transfer? How is Graetz number relevant in combined convection?

## Objective Type Questions

5.1 The ratio of buoyancy force to the viscous force acting on a fluid is called
(a) Reynolds number
(b) Grashof number
(c) Prandtl number
(d) Nusselt number

5.2 Grashof number is defined as
(a) $g\beta\theta L/\nu$ (b) $g\beta\theta L^2/\nu^2$
(c) $g\beta\theta L^3/\nu^2$ (d) $g\beta\theta L^3/\nu^3$

5.3 The characteristic length for computing Grashof number in the case of a horizontal cylinder is
(a) the length of the cylinder
(b) the diameter of the cylinder
(c) the perimeter of the cylinder
(d) the radius of the cylinder

5.4 In natural convection heat transfer, the Nusselt number is a function of
(a) Re and Pr (b) Re and Gr
(c) Gr and Pr (d) Gr and Bi

5.5 The maximum velocity in the laminar boundary layer in natural convection heat transfer is equal to

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(a) 0 (b) $\delta/3$
(c) $\delta/2$ (d) $\delta$

5.6 In natural convection heat transfer under uniform heat flux, the modified Grashof number, $Gr_x^*$ is defined as
(a) $Pr \times Gr$ (b) $Nu \times Gr$
(c) $Nu \times Gr/Pr$ (d) $Gr \times Pr/Nu$

5.7 For combined forced and natural convection, the relative magnitude of the following dimensionless parameter governs the relative importance of natural convection in relation to forced convection.
(a) Gr/Re (b) $Gr/Re^2$
(c) Gr.Pr/Re (d) $Re/Gr^2$

5.8 Forced convection dominates if
(a) $Gr/Re^2 << 1$ (b) $Gr/Re^2 >> 1$
(c) $Gr/Re^2 = 1$ (d) $Gr.Pr/Re^2 >> 1$

5.9 Natural convection dominates if
(a) $Gr/Re^2 << 1$ (b) $Gr/Re^2 >> 1$
(c) $Gr/Re^2 = 1$ (d) $Pr.Gr/Re^2 << 1$

5.10 Graetz number is defined as
(a) $Re_d \times Pr$ (b) $Re_d \times Pr(L/D)$
(c) $Re_d \times Pr\,(D/L)$ (d) $Gr \times Pr$

5.11 In natural convection heat transfer from a 3 cm horizontal diameter tube is given by the relation $Nu \propto (Gr)^{0.25}$, and the convective heat transfer coefficient is 100 W/m$^2$K. If the diameter of the tube is 12 cm, the value of '$h$' would be, with other parameters remaining the same,
(a) 100 W/m$^2$K (b) 90 W/m$^2$K
(c) 80 W/m$^2$K (d) 70.71 W/m$^2$K

5.12 **Assertion (A):** In natural convection turbulent flow over heated vertical plate, '$h$' is independent of the characteristic length.
**Reasoning (R):** In turbulent flow natural convection heat transfer over a heated vertical plate $Nu = C(Pr)^{1/3}$.
***Code:***
(a) Both A and R are false
(b) Both A and R are true
(c) A is true, R is also
(d) A is false, R is true

5.13 A 3.2 m high vertical pipe at 175ºC wall temperature is in a room with still air at 25ºC. The pipe supplies heat to room air at the rate of 8 kW by natural convection. Assuming laminar flow and other conditions remaining the same, the height of the pipe required to supply 1 kW will be
(a) 0.8 m (b) 0.4 m
(c) 0.2 m (d) 0.1 m

## Answers

| | | | | |
|---|---|---|---|---|
| 5.1 (b) | 5.2 (c) | 5.3 (b) | 5.4 (c) | 5.5 (b) |
| 5.6 (b) | 5.7 (b) | 5.8 (a) | 5.9 (b) | 5.10 (c) |
| 5.11 (d) | 5.12 (b) | 5.13 (c) | | |

## Open Book Problems

5.1 A metal plate, 0.609 m, high forms the vertical wall of an oven and is at a temperature of 161ºC. Within the oven is air at a temperature of 93ºC and one atmosphere. Assuming that natural conditions hold near the plate, estimate the mean heat transfer coefficient and the rate of heat transfer per unit width of the plate.

*Hints:* Find properties of air at $T_f = \dfrac{161+93}{2}$ = 127ºC = 400 K from the Appendix: $\nu$, $\alpha$, $k$ and Pr . $\beta = \dfrac{1}{T_f}$.

Rayleigh number,

$$Ra_L = Gr_L . Pr = \frac{g\beta L^3 (T_w - T_\infty)}{\nu^2} \cdot Pr$$

Then use Eq. (5.4),

$$\overline{Nu} = 0.68 + \frac{0.67 Ra_L^{1/4}}{[1+(0.492/Pr)^{9/16}]^{4/9}}$$ to find $\bar{h}$

and $Q = \bar{h}\, L(T_w - T_\infty)$.

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5.2 A thin 80 cm long and 8 cm wide horizontal plate is maintained at a temperature of 130ºC in a large tank full of water at 70ºC. Estimate the rate of heat transfer to the plate required to maintain the temperature of 130ºC.

*Hints:* The plate would be losing heat by free convection from both its upper and lower surfaces. The properties of water at $T_f = \frac{130+70}{2}$ = 100ºC are found from the Appendix: $\rho$, $c_p$, $\beta$, $k$, $\nu$ and $\alpha$.

Characteristic length, $L = \frac{A}{P} = \frac{w}{2}$ = 4 cm.

Find $\text{Ra}_L = \frac{g\beta\theta L^3}{\nu^2} \cdot \frac{\nu}{\alpha}$. For the top surface, use $\overline{\text{Nu}}_L = 0.14(\text{Ra}_L)^{1/3}$ to find $\bar{h}_L$, and for the bottom surface, use $\overline{\text{Nu}}_L = 0.27(\text{Ra}_L)^{1/4}$ to find $\bar{h}_B$. Then $Q = (\bar{h}_B + \bar{h}_L)\, w\, L(T_w - T_\infty)$.

5.3 Air flow through a long rectangular (30 cm width × 6 cm height) air conditioning duct maintains the outer duct surface temperature at 15°C. If the duct is unisulated and exposed to air at 25°C, calculate the heat gained by the duct per metre length, assuming it to be horizontal.

*Hints:* At film temperature $T_f = \frac{15+25}{2} = 20^\circ\text{C}$, the properties of air, viz., $\rho$, $c_p$, $\nu$, $k$, Pr and $\beta$ are obtained for air from the Appendix. Since the duct is laid horizontally, the heat gain by free convection is from the four vertical sides and the horizontal top and bottom. Now, find

Gr, Pr = $\text{Ra}_L = \frac{g\beta L^3(T - T_w)}{\nu^2} \cdot \text{Pr} = xL^3$,

(i) Vertical surfaces: use $\text{Nu}_L = 0.59(\text{Ra}_L)^{1/4}$ to find $\bar{h}_v$. (ii) Upper and lower surfaces: $L = A/p \simeq 2\,\omega = 0.30$ m. Find $\text{Ra}_L$.

Use $\bar{h}_{HL}\text{(top surface)} = \frac{k}{L} \times 0.54(\text{Ra}_L)^{1/4}$ and

$\bar{h}_{HB}\text{ (bottom) surface} = \frac{k}{L} \times 0.27(\text{Ra}_L)^{1/4}$

∴ Rate of heat gained per unit length, $Q = 2Q_v + Q_{H(B)} + (Q_H)_L$.

## Problems for Practice

5.1 A vertical plate 5 m high and 1.5 m wide has one of its surfaces insulated. The other surface maintained at a uniform temperature of 400 K is exposed to quiescent atmospheric air at 300 K. Calculate the total rate of heat loss from the plate [show that $\text{Gr}_L = 8.137 \times 10^{11}$ and use Eq. (5.45)]. Properties of air at 350 K are $\nu = 20.75 \times 10^{-6}$ m$^2$/s, Pr = 0.697 and $k$ = 0.03 W/m K.

(*Ans.* 4.133 W)

5.2 A square plate 0.5 m × 0.5 m with one surface insulated and the other surface maintained at a uniform temperature of 385 K which is placed in quiescent air at atmospheric pressure and 315 K. Calculate the average heat transfer coefficient for natural convection for the following orientations of the hot surface: (a) the plate is horizontal and the hot surface faces up, (b) the plate is vertical and (c) the plate is horizontal, and the hot surface faces down.

(*Ans.* (a) 6.18, (b) 5.0, (c) 2.29 W/m$^2$K)

5.3 A power amplifier is mounted vertically in air at 25°C. The case is made of anodized aluminium with a surface area of about 3800 mm$^2$ and a height of 40 mm. Determine (a) the heat transfer coefficient for natural convection cooling with a case temperature of 125°C and (b) the rate of total heat dissipation. (c) What is the percentage of total heat that is lost by natural convection? Properties of air at 75°C are $\nu = 2.06 \times 10^{-5}$ m$^2$/s, Pr = 0.697 and $k$ = 0.0299 W/m K.

(*Ans.* (a) 6.88 W/m$^2$ K, (b) 6.32 W, (c) 41%)

5.4 A vertical plate 10 cm high and 5 cm wide is cooled by natural convection. The rate of heat transfer is 5.55 W and the air

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temperature is 38°C. Estimate the maximum temperature of the plate. Assume uniform heat flux.

(*Ans.* $T_{max}$ = 175°C)

5.5 Estimate the heat transfer coefficient for the power amplifier of example 5.3 if it is mounted horizontally.

(*Ans.* 4.24 W/m$^2$ K)

5.6 A horizontal cylinder of 2.5 cm diameter and 0.6 m length is suspended in water at 20°C. Calculate the rate of heat transfer if the cylinder surface is at 55°C. What would be the rate of heat transfer if the cylinder is suspended in light oil at 55°C. The following properties (at 37.5°C) may be used

| | Water | Oil |
|---|---|---|
| $\rho$ (kg/m$^3$) | 992 | 905 |
| $\mu$ (kg/hm) | 2.47 | 82.0 |
| $k$ (W/m K) | 0.622 | 0.133 |
| $\beta$ (K$^{-1}$) | $3.96 \times 10^{-4}$ | $7.2 \times 10^{-4}$ |
| Pr | 4.64 | 324 |

(*Ans.* 1470 W, 171 W)

5.7 A horizontal pipe 0.3048 m in diameter is maintained at a temperature of 250°C in a room where the ambient air is at 15°C. Calculate the heat loss by natural convection per metre length.

(*Ans.* 1.49 kW/m)

5.8 A cube 20 cm on a side is maintained at 60°C and exposed to atmospheric air at 20°C. Calculate the heat transfer.

(*Ans.* 108.8 W)

5.9 Two 50 cm horizontal square plates are separated by a distance of 1 cm. The lower plate is maintained at a constant temperature 38°C and the upper plate is constant at 27°C. Water at atmospheric pressure occupies the space between the plates. Calculate heat lost by the lower plate. Use $k_{eff}/k = C(\text{Gr Pr})^n \, (L/\delta)^m$, where $C$ = 0.13, $n$ = 0.3 and $m$ = 0.

(*Ans.* 964 W)

5.10 Calculate the rate of convection heat loss from the top and bottom of a flat 1 m$^2$ horizontal restaurant grill heated to 227°C in ambient air at 27°C.

(*Ans.* $Q_{total}$ = 2268 W)

5.11 Estimate the electrical power required to maintain a vertical heater surface at 130°C in ambient air at 20°C. The plate is 15 cm high and 10 cm wide. Compare with results for a plate 450 cm high. The heat transfer coefficient for radiation is 8.5 W/m$^2$ K in both cases.

(*Ans.* 50.8 W, 1465 W)

5.12 At what temperature will a long, heated, horizontal steel pipe 1 m in diameter produce turbulent flow in (a) air and (b) a water bath, both at 27°C?

(*Ans.* $\Delta T_{air}$ = 12°C, $\Delta T_{water}$ = 0.05°C)

5.13 A 20 cm diameter steel shaft is heated to 400°C for heat treating. The shaft is then allowed to cool in air at 20°C while rotating about its own horizontal axis at 3 rpm. Compute the rate of convection heat transfer from the shaft when it has cooled to 100°C.

5.14 Air at 1 atm and 27°C is forced through a horizontal 25 mm diameter tube at an average velocity of 30 cm/s. The tube wall is maintained at a constant temperature of 140°C. Calculate the heat transfer coefficient for this situation if the tube is 0.4 m long. What would have been the percentage error if strictly laminar forced convection was considered?

(*Ans.* 9.4 W/m$^2$ K, 41% error)

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# Condensation and Boiling

# 6

Heat transfer by change of phase includes (a) condensation, (b) boiling, (c) melting or solidification and (d) sublimation, in which the respective latent heat is released. Since the phase change occurs at constant temperature, the heat transfer coefficient is high, and is one order of magnitude higher than that for a single-phase fluid. In this chapter, we will consider only condensation and boiling heat transfer.

Boiling and condensation involve fluid motion and are hence regarded as the convection mode of heat transfer. Since there is a phase change, heat transfer rates are achieved with small temperature differences. In addition to the latent heats $h_{fg}$, two other parameters are important, namely the surface tension $\sigma$ between liquid and vapour, and the density difference between the two phases, which induces buoyancy force proportional to $g(\rho_1 - \rho_v)$.

## 6.1 DIMENSIONLESS PARAMETERS IN BOILING AND CONDENSATION

Boiling and condensation processes are quite complex due to the many variables involved, and it is difficult to derive the governing equations. Dimensional analyses of the processes help to identify the relevant dimensionless groups and enhance understanding of the related physical mechanisms. Buckingham pi theorem will be used to obtain the appropriate dimensionless parameters. For condensation or boiling, the convection coefficient depends on the difference between surface and saturation temperatures $\Delta T = (T_w - T_{sat})$, the body force arising from the liquid–vapour density difference, $g(\rho_l - \rho_v)$, the latent heat $h_{fg}$, the surface tension $\sigma$, a characteristic length $L$ and the thermophysical properties of the liquid or vapour $\rho$, $c_p$, $\mu$ and $k$. Therefore,

$$h = h[\Delta T, g\,(\rho_l - \rho_v), h_{fg}, \sigma, L, \rho, c_p, k, \mu] \tag{6.1}$$

Since there are 10 variables in 5 dimensions (m, kg $s$, J, K) there will be (10 – 5) = 5 pi-groups, which can be expressed in the following forms:

$$\frac{hL}{k} = f\left[\frac{\rho g(\rho_l - \rho_v)L^3}{\mu^2}, \frac{c_p \Delta T}{h_{fg}}, \frac{\mu c_p}{k}, \frac{g(\rho_l - \rho_v)L^2}{\sigma}\right] \tag{6.2}$$

In dimensionless groups,

$$\mathrm{Nu}_L = \mathbf{f}\left[\frac{\rho \mathbf{g}\,(\rho_l - \rho_v)\mathbf{L}^3}{\mu^2}, \mathbf{Ja, Pr, Bo}\right] \tag{6.3}$$

The Nusselt and Prandtl numbers are familiar from earlier single-phase convection analyses. The new dimensionless parameters are the Jakob number Ja, the Bond number Bo, and a nameless parameter akin to Grashof number, which represents the effect of buoyancy-induced fluid motion on heat transfer. The Jakob number Ja = $(c_p\,\Delta T)/(h_{fg})$ is the ratio of the maximum sensible energy absorbed to the latent heat. In many cases, Ja has a small numerical value. The Bond number Bo = $g(\rho_l - \rho_v)L^2/\sigma$ is the ratio of the gravitational body force to the surface tension force.

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## 6.2 CONDENSATION HEAT TRANSFER

When a saturated vapour comes in contact with a surface the temperature of which is maintained below the saturation temperature at the vapour pressure, the vapour cannot but condense into liquid releasing the latent heat of condensation at that pressure with a coolant (cooling water) carrying away this heat (Fig. 6.1). There are two modes in which condensation can take place on a cooling surface.

1. Dropwise condensation
2. Filmwise condensation

In film condensation, a stable coherent film of liquid condensate is formed on the surface through which the heat released during condensation is conducted into the surface (Fig. 6.1). On a wettable cooling surface, film condensation takes place.

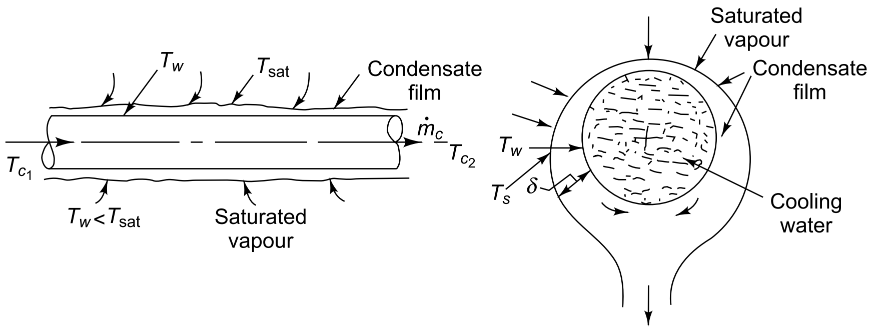


**Fig. 6.1** *Condensing of saturated vapour*

## 6.3 DROPWISE CONDENSATION

In dropwise condensation, vapour condenses on the surface in the form of drops, and consequently a large part of cooling surface is always bare to vapour for undergoing condensation (Fig. 6.2). The rate of heat transfer is many times larger than what is achieved in film condensation. Dropwise condensation occurs on a nonwettable cooling surface where the liquid condensate drops do not spread.

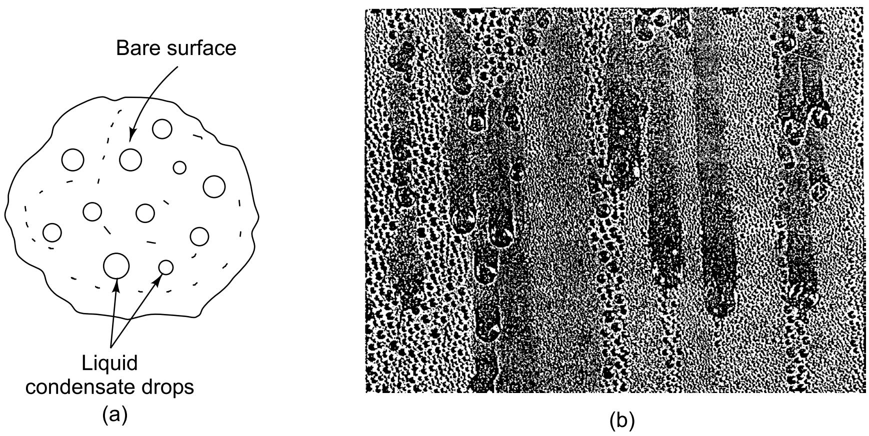


**Fig. 6.2** *(a) Drop condensation (b) Steam condensing dropwise on a vertical surface of venyl plastic (Rohsenow and Choi)*

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Let us explain briefly what is a wettable or a nonwettable surface. The surface of a liquid always tends towards a minimum. A freely suspended drop of liquid always takes the shape of a sphere which is of the geometrical shape having the minimum surface area for the same volume. This is due to the effect of surface tension. Surface tension always exists whenever there is a discontinuity in the material medium. Mercury in contact with air has a certain surface tension. With water, mercury has another surface tension. Let us consider the equilibrium of a liquid drop on a solid surface (Fig. 6.3), $\sigma$ being the surface tension as shown.

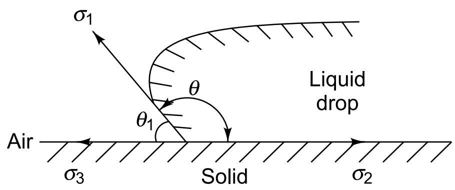


**Fig. 6.3** *Equilibrium of a liquid drop on a solid surface*

If $\sigma_1 \cos \theta_1 + \sigma_3 = \sigma_2$, the liquid drop remains in equilibrium and does not spread. The surface is nonwettable (e.g. mercury in glass).

$$\cos \theta_1 = \frac{\sigma_2 - \sigma_3}{\sigma_1} = -\cos \theta$$

or
$$\cos \theta = \frac{\sigma_3 - \sigma_2}{\sigma_1} \quad \text{where } \theta \text{ is the angle of contact.}$$

If $(\sigma_1 \cos \theta_1 + \sigma_3) > \sigma_2$, the liquid drop spreads and the surface is wettable (e.g. water in glass). When $\theta$ is obtuse, the surface is nonwettable, and if $\theta$ is acute, the surface is wettable.

Dropwise condensation is much desirable because of its higher heat transfer rates. However, it hardly occurs on a cooling surface. When the surface is coated with some promoter like teflon, grease, mercaptan, oleic acid and so on, drop condensation can occur for some time. But the effectiveness of the promoter gradually decays due to fouling, oxidation or its slow removal by the flow of the condensate. Condensers are usually designed on the basis that film condensation would prevail.

## 6.4 LAMINAR FILM CONDENSATION ON A VERTICAL PLATE

Numerous experimental and theoretical investigations have been conducted to determine the heat transfer coefficient for film condensation on surfaces. The first fundamental analysis in this aspect was given by Nusselt in 1916 [1]. Over the years, improvements have been made on Nusselt's theory. But with the exception of liquid metals, this theory is quite successful and is still widely used. Nusselt's theory of film condensation of pure vapours on a vertical plate is presented below. It serves as a basis to better understand heat transfer during condensation.

### 6.4.1 Nusselt's Theory

Let us consider condensation of a vapour on a vertical plate as shown in Fig. 6.4. Here $x$ is the axial coordinate, measured downward along the plate, and $y$ is the coordinate normal to the condensing surface. The condensate thickness is represented by $\delta(x)$. Nusselt made the following assumptions:

1. The vapour is pure, dry and saturated.
2. The condensate flow is under the action of gravity and is laminar.
3. The vapour at the liquid–vapour interface is stagnant so that there is no shear stress or drag on the flow of condensate.
4. The plate is maintained at a uniform temperature $T_w$ that is less than the saturation temperature of the vapour $T_s$.
5. The liquid temperature at the interface is that of saturated vapour.
6. Fluid properties are constant.

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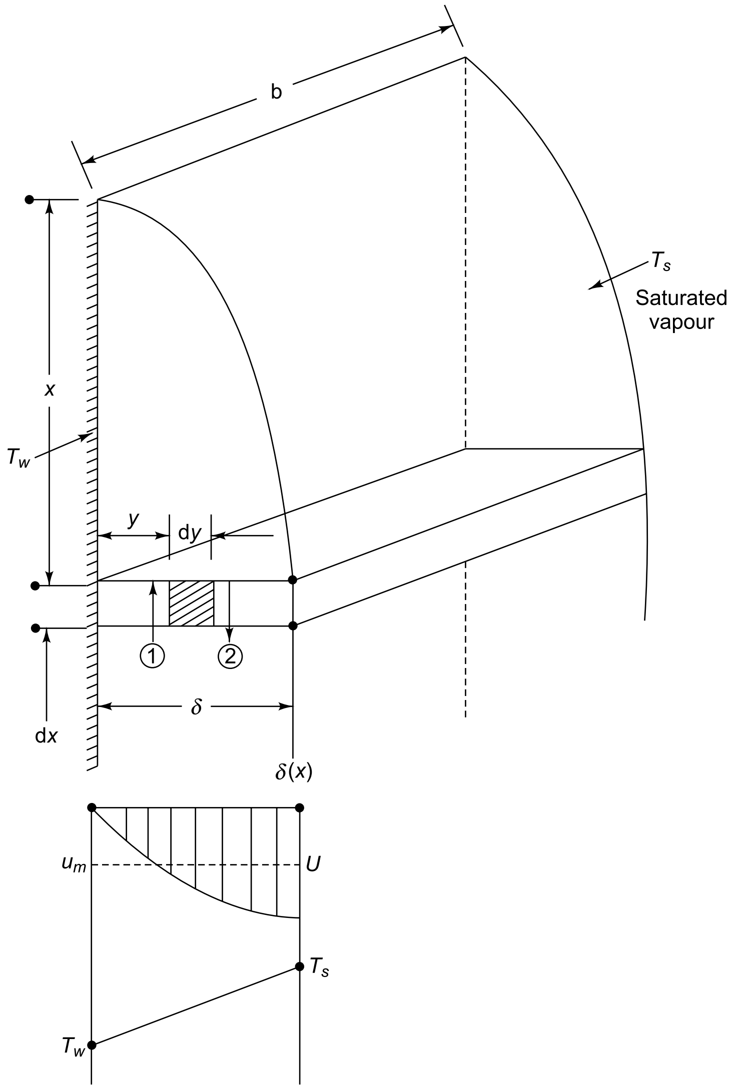
7. Heat transfer across the condensate layer is by pure conduction, and the liquid temperature profile is linear.
8. Heat transfer is at steady state.

Let us take a lump of condensate of thickness d$x$ at a distance $x$ from the top of the plate and consider d$y$ portion of this lump. The condensate drains out under the action of mechanical forces. Making a force balance on the element.

$$\Sigma F = F_g + F_\tau + F_p = 0$$

where $F_g$ is the gravity force, $F_\tau$ is the shear force and $F_p$ is the pressure force exerted on the element. Here,

$$F_g = \rho\,(b\,\mathrm{d}x\,\mathrm{d}y)g$$




(a)

**Fig. 6.4** *Film condensation on a vertical surface*

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$$F_\tau = \left[-F_{\tau_1} + F_{\tau_1} + \frac{\partial}{\partial y}(F_{\tau_1})\,\mathrm{d}y\right]$$

$$= \frac{\partial}{\partial y}\left[\left(\mu_1 \frac{\partial u}{\partial y}\right) b\,\mathrm{d}x\right]\mathrm{d}y$$

$$F_p = pb\,\mathrm{d}y - pb\,\mathrm{d}y - \frac{\partial p}{\partial x}\,\mathrm{d}xb\,\mathrm{d}y$$

$$= -\frac{\partial p}{\partial x}\cdot \mathrm{d}xb\,\mathrm{d}y$$

On substitution,

$$\rho b\,\mathrm{d}x\,\mathrm{d}yg + \mu_1 \frac{\partial^2 u}{\partial y^2}\, b\,\mathrm{d}x\,\mathrm{d}y - \frac{\partial p}{\partial x}\, b\,\mathrm{d}x\,\mathrm{d}y = 0$$

or

$$\mu_1 = \frac{\partial^2 u}{\partial y^2} - \rho g + \frac{\partial p}{\partial x} \tag{6.4}$$

or

$$\mu_1 \frac{\partial u}{\partial y} = \left(\frac{\partial p}{\partial x} - \rho g\right) y + B_1 \tag{6.5}$$

When $y = \delta$, $u = U$, $\mathrm{d}u/\mathrm{d}y = 0$.

$\therefore$

$$0 = \left(\frac{\partial p}{\partial x} - \rho g\right) + B_1$$

$$B_1 = -\left(\frac{\partial p}{\partial x} - \rho g\right)$$

$$\mu_1 \frac{\mathrm{d}u}{\mathrm{d}y} = \left(\frac{\partial p}{\partial x} - \rho g\right)(y - \delta)$$

For $y > \delta$, $p = p_v$, $\rho = \rho_v$, $\mathrm{d}u/\mathrm{d}y = 0$, $\dfrac{\partial p_v}{\partial x} = \rho_v g$.

$$\mu_1 \frac{\mathrm{d}u}{\mathrm{d}y} = (\rho_v - \rho) g\,(y - \delta)$$

$$\mu_1 u = (\rho_v - \rho) g \left(\frac{y^2}{2} - \delta y\right) + B_2$$

When $y = 0$, $u = 0$, and therefore $B_2 = 0$.

$$\mu_1 u = (\rho_v - \rho) g \left(\frac{y^2}{2} - \delta y\right) \tag{6.6}$$

At $y = \delta$, $u = U$ and since $\rho = \rho_1$

$$\mu_1 U = (\rho_l - \rho_v)\, g \left(\delta^2 - \frac{\delta^2}{2}\right)$$

$$= (\rho_l - \rho_v)\, g\, \frac{\delta^2}{2} \tag{6.7}$$

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Dividing Eq. (6.6) by Eq. (6.7)

$$\frac{u}{U} = \frac{\delta y - (y^2/2)}{\delta^2/2} = \frac{2y}{\delta} - \frac{y^2}{\delta^2} \tag{6.8}$$

This is the velocity profile of the condensate across its thickness, which is parabolic.

Let $\Gamma$ = flow rate of condensate per unit width of plate,

$$= \rho_1 u_m \delta,$$

where $u_m$ is the mean velocity of the condensate.

From Eq. (6.6),

$$u_m = \int_0^\delta u\,dy = \frac{1}{\delta}\int_0^\delta \frac{(\rho_1 - \rho_v)}{\mu_1} g \left(\delta y - \frac{y^2}{2}\right) dy$$

$$= \frac{1}{\delta}\frac{(\rho_1 - \rho_v)}{\mu_1} g \left(\delta \frac{\delta^2}{2} - \frac{1}{2}\frac{\delta^3}{3}\right)$$

$$= \frac{1}{3}(\rho_l - \rho_v) g \frac{\delta^2}{\mu_1}$$

$$\Gamma = \rho_l \cdot \frac{1}{3}(\rho_l - \rho_v) g \frac{\delta^2}{\mu_1} . \delta$$

$$= \frac{1}{3}\rho_1(\rho_1 - \rho_v)\frac{g\delta^3}{\mu_1} \tag{6.9}$$

$$d\Gamma = \rho_l(\rho_l - \rho_v)\frac{g\delta^2}{\mu_1}\,d\delta$$

Now, $$dQ = -kb\,dx\,\frac{T_w - T_s}{\delta} = d\Gamma b h_{fg}$$

where $h_{fg}$ is the latent heat of condensation.

Let $\theta = T_s - T_w$,

$$\frac{k\theta\,dx}{\delta} = h_{fg}\,\rho_l(\rho_l - \rho_v) g \frac{\delta^2}{\mu_1}\,d\delta$$

$$\int_0^\delta \delta^3 d\delta = \int_0^x - \frac{k\theta\mu_1}{g\rho_l(\rho_1 - \rho_v)h_{fg}}\,dx$$

$$\frac{\delta^4}{4} = \frac{k\theta\mu_1 x}{g\rho_l(\rho_l - \rho_v)\,h_{fg}}$$

$$\delta = \left[\frac{\mathbf{4k\mu_1\theta x}}{\mathbf{g\rho_1(\rho_1 - \rho_v)h_{fg}}}\right]^{1/4} = \delta(x) \tag{6.10}$$

This is the local film thickness of condensate layer. As $x$ increases, $\delta$ increases.

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Now,

$$h_x = \frac{k}{\delta} = \left[\frac{k^4 g\rho_l(\rho_l - \rho_v)h_{fg}}{4k\mu_1\theta x}\right]^{1/4}$$

$$= \left[\frac{\mathbf{k^3}\rho_\mathbf{l}(\rho_\mathbf{l} - \rho_\mathbf{v})\mathbf{h_{fg}}}{\mathbf{4k}\mu_\mathbf{1}\theta\mathbf{x}}\right]^{\mathbf{1/4}} \tag{6.11}$$

This is the local heat transfer coefficient. As $x$ increases, $h_x$ decreases. It can also be observed that as $\theta$ decreases, $h_x$ increases. The local Nusselt number at $x$ is

$$\mathrm{Nu}_x = \frac{h_x x}{k} = \left[\frac{\rho_l(\rho_l - \rho_v)gh_{fg}x^3}{4\mu_1 k\theta}\right]^{1/4} \tag{6.12}$$

The average heat transfer coefficient is given by

$$h_m = \int_0^L h_x \mathrm{d}x = \frac{1}{L}\int_0^L \left[\frac{k_1^3\rho_l(\rho_l - \rho_v)\, gh_{fg}}{4\mu_1\theta x}\right]^{1/4} \mathrm{d}x$$

$$= \left[\frac{k_1^3\rho_l(\rho_l - \rho_v)gh_{fg}}{\mu_1\theta}\right]^{1/4} \frac{1}{L}\frac{1}{\sqrt{2}}\int_0^L x^{-1/4}\, \mathrm{d}x$$

$$= \left[\frac{k_1^3\rho_l(\rho_l - \rho_v)gh_{fg}}{\mu_1\theta}\right]^{1/4} \frac{1}{\sqrt{2}\, L}\frac{4}{3}L^{3/4}$$

$$= 0.943\left[\frac{\mathbf{k_1^3}\rho_\mathbf{1}(\rho_l - \rho_v)\mathbf{gh_{fg}}}{\mu_\mathbf{1}\theta\mathbf{L}}\right]^{\mathbf{1/4}} \tag{6.13}$$

where the subscript $l$ represents the liquid condensate.

Since $\rho_1 >> \rho_v$,

$$h_m = 0.943\left(\frac{\mathbf{k_1^3}\rho_\mathbf{1}^\mathbf{2}\mathbf{gh_{fg}}}{\mu_\mathbf{1}\theta\mathbf{L}}\right)^{\mathbf{1/4}} \tag{6.14}$$

Equations (6.13) and (6.14) are the *Nusselt's equations* for laminar film condensation on a vertical plate, which can also be applied to condensation outside a tube of large diameter. These give conservative values of heat transfer coefficient. McAdams [2] suggested 20% increase over this value so that

$$h_m = 1.13\left(\frac{\mathbf{k_1^3}\rho_\mathbf{1}^\mathbf{2}\mathbf{gh_{fg}}}{\mu_\mathbf{1}\theta\mathbf{L}}\right)^{\mathbf{1/4}} \tag{6.15}$$

The bulk temperature of the condensate is always less than saturation temperature and hence, subcooled. If $T_B$ is the bulk temperature, then by energy balance,

$$(\dot{m}h)_{\mathrm{in}} = (\dot{m}h)_{\mathrm{out}}$$

$$\int_0^\delta ub\,\mathrm{d}y c_p T = \int_0^\delta ub\,\mathrm{d}y c_p T_B$$

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$$\therefore \qquad \mathbf{T_B} = \frac{\int_0^\delta \mathbf{Tu\ dy}}{\int_0^\delta \mathbf{u\ dy}} \tag{6.16}$$

Assuming a linear temperature profile (Fig. 6.4),

$$T = my + C \tag{6.17}$$

and since the velocity distribution [Eq. (6.5)] is given by

$$u = \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right) U$$

Equation (6.16) becomes

$$T_B = \frac{\int_0^\delta (my + c) U \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right) \mathrm{d}y}{\int_0^\delta \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right) \mathrm{d}y} \tag{6.18}$$

Numerator $$= \int_0^\delta \left(\frac{2my^2}{\delta} - \frac{my^3}{\delta^2} - \frac{2cy}{\delta} - \frac{cy^2}{\delta^2}\right) \mathrm{d}y$$

$$= \frac{2m}{\delta}\frac{\delta^3}{3} - \frac{m}{\delta^2}\frac{\delta^4}{4} + \frac{2c}{\delta}\frac{\delta^2}{2} - \frac{c}{\delta^2}\frac{\delta^3}{3}$$

$$= \frac{5}{12} m\delta^2 + \frac{2}{3} c\delta$$

Denominator $$= \frac{2}{\delta}\frac{\delta^2}{2} - \frac{1}{\delta^2}\frac{\delta^3}{3} = \delta - \frac{\delta}{3} = \frac{2}{3}\delta$$

$$T_B = \frac{\frac{5}{12} m\delta^2 + \frac{2}{3} c\delta}{\frac{2}{3}\delta} = \frac{5}{8} m\delta + c \tag{6.19}$$

Again, $T = my + c.$

When $y = 0,\ T = T_w$

$\therefore$ $c = T_w$

When $y = \delta,\ T = T_{sat} = T_s.$

$\therefore$ $T_s = m\delta + c = m\delta + T_w$

$$m = \frac{T_s - T_w}{\delta}$$

Substituting in Eq. (6.19),

$$T_B = \frac{5}{8}(T_s - T_w) + T_w = \frac{5}{8} T_s + \frac{3}{8} T_w$$

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$$= T_s - \frac{3}{8} T_s + \frac{3}{8} T_w = T_s - \frac{3}{8} (T_s - T_w)$$

$$= T_s - \frac{3}{8} \theta$$

or $$\mathbf{T_s - T_B = \frac{3}{8} \theta} \tag{6.20}$$

The average enthalpy change during condensation with subcooling,

$$\mathbf{h'_{fg} = h_{fg} + c_{p1} (T_s - T_B) = h_{fg} + \frac{3}{8} c_{p1} \theta} \tag{6.21}$$

Substituting in Eq. (6.13),

$$h_m = 0.943 \left[ \frac{k_1^3 \rho_1 (\rho_1 - \rho_v)\, g h'_{fg}}{\mu_1 \theta L} \right]^{1/4} \tag{6.22}$$

If the surface is inclined at an angle $\psi$ with the horizontal (Fig. 6.5), the average coefficient is

$$h_m = 0.943 \left[ \frac{k_1^3 \rho_1 (\rho_1 - \rho_v) g h'_{fg} \sin \psi}{\mu_1 \theta L} \right]^{1/4} \tag{6.23}$$

A modified integral analysis was developed by Rohsenow [3] which shows the temperature distribution slightly curved, the film thickness slightly greater, but the heat transfer rate slightly larger, agreeing better with experimental data.

For $Pr > 0.5$ and $Ja = (c_{p1}\ \theta / h'_{fg}) < 1.0$, where Ja is the Jakob number, it yields results similar to Eqs (6.13) – (6.15) and (6.22), except that $h'_{fg}$ is replaced by

$$h_{fg} + 0.68\, c_{p1}\, (T_s - T_w),$$

so that

$$\mathbf{h_m = 0.943 \left[ \frac{k_1^3 \rho_1 (\rho_1 - \rho_v) g\, h_{fg} (1 + 0.68\ Ja)}{\mu_1 \theta L} \right]^{1/4}} \tag{6.24}$$

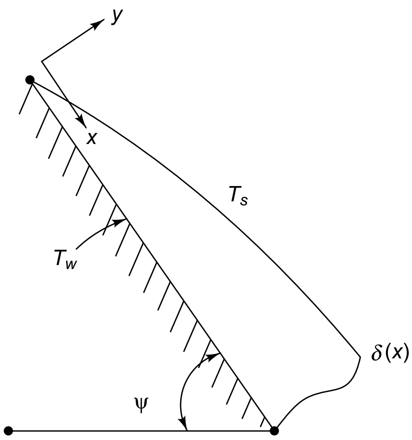


**Fig. 6.5** *Filmwise condensation on an inclined plane surface*

Some insight can be gained by writing Eq. (6.13) in terms of commonly used dimensionless products

$$\overline{Nu} = 0.943 \left[ \frac{Gr_L\, Pr_l}{Ja} \right]^{1/4}$$

where $Gr_L$ is based on the plate length

$$Gr_L = \frac{\rho_l (\rho_l - \rho_v) g L^3}{\mu_l^2},$$

$Pr_l$ is the liquid Prandtl number, and Ja is the Jakob number defined as

$$\mathbf{Ja = \frac{c_{Pl} (T_s - T_w)}{h_{fg}} = \frac{sensible\ heat}{latent\ heat}}$$

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The Jakob number is a measure of the importance of subcooling, expressing the change in the sensible heat per unit mass of condensed liquid in the film relative to the enthalpy associated with the phase change. The value of Ja is usually small, e.g. for condensation of steam it is of the order of 0.01.

The density change in Grashof number $Gr_L$ is associated with the density change resulting from the change in phase whereas in the conventional Grashof number in natural convection, the density change is associated purely with the temperature changes in the fluid. Obviously, the former is much larger than the latter.

The above analysis is also valid for the inside and outside surfaces of vertical tubes if the tube diameter is larger than the film thickness (Fig. 6.6). It cannot, however, be extended to inclined tubes, where the condensate flow will not be parallel to the tube axis, and the angle of inclination would vary with $x$.

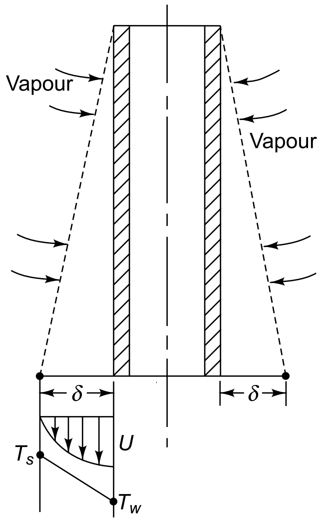


**Fig. 6.6** *Film condensation on the outside surface of a vertical tube*

## 6.5 CONDENSATION ON HORIZONTAL TUBES

The condensate film on the outside of horizontal tubes flows around the tube and off the bottom in a sheet, as shown in Fig. 6.7. The liquid film is very thin so that the above analysis applies here except that $g$ is replaced by $g \sin \phi$ and the average value of $h$ follows from integration over the range of values from 0 to 180°, as given below:

$$h_m = 0.725\left[\frac{k_1^3 \rho_1(\rho_l - \rho_v) g h'_{fg}}{\mu_1 D_0 \theta}\right]^{1/4} \tag{6.25}$$

where $D_0$ is the outside diameter of the tube. This is the *Nusselt's equation* for film condensation on a horizontal tube.

Therefore, $h_m \propto D_0^{-\frac{1}{4}}$. Smaller $D_0$ would yield higher $h_m$. But smaller $D_0$ means less surface area $\pi D_0 L$ exposed for condensation, and so less heat transfer. Thus there is an optimum tube diameter.

The heat transfer coefficient on a horizontal tube decreases from a maximum value at $\phi = 0$ to essentially zero for $\phi = 180°$. The condensing rate on the upper half of the tube is 46% greater than on the lower half.

Dividing Eq. (6.25) by Eq. (6.13),

$$\frac{(h_m)_H}{(h_m)_V} = \frac{0.725}{0.943}\left(\frac{L}{D_0}\right)^{1/4} = 0.77\left(\frac{L}{D_0}\right)^{1/4} \tag{6.26}$$

If $L/D_0 = 2.87$, then $(h_m)_H = (h_m)_V$ and, if $L > 2.87\, D_0$, $(h_m)_H > (h_m)_V$.

Therefore, condenser tubes are usually horizontal. Vertical condenser is highly uncommon.

For a bank of horizontal tubes in a vertical tier (Fig. 6.8),

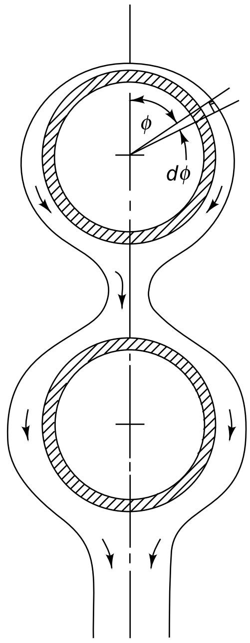


**Fig. 6.7** *Condensate film on horizontal tubes*

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$$\mathbf{h_m = 0.725\left[\frac{k_l^3\rho_l(\rho_l-\rho_v)gh'_{fg}}{\mu_l N D_0 \theta}\right]^{1/4}} \tag{6.27}$$

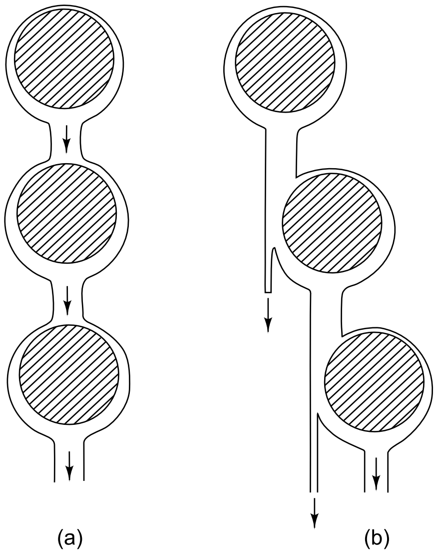


**Fig. 6.8** *Film condensation on horizontal tubes (a) in a vertical tier and (b) in staggered arrangement*

where $N$ is the number of tubes in the tier. Thus, $h_m \propto 1/N^{1/4}$. Therefore, as $N$ increases, $h_m$ decreases. If the tubes are arranged in a vertical tier [Fig. 6.8(a)], the condensate from one tube drips on to that of the next, and so on. Thus, the film thickness is greater for the lower tubes, increasing the resistance to heat transfer. The staggered arrangement of tubes [Fig. 6.8(b)], however, gives improved performance.

For condensation on a sphere, it can similarly be shown,

$$h_m = 0.815\left[\frac{k_l^3\rho_l(\rho_l-\rho_v)gh'_{fg}}{\mu_l D\theta}\right]^{1/4} \tag{6.28}$$

where $D$ is the diameter of sphere.

The total heat transfer to the surface is

$$Q = h_m A\,(T_s - T_w) \tag{6.29}$$

and the rate of condensation is

$$\dot{m} = \frac{Q}{h'_{fg}} \tag{6.30}$$

The physical properties of the liquid film should be evaluated at an effective film temperature

$$T_{\text{film}} = T_w + 0.25\,(T_s - T_w) \tag{6.31}$$

For simplicity, properties of the condensate are often evaluated at the average temperature

$$T_{\text{film}} = \frac{T_w + T_s}{2}$$

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## 6.6 CONDENSATION NUMBER

Reynolds number of the condensate flow is very important in determining condensation behaviour. It is often convenient to express the heat transfer coefficient directly in terms of Re.

$$h_m = \frac{1}{L}\int_0^L h_x \mathrm{d}x = \frac{4}{3}h_x = \frac{4}{3}\frac{k_1}{\delta} \tag{6.32}$$

From Eq. (6.9), since $\rho_1 >> \rho_v$

$$\Gamma = \frac{1}{3}\rho_1^2 \frac{g\delta^3}{\mu_1}$$

$$\therefore \qquad \delta = \left(\frac{3\Gamma\mu_1}{\rho_1^2 g}\right)^{1/3} \tag{6.33}$$

Substituting in Eq. (6.32),

$$h_m = \frac{4}{3}\left(\frac{k_1^3\rho_1^2 g}{3\Gamma\mu_1}\right)^{1/3}$$

The Reynolds number of the condensate film, Re, when based on the hydraulic diameter can be written as

$$\mathbf{Re} = \frac{\mathbf{u_m D}\rho}{\mu} = \mathbf{u_m}\frac{\mathbf{4A}}{\mathbf{P}}\frac{\rho}{\mu} = \frac{4\omega}{\mathbf{P}\mu} = \frac{4\Gamma}{\mu}$$

where the hydraulic diameter $D = 4A/p$; $P$ = wetted perimeter which is $\pi D$ for vertical tube of outside diameter $D$, $2L$ for horizontal tube of length $L$ and $b$ for vertical or inclined plate of width $b$; and $A$ is the cross-sectional area for condensation flow.

The expression for $h_m$ can be written as

$$h_m = \frac{4}{3}\left(\frac{k_1^3\rho_1^2 g}{3\Gamma\mu_1}\frac{4\mu_1}{4\mu_1}\right)^{1/3}$$

Substituting $\quad \mathrm{Re} = 4\Gamma_1/\mu_1$,

$$h_m = \frac{4}{3}\left(\frac{k_1^3\rho_1^2 g}{\mu_1^2}\right)^{1/3}\left(\frac{4}{3}\right)^{1/3}(\mathrm{Re})^{-1/3}$$

$$\therefore \qquad \mathbf{h_m}\cdot\left(\frac{\mu_1^2}{\mathbf{k_1^3}\rho_1^2\mathbf{g}}\right)^{1/3} = \mathbf{1.47\ Re^{-1/3}} \tag{6.34}$$

The left hand side of Eq. (6.34) is a dimensionless group, called the *condensation number* Co.

$$\mathbf{Co} = \mathbf{h_m}\left[\frac{\mu_1^2}{\mathbf{k_1^3}\rho_1(\rho_1 - \rho_v)\mathbf{g}}\right]^{1/3} \tag{6.35}$$

For a vertical plate,

$$\mathbf{Co = 1.47\ Re^{-1/3}} \tag{6.36}$$

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Similarly, for a horizontal tube, Eq. (6.25) can be expressed in the following form,

$$\mathbf{h_m}\left(\frac{\mu_1^2}{k_1^3\rho_1^2\ \mathbf{g}}\right)^{1/3} = \mathbf{1.51\ Re^{-1/3}} \tag{6.37}$$

## 6.7 TURBULENT FILM CONDENSATION

Just as a fluid flowing over a surface undergoes a transition from laminar to turbulent flow, in the same way the motion of the condensate becomes turbulent when its Reynolds number exceeds a critical value of 2000.

For Re > 2000, Colburn's relation [4] can be used

$$\mathbf{h_x = 0.056\left(\frac{4\Gamma}{\mu_1}\right)^{0.2}\left(\frac{k_1^3\rho_1^2 g}{\mu_1^2}\right)^{1/3}(Pr_1)^{1/2}} \tag{6.38}$$

We obtain average values of heat transfer coefficient, using Eq. (6.13) for $4\Gamma/\mu_1 < 2000$ and Eq. (6.38) for $4\Gamma/\mu_1 > 2000$. The results are plotted as solid lines in Fig. 6.9, where some experimental data are also shown. The heavy dashed line is an empirical curve recommended by McAdams [2].

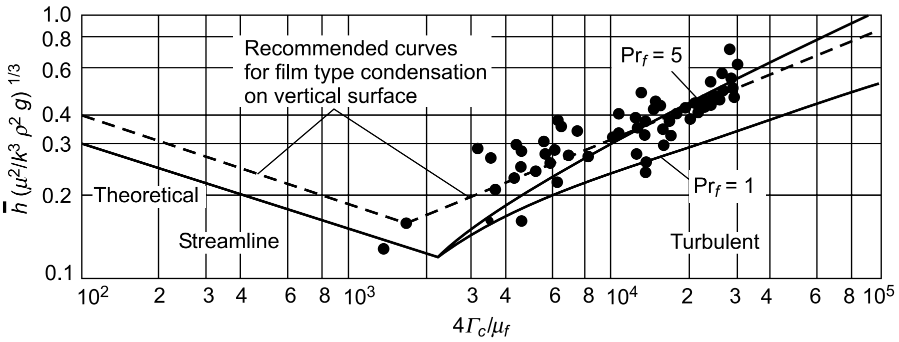


**Fig. 6.9** *Effect of turbulence in film on heat transfer with condensation*

Turbulent flow of condensate is hardly ever reached on a horizontal tube, where flow is almost always laminar, but it may occur on the lower part of a vertical surface, when $h_m$ becomes larger due to turbulence with larger length. Kirkbride [5] proposed the following empirical correlation for film condensation on a vertical plate after the start of turbulence.

$$h_m\left(\frac{\mu_1^2}{k_1^3\rho_1^2 g}\right)^{1/3} = 0.0077\ (\mathrm{Re})^{0.4} \tag{6.39}$$

## 6.8 EFFECT OF HIGH VAPOUR VELOCITY

Nusselt's assumption on condensation film theory is that the vapour is stagnant at the interface, and there is no frictional drag between the condensate and the vapour. This approximation is not valid when the vapour velocity is substantial compared with the condensate velocity at the liquid–vapour interface. When the vapour flows upward, it adds a retarding force to the viscous shear and causes the film thickness to increase. When the vapour flows downward, the film thickness decreases and the heat transfer coefficient increases and is

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larger than that predicted from Eq. (6.13). In addition, the transition from laminar to turbulent flow occurs at about $\mathrm{Re}_d = 300$ when the vapour velocity is high. Carpenter and Colburn [6] correlated their experimental data for condensation with high vapour velocity,

$$\frac{h_m}{c_{p_1} G_m} \mathrm{Pr}_1^{1/2} = 0.046 \left(\frac{\rho_1}{\rho_v}\right) f \tag{6.40}$$

where $G_m$ = mean value of mass velocity of the vapour, kg/m$^2$ s

$$= \left[\frac{G_1^2 + G_1 G_2 + G_2^2}{3}\right]^{1/2}$$

$G_1$ = mass velocity at the top of the tube,

$G_2$ = mass velocity at the bottom of the tube,

$f$ = pipe friction coefficient evaluated at the average vapour velocity = $\tau_w/(G_m^2/2\rho_v)$

$\tau_w$ = wall shear stress, N/m$^2$

All physical properties of Eq. (6.40) are evaluated at a reference temperature of $(0.25T_s +| \, 0.75T_w)$.

## 6.9 EFFECT OF SUPERHEATED VAPOUR

When the vapour is superheated and the wall temperature of the cold surface is also above the saturation temperature, no condensation occurs, and the vapour simply cools down, becoming less superheated. However, if the wall temperature is less than saturation temperature and the liquid–vapour interface remains at saturation, the condensation rate is only slightly increased by greater superheat since the enthalpy change is larger as given below.

$$h''_{fg} = c_v (T_v - T_s) + h_{fg} + \frac{3}{8} c_1 (T_s - T_w) \tag{6.41}$$

where $c_v$ is the specific heat of vapour. For superheated vapour, in Eqs (6.22) and (6.23), $h'_{fg}$ is replaced by $h''_{fg}$, which yields slightly higher value of heat transfer coefficient. The heat transfer rate is calculated by

$$\frac{Q}{A} = h(T_s - T_w) \tag{6.42}$$

since $(T_s - T_w)$ is still the driving force for heat transfer across the liquid condensate film. The condensate rate is calculated from

$$\frac{w}{A} = \frac{1}{h''_{fg}}\left(\frac{Q}{A}\right) \tag{6.43}$$

## 6.10 EFFECT OF NON-CONDENSABLE GAS

When a condensable vapour is condensing in the presence of a non-condensable gas, the vapour must diffuse through the gas, requiring a decrease in vapour partial pressure toward the liquid–vapour interface. Using Dalton's law of partial pressure,

$$p = p_v + p_a$$

where $p_v$ is the vapour pressure and $p_a$ is the partial pressure of say, air, which is the non-condensable gas. Thus, the interface saturation temperature is significantly below the temperature of the main vapour–gas mixture (Fig. 6.10). The combined mass and heat transfer process requires a lengthy trial-and-error calcula-

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tion to account for this effect (Bailey *et al*). As the mixture of saturated vapour and air comes in contact with the cooling surface, vapour condenses into liquid, but air being non-condensable remains in the gas phase in the form of an air film adjacent to the condensate film, through which heat released gets conducted and the vapour diffuses to the cooling surface, thus affecting both heat transfer and mass transfer. The heat transfer coefficient decreases sharply from the presence of this non-condensable gas; even 0.5% mass of air can decrease it by a factor of 2. The non-condensable gas like air not only blankets the cooing surface and offers a high thermal resistance ($k$ of air being low), but also inhibits the mass transfer of vapour by offering diffusional resistance. In a steam condenser of a power plant, air which leaks into the shell is continuously driven out by a steam jet air ejector to improve the condenser performance.

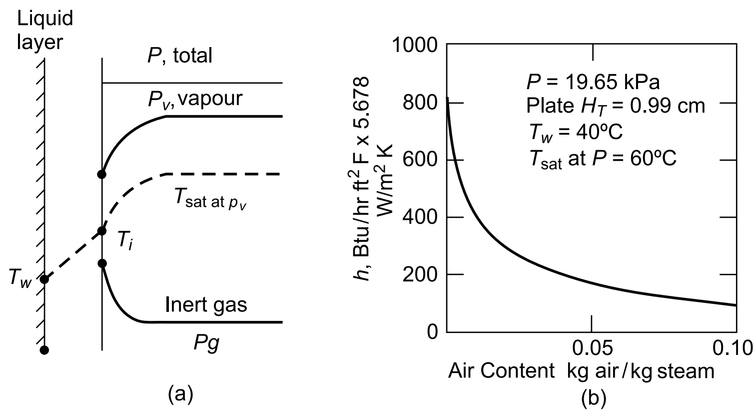


**Fig. 6.10** *Effect of non-condensable gas on heat transfer coefficient for condensing vapour*

## 6.11 FILM CONDENSATION INSIDE HORIZONTAL TUBES

For refrigeration and air conditioning systems, condensers often involve condensation inside horizontal or vertical tubes. Conditions within the tube are complicated and greatly depend on the vapour velocity inside the tube. If the vapour velocity is small, the condensate flow is from the upper portion of the tube to the bottom, from which it flows in a longitudinal direction with the vapour [Fig. 6.11(a)]. For low vapour velocities such that

$$\mathrm{Re}_{v,\,i} = \left(\frac{\rho_v u_{m,v} D}{\mu_v}\right) < 35{,}000$$

where $i$ refers to the tube inlet, Chato [7] recommends the following equation

$$\bar{h}_D = 0.555\left[\frac{k_1^3 \rho_1 (\rho_l - \rho_v) g h'_{fg}}{\mu_1 \theta D}\right]^{1/4} \tag{6.44}$$

where $$h'_{fg} = h_{fg} + \frac{3}{8} c_{p_1}\, \theta.$$

At higher vapour velocities the two-phase flow regimes become annular [Fig. 6.11(b)]. The vapour occupies the core of the annulus, diminishing in diameter as the thickness of the outer condensate layer increases in the flow condition, the details being given by Rohsenow [8].

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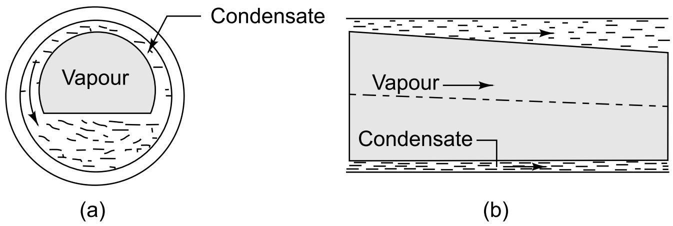


**Fig. 6.11** *Film condensation in a horizontal tube. (a) Cross-section of condensate flow for low vapour velocities. (b) Longitudinal section of condensate flow for large vapour velocities*

## 6.12 BOILING HEAT TRANSFER

Heat transfer to boiling liquids is a convection process which involves a change of phase from liquid to vapour. In a boiling process, the average liquid temperature may remain well below the saturation temperature ($T_s$) with the wall temperature ($T_w$) above saturation thus producing "local" boiling at the wall with subsequent condensation in the colder bulk of the liquid. This is known as *subcooled boiling*. Boiling in a liquid at saturation temperature is known as *saturated or bulk boiling*.

In some designs of evaporators the heating surface is submerged beneath a free surface of liquid. This is known as *pool boiling*. When the liquid flows through a tube with subcooled or saturated boiling, this bounded convection process is called *forced-convection boiling*, even though circulation may occur either by density difference or by a pump in a fluid circuit.

We begin with a discussion of the different regimes of boiling, without superimposed forced convection. Although boiling is a familiar phenomenon, it is an extremely complicated process, difficult to analyse due to the many variables involved. We then focus our attention on nucleate boiling, including peak heat flux, which is of great engineering interest. There has been considerable progress in gaining a physical understanding of the boiling mechanism. By observing the boiling phenomena with the aid of high-speed photography it has been found that there are distinct regimes of boiling in which heat transfer mechanisms differ radically.

## 6.13 REGIMES OF BOILING

The existence of different regimes of boiling was first discussed by Nukiyama [9]. To acquire a physical understanding of the various regimes of boiling we would consider the heating of distilled water at one atmosphere on an electrically heated nichrome tube (Fig. 6.12). The heating surface is submerged in the liquid. The boiling process in this situation is referred to as *pool boiling*. Heat flux $q_w$ is determined from measured voltage and current and the surface area of the tube, and temperatures of the tube wall surface ($T_w$) and the bulk liquid ($T$) are measured by thermocouples as shown. The electrical energy input is controlled by the variable rheostat. A typical boiling curve is shown in Fig. (6.13). As current flows, the wall temperature $T_w$ increases. Heat is transferred from wall to the fluid.

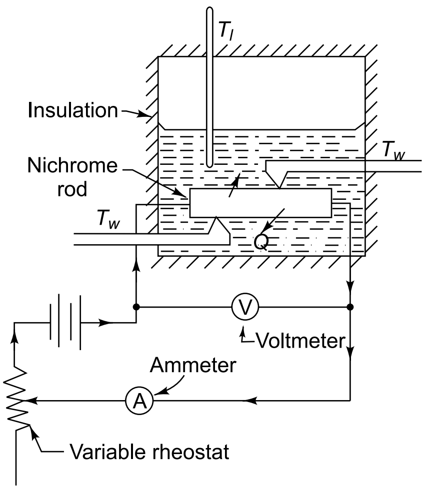


**Fig. 6.12** *Pool boiling heat transfer*

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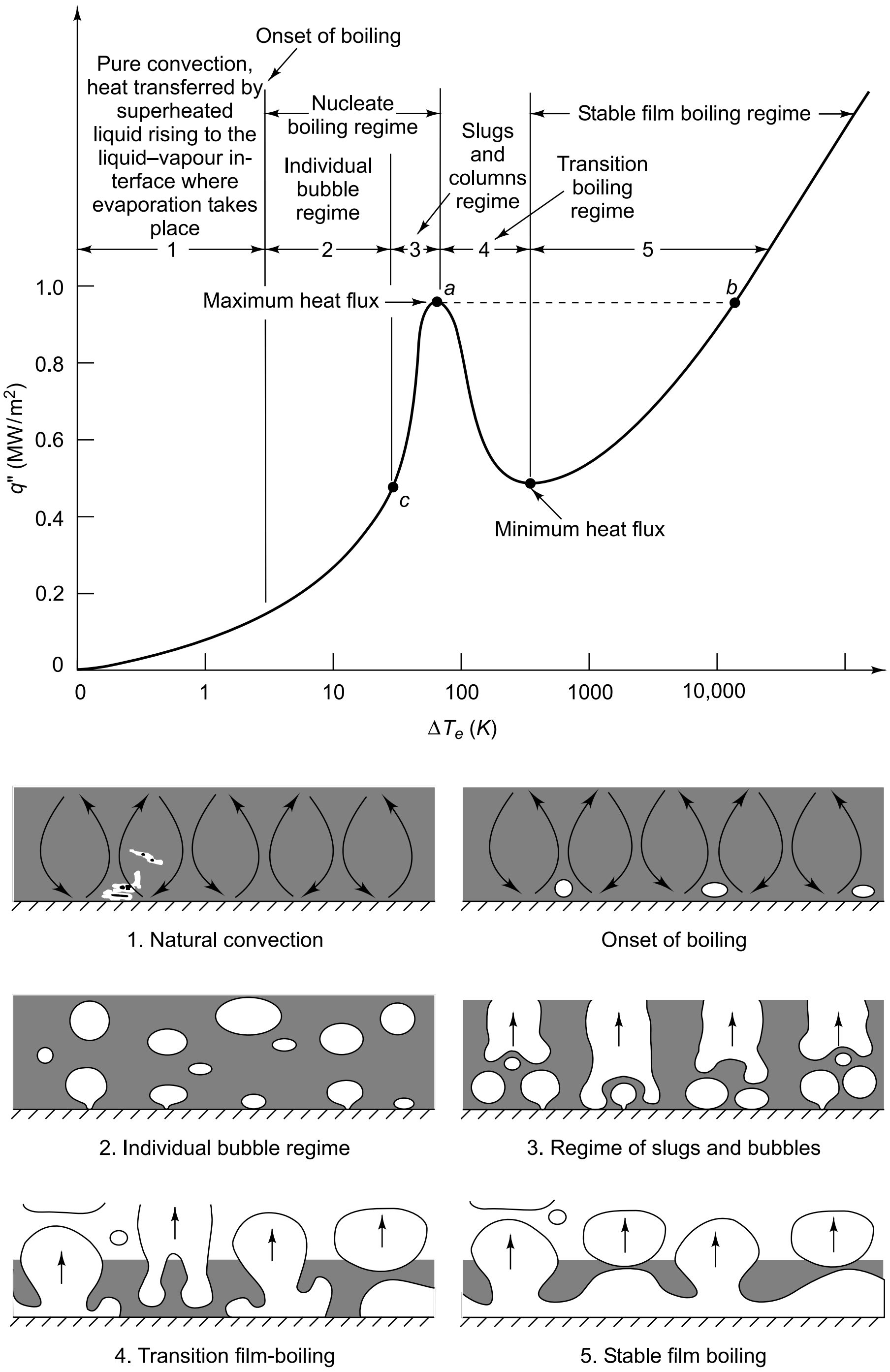


**Fig. 6.13** *Typical boiling curve for pool boiling of water at saturation temperature and atmospheric pressure with schematic representation of each boiling regime*

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The density of the fluid at the surface decreases. Hot fluid moves up and cold fluid descends. A circulation current is set up by density difference. Heat transfer is by natural convection. If there is more energy input, there is more temperature difference or excess temperature $\Delta T_e$ between the surface and the fluid.

When $T_w > T_{sat}$ (100°C for water) the liquid layer adjacent to the heating surface is superheated, which is carried away by convection current, and evaporation of liquid occurs from the free surface.

When $T_w$ is increased still further, a point is soon reached when the energy levels of some liquid molecules become so high that they break away from the surrounding molecules, get transformed from liquid to vapour nuclei and ultimately form vapour bubbles at some favoured spots, called *nucleation sites*. At first the number and size of bubbles are small, and these bubbles rise up and condense in the liquid before reaching the interface.

With more electrical energy input when $T_w$ is increased further, the bubbles gradually grow in size and become more in number. They rise up and reach the interface before condensing. There will be evaporation of liquid at the interface.

When $T_w$ is further increased, the bubbles coalesce with one another on the heating surface, to form an unstable vapour film which continually collapses and re-forms.

With further increase in temperature $T_w$, the bubble formation becomes vigorous and a stable vapour film is developed on the heating surface through which heat is conducted. As temperature $T_w$ still increases, the properties of vapour change, to slowly increase $h$ and hence heat flux $q_w$. With still increasing temperature, radiation gradually predominates, till the heating element reaches the burnout point and melts away to snap the electrical circuit.

### 6.13.1 Modes of Pool Boiling

The different modes or regimes of pool boiling are shown in Fig. 6.14, which is known as *Farber-Scorah boiling curve* [10]. It pertains to water at 1 atm, although similar trends characterise the behaviour of other fluids. Since $q_w = h\,(T_w - T_s) = h\,\Delta T_e$, where $\Delta T_e$ is the *excess temperature* of the wall over the saturation temperature ($T_s$), different boiling regimes may be delineated according to the value of $\Delta T_e$.

#### Natural Convection Boiling (0–A)

It is said to exist if $\Delta T_e < \Delta T_{e,\,A}$, where $\Delta T_{e,\,A} \cong 5$ºC. Below point $A$, referred to as the *onset of nucleate boiling* (ONB), the fluid motion is mainly due to density difference or buoyancy effect. According to whether the flow is laminar or turbulent, $h \infty (\Delta T_e)^{1/4}$ or $(\Delta T_e)^{1/3}$, respectively, in which case $q_w$ varies as $(\Delta T_e)^{5/4}$ or $(\Delta T_e)^{4/3}$.

#### Nucleate Boiling (A–C)

It exists in the range $\Delta T_{e,\,A} \le \Delta T_{e,\,A} \le \Delta T_{e,\,C}$, where $\Delta T_{e,\,C} \cong 30$°C. In this range, two different flow regimes may be distinguished. In region $A$–$B$, isolated bubbles form at nucleation sites and get detached from the surface. This separation induces good fluid mixing near the surface, substantially increasing $h$ and $q_w$. As $\Delta T_e$ is increased beyond $\Delta T_{e,\,B}$, more nucleation sites become active and increased bubble formation causes bubble interference and coalescence. In the region $B$–$C$, the vapour escapes as *jets or columns*, which later merge into slugs of the vapour. Interference between the densely populated bubbles inhibits the motion of the liquid near the surface. Point $P$ corresponds to an inflection in the boiling curve at which the heat transfer coefficient is maximum. Then $h$ starts decreasing with increasing $\Delta T_e$, although $q_w$ (= $h\,\Delta T_e$) continues to increase. This trend occurs because for $\Delta T_e > \Delta T_{e,\,p}$, the relative increase in $\Delta T_e$ exceeds the relative reduction in $h$. At point $C$, however, further increase in $\Delta T_e$ is balanced by the decrease in $h$.

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The maximum heat flux $q_{w,\,C} = q_{max}$ is termed as the *critical heat flux* (CHF), and in water at atmospheric pressure it exceeds 1 MW/m$^2$.

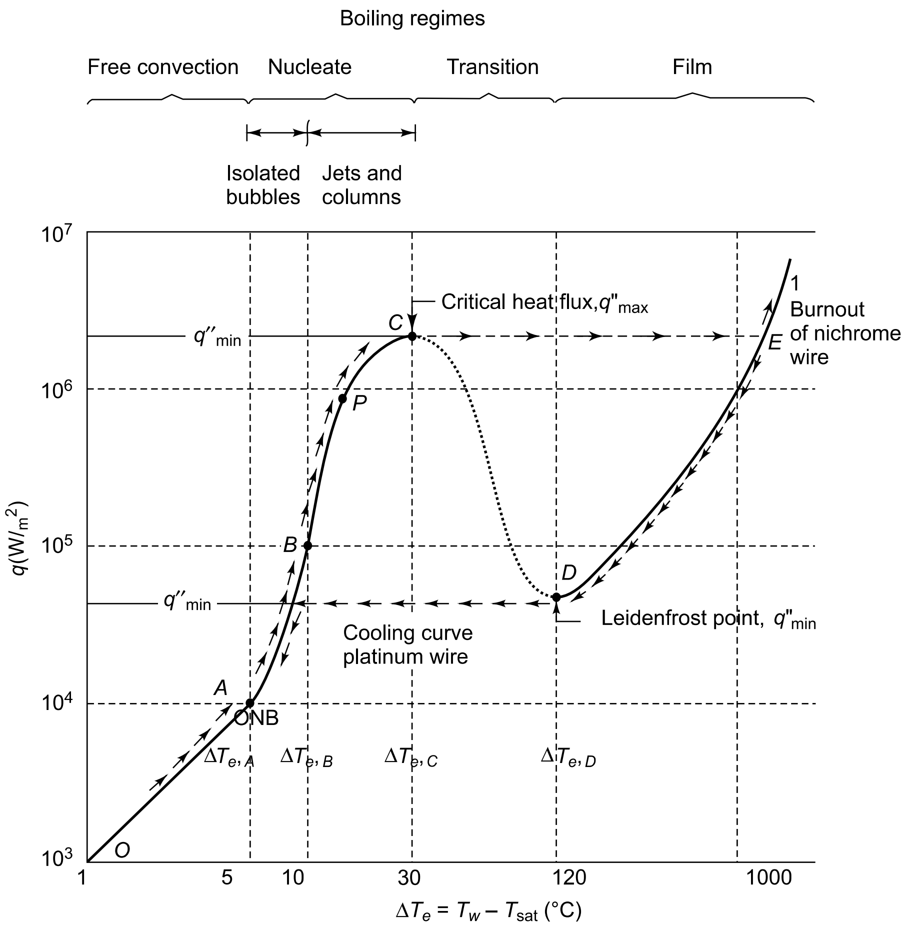


**Fig. 6.14** *Farber-Scorah boiling curve showing different boiling regimes*

With further increase in power, the heat flux increased to very high level, for a value slightly larger than $q''_{max}$, the wire temperature jumped to the melting point and burnout occurred. However, repeating the experiment with a platinum wire having a higher melting point of 2045 K compared to that of nichrome at 1500 K, Nukiyama was able to maintain heat fluxes above $q''_{max}$ without burnout. When he subsequently reduced the power, the variation of $\Delta T_e$ with $q''$ followed the *cooling curve* of Fig. 6.14. When the heat flux reached the minimum point $q''_{min}$, a further decrease in power caused the excess temperature to drop abruptly, and the process followed the original heating curve back to the saturation point.

### Partial Film Boiling (C–D)

The heat flux rate is very high in nucleate boiling because of agitating motion of bubbles. The formation of increasing number of bubbles forms an unstable film, the thermal conductivity of which is very low. The portion of the surface covered by vapour bubbles at any instant is effectively insulated.

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So long as the agitating motion of bubbles predominates over the insulating effects of the film, the heat flux rate continues to increase. But when the insulating effects overshadow the effect of fluid agitation, the heat flux rate decreases with increasing temperature, which happens in *partial film boiling, transition boiling* or *unstable film boiling* (*C–D*). In this region $\Delta T_{e,\,C} \le \Delta T_{e,} \le \Delta T_{e,\,D}$, where $\Delta T_{e,\,D} \simeq 120°C$. Bubble formation is so rapid that bubbles coalesce to form a vapour film on the surface. This vapour film is unstable, continually forming, collapsing and re-forming, and the conditions oscillate between film boiling and nucleate boiling. The fraction of the total surface covered by the film increases with increasing $\Delta T_e$. Since the thermal conductivity of the vapour is much less than that of the liquid, $h$ and $q_w$ must decrease with increasing $\Delta T_e$.

### Film Boiling (D–E)

It exists for $\Delta T_e \ge \Delta T_{e,\,D.}$ At point $D$ of the boiling curve, called the *Leidenfrost point*, the heat flux is a minimum, $q_D = q_{\min}$, and the surface is completely covered by a vapour blanket. Heat transfer from the surface to the liquid occurs by conduction through the stable vapour film. Water drops falling on a very hot solid surface get immediately separated from the latter by a vapour film and move about it before slowly boiling away—a phenomenon first observed by Leidenfrost in 1756. As the surface temperature is increased, heat flux slowly increases because of increasing $h$. Then the heating surface becomes glowing and radiation effect gradually predominates, the heat flux rapidly increasing with $\Delta T_e$.

Figure 6.15 illustrates the nature of the vapour formation and bubble dynamics associated with nucleate boiling, transition boiling and film boiling of methanol on a horizontal tube.

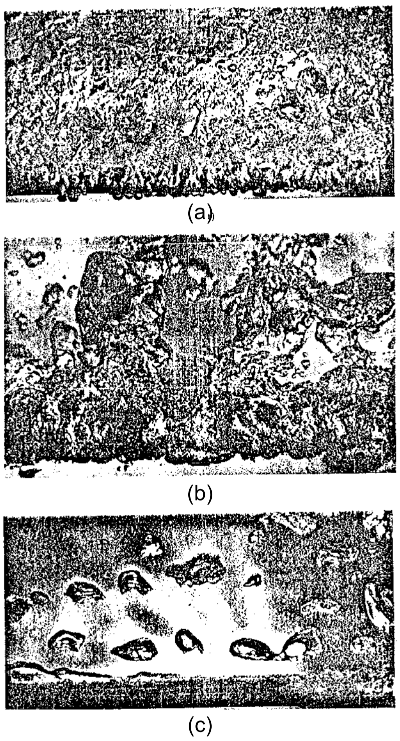


**Fig. 6.15** *Boiling of methanol on a horizontal tube: (a) Nucleate boiling in the jets and columns, (b) Transition boiling and (c) Film boiling*

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The foregoing discussion of the boiling curve assumes that control may be maintained over $T_w$ by controlling $q_w$ (as in a nuclear reactor or in an electric resistance heating device). If we start at point $B$, say, in Fig. 6.16 and gradually increase $q_w$, the value of $\Delta T_e$, and hence the value of $T_w$, will also increase, following the boiling curve to point $C$. However, any increase in $q_w$ beyond this point will induce a sharp departure from the boiling curve in which surface conditions change abruptly from $\Delta T_{e,C}$ to $\Delta T_{e,E} = T_{w,E} - T_s$, $T_s$ being the saturation temperature. Since $T_{w,E}$ may exceed the melting point of the solid, destruction or failure of the system may occur. For this reason point $C$ is often called the *burnout point* or the *boiling crisis* indicating the onset of *departure from nucleate boiling* (DNB). The accurate knowledge of *critical heat flux (CHF)*, $q_{w,C} = q_{max}$ is required. It is desired to operate the heat transfer surface close to this value, but it is dangerous to exceed it.

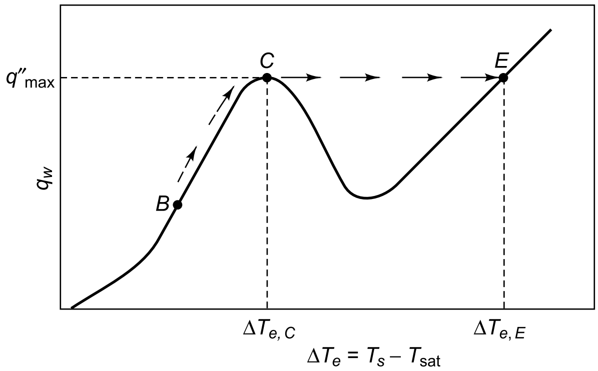


**Fig. 6.16** *Onset of the boiling crisis*

## 6.14 NUCLEATE BOILING

Nucleate boiling is of most importance in boiling heat transfer. Film boiling is always to be avoided. Nucleate boiling involves two separate processes—the formation of bubbles (nucleation) and the subsequent growth and motion of these bubbles.

Two conditions are required to be fulfilled for bubbles to form:

1. The liquid at the heating surface must be superheated.
2. There must be dissolved gases present to form the nuclei of bubbles.

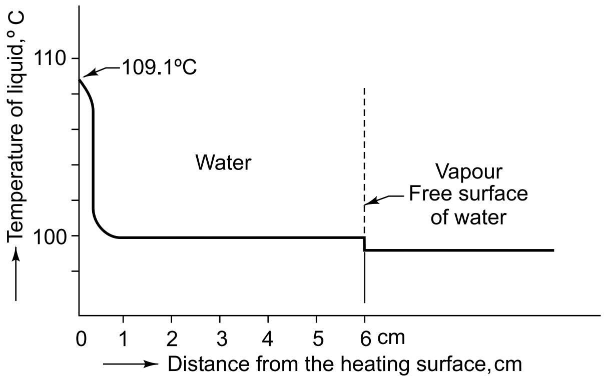


**Fig. 6.17** *Pool boiling of water at 1 atm: Liquid superheat*

Nucleation starts when $T_w$ is only a few degrees above $T_{sat}$. Bubbles form at some favoured spots, called nucleation sites, on the heating surface. The liquid layer immediately adjacent to the heating surface will be at the same temperature and hence be superheated (Fig. 6.17), since $T_w > T_{sat}$. Therefore, nucleation requires liquid to be superheated. Bubbles always originate on the heating surface where the liquid superheat is maximum.

Let us consider a vapour dubble in thermal equilibrium with a liquid at uniform temperature (Fig. 6.18). The bubble of radius $r$ is split into two halves and let us consider the equilibrium of one half.

Forces acting on the plane 1–1 are due to vapour pressure $p_v$, the liquid pressure $p_1$ and the surface tension $\sigma$, so that

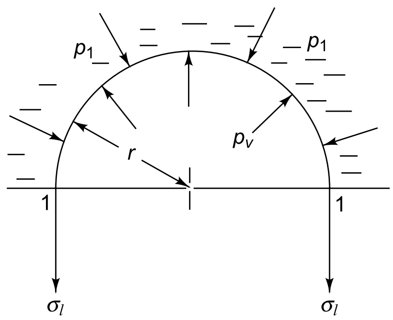


**Fig. 6.18** *Force balance on a half-bubble*

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$$\pi r^2 (p_v - p_1) = 2\pi r\sigma$$

or
$$\mathbf{p_v - p_1 = \frac{2\sigma}{r}} \tag{6.45}$$

If $T_v$ is the saturation temperature of the vapour, then at equilibrium, since $p_1 < p_v$ and $T_{liq} = T_v$, the liquid temperature must be superheated with respect to the liquid pressure. In other words,

$$\begin{aligned} p_1 &< p_v \\ (T_1)_{sat} &< (T_v)_{sat} \\ T_1 &> (T_1)_{sat} \end{aligned} \tag{6.46}$$

The Clausius–Clapeyron equation with the ideal gas approximation relates $T$ and $v$ along the saturation line as

$$\mathbf{\frac{dp}{dT} \cong \frac{h_{fg}\rho_v}{T_v} \cong \frac{h_{fg}p_v}{R_v T_v^2}} \tag{6.47}$$

where $R_v$ is the gas constant of the vapour.

Since $(p_v - p_1) \cong (T_v - T_{sat})\, dp/dT$, we can combine Eqs (6.45) and (6.47) to get

$$\mathbf{T_v - T_{sat} = \frac{2R_v T_{sat}^2 \sigma}{h_{fg} P_i r}} \tag{6.48}$$

If $(T_{liq} - T_{sat})$ is greater than $(T_v - T_{sat})$ calculated from Eq. (6.48), a bubble of radius $r$ will grow; if smaller, the bubble will collapse.

It is seen from Eq. (6.45) that smaller the radius of the bubble, more will be $(p_v - p_1)$, and more will be the amount of superheat. In other words, liquid superheat at the heating surface is maximum and the bubble size is minimum. As liquid superheat decreases towards interface, bubbles grow in size. If $r = 0$, $(p_v - p_1) = \infty$. In order for a bubble to grow without a nucleus, the initial pressure difference must be infinitely large. Thus, it is *impossible for a bubble to form without a nucleus*. Air or dissolved (trapped) gases in the liquid become the nuclei of bubbles.

Inert gas molecules like air are usually present in a liquid. A surface contains many cavities, and bubbles form at a heated surface from cavities which already have some gas or vapour present which are called *active cavities*. When heat is added, the vapour pocket in an active cavity grows by evaporation at the liquid–vapour interface near the heated wall, as shown in Fig. 6.19(a). In a liquid at or near the saturation temperature, the bubble grows and detaches, trapping vapour in the cavity, as shown in Fig. 6.19(b). This trapped vapour is the nucleus of the next bubble. Usually a bubble encompasses many cavities before it detaches. Thus, this vapour-trapping process can induce an inactive cavity into activity.

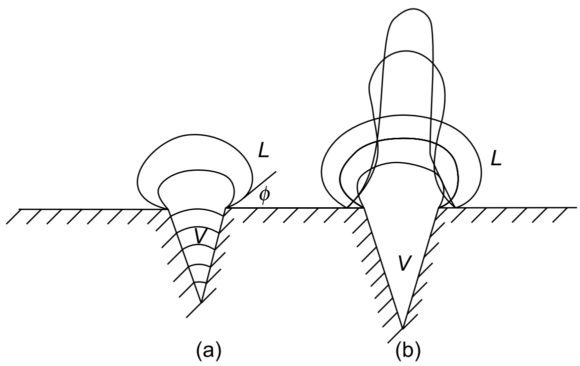

**Fig. 6.19** *Nucleation from cavities*

If in addition to the vapour at $p_v$, a gas at partial pressure $p_g$ is present in a bubble at equilibrium, Eqs (6.45) and (6.48) are modified as follows:

$$p_v - p_1 = \frac{2\sigma}{r} - p_g \tag{6.49}$$

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$$\mathbf{T_v - T_{sat} \cong \frac{RT_{sat}^2}{h_{fg}p_1}\left(\frac{2\sigma}{r} - P_g\right)} \tag{6.50}$$

Equation (6.50) indicates that less liquid superheat is required when non-condensable gas is present in the bubble. It is for this reason that ordinary tap water boils more easily than distilled water.

## 6.15 CORRELATIONS OF BOILING HEAT TRANSFER DATA

The phenomena of boiling heat transfer are more complex than those associated with single-phase convection. A single equation could not possibly correlate the data over the entire range of $\Delta T_e$ represented in Fig. 6.14, since the fluid flow patterns differ so radically in the various regimes.

The analysis of nucleate boiling requires prediction of the number of nucleation sites and the rate at which bubbles originate from each site. Yamagata *et al*. [12] showed that

$$q_w = C(\Delta T_e)^a\, n^b \tag{6.51}$$

where $n$ is the site density (active nucleation sites per unit area) and the exponents are approximately $a = 1.2$ and $b = 1/3$.

Using experimental data on pool boiling as a guide, Rohsenow [12] correlated the relevant data in the following form to predict the heat flux for nucleate pool boiling.

$$\mathbf{q_w = \mu_1\, h_{fg}\left[\frac{g(\rho_1 - \rho_v)}{\sigma}\right]^{1/2}\left(\frac{c_{p,1}\,\Delta T_e}{C_{sf}\,h_{fg}\,Pr_1^n}\right)^3} \tag{6.52}$$

where $c_{p,\,1}$ = specific heat of saturated liquid, J/kg K; $h_{fg}$ = latent heat of vaporisation, J/kg; $g$ = gravitational acceleration, m/s$^2$; $q_w$ = heat flux, W/m$^2$; $\rho_1$ = density of saturated liquid, kg/m$^3$; $\rho_v$ density of saturated vapour, kg/m$^3$; $\sigma$ = surface tension of the liquid–vapour interface, N/m; $Pr_1$ = Prandtl number of saturated liquid; $\mu_1$ = viscosity of the liquid, kg/ms; $n$ = 1.0 for water, 1.7 for other fluids; and $C_{sf}$ = empirical constant that depends on the nature of the heating surface–fluid combination.

The coefficient $C_{sf}$ also depends on the surface roughness, i.e., the number of nucleation sites, the values of which along with the exponent $n$ are given in Table 6.1 for various surface–fluid combinations.

**Table 6.1** *Values of $C_{sf}$ for various surface-fluid combinations*

| *Surface–fluid combination* | $C_{sf}$ | $n$ |
|---|---|---|
| Water–copper | | |
| Rough | 0.0068 | 1.0 |
| Polished | 0.0130 | 1.0 |
| Water–Stainless steel | | |
| Chemically etched | 0.0130 | 1.0 |
| Mechanically polished | 0.0130 | 1.0 |
| Ground and polished | 0.0060 | 1.0 |
| Water–brass | 0.0060 | 1.0 |
| Water–nickel | 0.0060 | 1.0 |
| Water–platinum | 0.0130 | 1.0 |
| *n*-Pentane–copper | | |
| Polished | 0.0154 | 1.7 |
| Lapped | 0.0049 | 1.7 |
| Benzene–chromium | 0.101 | 1.7 |
| Ethylalcohol–chromium | 0.0027 | 1.7 |

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The Rohsenow correlation is valid only for clear surfaces. Any contamination would affect $C_{sf}$ and $n$. Since $q_w \propto (\Delta T_e)^3$ a small error in $\Delta T_e$ causes a large error in $q_w$. Figure 6.20 shows Rohsenow's correlation of Addoms' data [13] for boiling of water using Eq. (6.52). Collier [14] recommends the following correlation which is simpler to use:

$$q_w = 0.000481\ \Delta T_e^{3.33}\ p_{cr}^{2.3}\left[1.8\left(\frac{p}{p_{cr}}\right)^{0.17} + 4\left(\frac{p}{p_{cr}}\right)^{1.2} + 10\left(\frac{p}{p_{cr}}\right)^{10}\right]^{3.33} \tag{6.53}$$

where the excess temperature $\Delta T_e$ is in °C, $p$ is the operating pressure in atm, $p_{cr}$ is the critical pressure in atm and $q_w$ is in W/m$^2$.

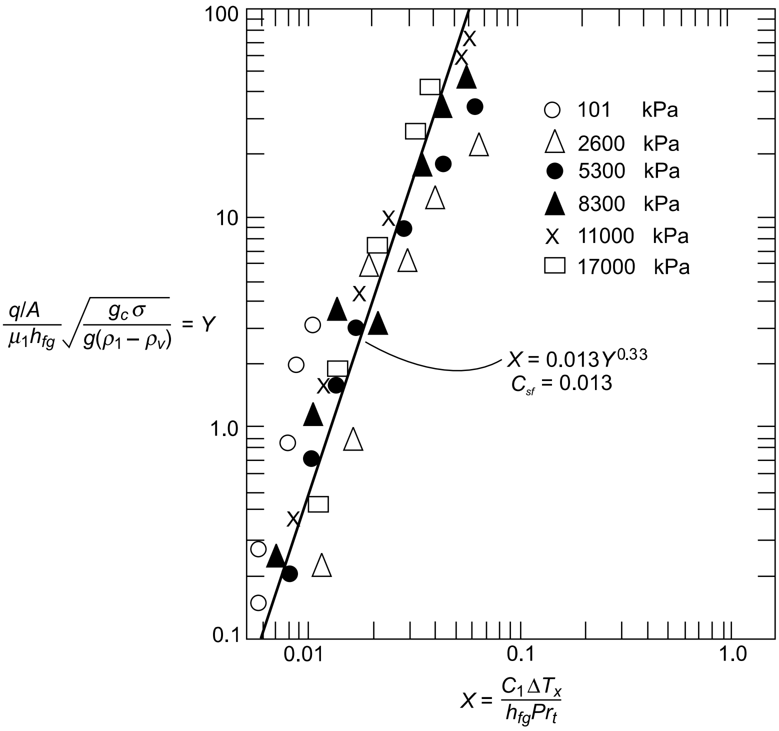


**Fig. 6.20** *Correlation of pool boiling heat transfer data of Addoms by the method of Rohsenow [10]*

### Critical Heat Flux

The critical heat flux (CHF) $q_{w,C} = q_{max}$ represents an important point on the boiling curve. Following Zuber [15], the following expression can be used to estimate $q_{max}$.

$$\mathbf{q_{max}} = \mathbf{0.149\ h_{fg}}\ \rho_v\left[\frac{\sigma g(\rho_1 - \rho_v)}{\rho_v^2}\right]^{1/4} \tag{6.54}$$

which greatly depends on pressure.

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### Minimum Heat Flux

The transition boiling regime is of little practical interest, as it may be obtained only by controlling the surface heater temperature. The onset of stable vapour film and the minimum heat flux condition are important, because if the heat flux drops below this minimum, the film will collapse and nucleate boiling will be re-established. Zuber [15] used stability theory to derive the following expression

$$q_{\min} = C\rho_v\, h_{fg}\left[\frac{g\sigma(\rho_l - \rho_v)}{(\rho_l + \rho_v)^2}\right]^{1/4} \tag{6.55}$$

where $C = 0.09$, as determined by Berenson [17].

### Film Pool Boiling

At excess temperature ($\Delta T_e$) beyond the Leidenfrost point, a continuous vapour film blankets the surface through which heat is conducted. Because of high surface temperatures encountered, film boiling is avoided. The conditions in film boiling are similar to those of laminar film condensation. For film boiling on tubes, Bromley [16] recommends the following correlation due to conduction alone:

$$\mathbf{\bar{h}_c = 0.62\left[\frac{g(\rho_l - \rho_v)\rho_v k_v^3 (h_{fg} + 0.68\, c_{p_v}\, \Delta T_e)}{D\mu_v \Delta T_e}\right]^{1/4}} \tag{6.56}$$

To include radiation from the surface [Berensen (17)],

$$\bar{h}_{\text{total}} = \bar{h}_c + 0.75\,\bar{h}_r \tag{6.57}$$

where $h_r$ is estimated from

$$\bar{h}_r = \sigma\,\varepsilon_s\,\frac{(T_w^4 - T_{\text{sat}}^4)}{T_w - T_{\text{sat}}} \tag{6.58}$$

where $\varepsilon_s$ is the surface emissivity.

## 6.16 FORCED CONVECTION BOILING

In *pool boiling*, fluid flow is mainly due to the buoyancy driven motion of bubbles originating from the heated surface. In *forced convection boiling*, flow is due to directed motion of the fluid as well as due to buoyancy effects. Conditions depend strongly on geometry, which may involve external flow over heated plates and cylinders or internal flow. Internal forced convection boiling is commonly referred to as *two phase flow* and is characterised by rapid changes from liquid to vapour in the flow direction.

### 6.16.1 External Forced Convection Boiling

For external flow over a heated plate, the heat flux can be estimated by standard forced convection correlations upto the start of boiling. As the temperature of the heater plate is increased nucleate boiling will occur, causing the heat flux to increase. If vapour generation is not extensive and the liquid is subcooled, Bergles and Rohsenow [18] suggest a method for estimating the total heat flux in terms of components associated with pure forced convection and pool boiling.

Both forced convection and subcooling increase the CHF for nucleate boiling, which can be as high as 35 MW/m$^2$ compared to 1.3 MW/m$^2$ for pool boiling of water at 1 atm. For liquids flowing in cross flow over a cylinder, Lienhard and Eichorn [19] gave correlations for low and high velocity flows.

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### 6.16.2 Two-Phase Flow

Internal forced convection boiling is associated with bubble formation at the inner surface of a heated tube through which a liquid is flowing. Bubble growth and separation strongly depend on flow velocity. Let us consider flow development in a vertical tube (Fig. 6.21). Heat transfer to the subcooled liquid entering the tube ($T_w > T_{sat}$) is initially by forced convection and may be predicted from the correlations in Chapter 4. However, once boiling is initiated, bubbles appear at the surface and are carried into the mainstream of the liquid. This is known as *bubbly flow*. There is a sharp increase in the convection heat transfer coefficient associated with this bubbly flow regime. As the volume fraction of bubbles increases individual bubbles coalesce to form larger bubbles or slugs of vapour. The volume occupied by bubbles keeps on increasing till a dryness fraction of about 0.1 is reached. The *slug-flow regime* is followed by an *annular-flow regime*

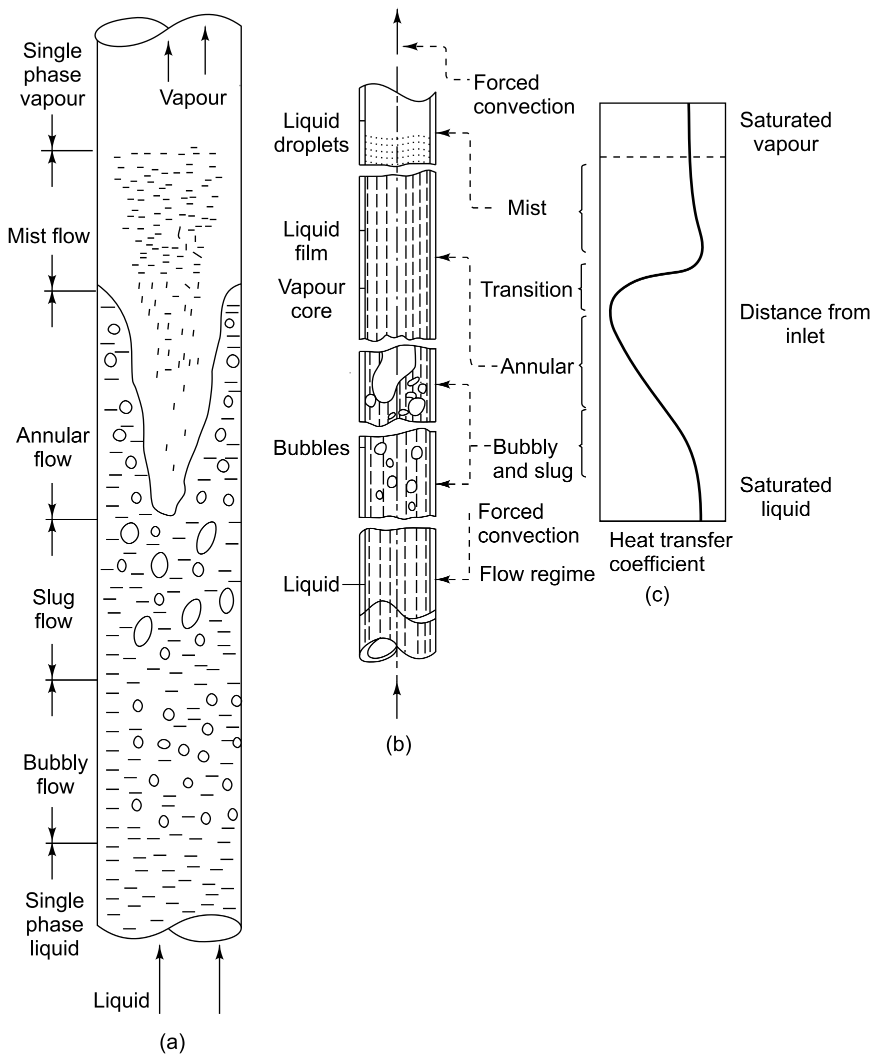


**Fig. 6.21** *Flow regimes for forced convection boiling in a vertical tube*

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in which liquid forms a film. This moves along the inner surface, while vapour moves at a larger velocity through the core of the tube. The heat transfer coefficient continues to increase through the bubbly flow and much of the annular flow regimes. However, *dry spots* eventually appear on the inner surface, at which point $h$ begins to decrease. The *transition regime* is characterised by the growth of dry spots, until the surface is completely dry and all remaining liquid is in the form of droplets appearing in the vapour core. The convection coefficient continues to decrease through this regime. There is little change in this coefficient through the *mist-flow regime* which persists until all the droplets are converted to vapour. The vapour is then *superheated* by forced convection from the surface [20, 21].

## Solved Examples

**Example 6.1** Saturated steam at 54.5°C condenses on the outside surface of a 25.4 mm outer diameter 3.66 m long vertical tube maintained at a uniform temperature of 43.3°C. Because of the occurrence of ripples on the surface of the condensate film the actual heat transfer coefficient is about 20% higher than that obtained by Nusselt's equation. Determine the average condensation heat transfer coefficient over the entire length of the tube and the rate of condensate flow at the bottom of the tube. Check that the flow is laminar. The properties of condensate at 48.9°C are $h_{fg}$ = 2372.4 kJ/kg, $k$ = 0.642 W/mK, $\rho$ = 988.4 kg/m$^3$ and $\mu = 0.558 \times 10^{-3}$ kg/ms.

**Solution** By using Nusselt's equation with 20% excess and neglecting density of vapour,

$$h_m = 1.2 \times 0.943 \left( \frac{k_1^3 \rho_1^2 g h_{fg}}{\mu_1 \theta L} \right)^{1/4}$$

$$= 1.13 \left[ \frac{(0.642)^3 \times (988.4)^2 \times 9.81 \times (2372.4 \times 10^3)}{(0.558 \times 10^{-3})\,(54.5 - 43.3) \times 3.66} \right]^{1/4}$$

$$= 4559 \text{ W/m}^2\text{ K} \quad \textit{Ans.}$$

Condensate flow rate,

$$= \frac{\pi DL\, h_m (T_s - T_w)}{h_{fg}}$$

$$= \frac{\pi \times 25.4 \times 10^{-3} \times 3.66 \times 4559\,(54.5 - 43.3)}{2372.4 \times 10^3}$$

$$= 6.24 \times 10^{-3} \text{ kg/s} \quad \textit{Ans.}$$

Reynolds number,

$$\text{Re} = \frac{4\Gamma}{\mu} = \frac{4 \times 6.24 \times 10^{-3}}{(0.558 \times 10^{-3})\,\pi \times 25.4 \times 10^{-3}}$$

$$= 560.2$$

Since Re < 2000, the flow is laminar and the above calculations are valid.

**Example 6.2** Steam flowing at a rate of 10,000 kg/h and at 120°C is condensed using cooling water at an average temperature of 20°C. The condenser contains 800 tubes which have an outer diameter of 18 mm. Estimate the heat transfer coefficient (a) if the condenser is vertical and (b) if the condenser is horizontal with tubes 3 m long and condensation takes place outside the tubes.

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**Solution** From Eq. (6.34),

$$h_m \left( \frac{\mu_l^2}{k_l^3 \rho_l^2 g} \right)^{1/3} = 1.47 \left( \frac{4\Gamma}{\mu_l} \right)^{-1/3} \tag{1}$$

At the average temperature of (120 + 20)/2 or 70°C, for condensate film $\rho = 977.8$ kg/m$^3$, $k = 0.66$ W/m K and $\mu = 404.7 \times 10^{-6}$ kg/ms.

$$\Gamma = \text{mass rate of flow of condensate per unit length of perimeter}$$

$$= \frac{10,000/3600}{800 \times \pi \times 0.018}$$

$$= 6.14 \times 10^{-2} \text{ kg/s per metre of perimeter.}$$

Reynolds number, $\text{Re} = \dfrac{4\Gamma}{\mu} = \dfrac{4 \times 6.14 \times 10^{-2}}{404.7 \times 10^{-6}} = 607$

Flow in the film is thus laminar and the equation (1) above is valid

$$h_m \left[ \frac{(404.7 \times 10^{-6})^2}{(0.66)^3 (977.8)^2 \times 9.81} \right]^{1/3} = 1.47\ (607)^{-1/3}$$

$$h_m\ (3.931 \times 10^{-5}) = 1.47 \times 0.116$$

$$h_m = \frac{0.174}{3.931 \times 10^{-5}} = 4426 \text{ W/m}^2 \text{ K} \quad \textit{Ans.} \text{ (a)}$$

(b) Horizontal tubes

$$h_m \left( \frac{\mu_l^2}{k_l^3 \rho_l^2 g} \right)^{1/3} = 1.51 \left( \frac{4\Gamma'}{\mu_l} \right)^{-1/3}$$

where $\Gamma'$ is defined as the mass rate of flow per unit length of tube

$$\Gamma' = \frac{10,000/3600}{3 \times 800} = 1.15 \times 10^{-3} \text{ kg/s per metre}$$

$$\frac{4\Gamma'}{\mu} = \frac{4 \times 1.15 \times 10^{-3}}{404.7 \times 10^{-6}} = 11.4$$

The flow is laminar and the equation is valid.

$$h_m \times 3.931 \times 10^{-5} = 1.51 \times (11.4)^{-1/3}$$

$$h_m = \frac{0.670}{3.931 \times 10^{-5}} = 17,048 \text{ W/m}^2 \text{ K} \quad \textit{Ans.} \text{ (b)}$$

This illustrates the advantage of using horizontal rather than vertical condensers.

**Example 6.3** Saturated steam at 110°C condenses on the outside of a bank of 64 horizontal tubes of 25 mm outer diameter, 1 m long arranged in a 8 × 8 square array. Calculate the rate of condensation if the tube surface is maintained at 100°C. The properties of saturated water at 105°C are $\rho$ = 954.7 kg/m$^3$, $k$ = 0.684 W/m$^2$ K, $\mu$ = 271 × 10$^{-6}$ kg/ms and $h_{fg}$ = 2243.7 kJ/kg.

Had the condenser been vertical, what would have been the rate of condensation?

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**Solution** Using Nusselt's equation for condensation on a bank of horizontal tubes,

$$(h_m)_H = 0.725\left(\frac{k_1^3\rho_1^2 g h_{fg}}{N\mu D_0\theta}\right)^{1/4}$$

$$= 0.725\left[\frac{(0.684)^3(954.7)^2\times 9.81\times(2243.7\times 10^3)}{8\times 271\times 10^{-6}\times 25\times 10^{-3}\times 10}\right]^{1/4}$$

$$= 0.725\ (10.432\times 10^3) = 7563.5\ \text{W/m}^2\ \text{K}$$

Rate of condensation, $\omega$

$$= \frac{7563.5\times\pi\times 25\times 10^{-3}\times 1\times 10\times 64}{2243.7\times 10^3}$$

$$= 0.1694\ \text{kg/s}\quad \textit{Ans.}$$

For a vertical condenser,

$$(h_m)_V = 0.943\left(\frac{k_1^3\rho_1^2 g h_{fg}}{\mu_1\theta L}\right)^{1/4}$$

$$= 0.943\left[\frac{(0.684)^3\ (954.7)^2\times 9.81\times 2243.7\times 10^3}{271\times 10^{-6}\times 10\times 1}\right]^{1/4}$$

$$= 0.943\ (69.77\times 10^2) = 6579\ \text{W/m}^2\ \text{K}$$

Rate of condensation for a vertical condenser,

$$= \frac{6579\times 64\times\pi\times 0.025\times 1\times 10}{2243.7\times 1000}$$

$$= 0.147\ \text{kg/s}\quad \textit{Ans.}$$

**Example 6.4** The outer surface of a vertical cylindrical drum of 350 mm diameter is exposed to saturated steam at 2.0 bar for condensation. If the surface temperature of the drum is maintained at 80°C, calculate (i) the length of the drum and (ii) the thickness of the condensate layer to condense 70 kg/h of steam.

**Solution** Given: $D = 0.35$ m, $T_w = 80$°C, $\dot{m} = 70$ kg/h

Corresponding to 2 bar, from steam tables,

$$T_{sat} = 120.2°\text{C},\ \rho_g = \frac{1}{v_g} = \frac{1}{0.885} = 1.13\ \text{kg/m}^3$$

$$h_{fg} = 2201.6\ \text{kJ/kg}$$

The properties of saturated water at the mean film temperature,

$$T_f = \frac{120.2+80}{2} \cong 100°\text{C}$$

$$\rho_f = 956.4\ \text{kg/m}^3,\ k = 68.23\times 10^{-2}\ \text{W/mK},$$

$$\mu = 283\times 10^{-6}\ \text{kg/m-s}$$

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Assuming film condensation and laminar flow, the film thickness at the bottom edge,

$$\delta_L = \left[\frac{4k\mu\theta x}{\rho_f(\rho_f - \rho_g)gh_{fg}}\right]_{x=L}^{1/4}$$

$$\therefore \quad \delta_L = \left[\frac{4\times 68.23\times 10^{-2}\times 283\times 10^{-6}(120.2-80)\times L}{958.4(958.4-1.13)\times 9.81\times 2201.6\times 10^3}\right]^{1/4}$$

$$= 1.988 \times 10^{-4} \times (L)^{1/4}$$

Average heat transfer coefficient is given by

$$\bar{h} = \frac{4}{3}\frac{k_f}{\delta_L} = \frac{4}{3}\frac{68.23\times 10^{-2}}{1.988\times 10^{-4}L^{1/4}}$$

$$= 3432.09\,(L)^{-1/4}$$

Heat transfer rate, $Q$ (using McAdam's correction of 20% excess of $h$)

$$= \bar{h}A_s(T_{sat} - T_w) = \dot{m}\,h_{fg}$$

$$\therefore \quad 1.2 \times 3432.09 \times L^{1/4} \times \pi \times 0.35 \times L \times (120.2 - 80)$$

$$= \frac{70}{3600} \times 2201.6 \times 10^3$$

$$182046.8(L)^{3/4} = 42808.88$$

$$\therefore \quad L = 0.1452 \text{ m} = 145.2 \text{ mm} \quad \textit{Ans. (i)}$$

$$\therefore \quad \delta_L = 1.988 \times 10^{-4} \times (0.1452)^{1/4}$$

$$= 1.227 \times 10^{-4} \text{ m} = 0.1227 \text{ mm} \quad \textit{Ans. (ii)}$$

Let us now check whether the condensate flow is laminar or not.

$$\text{Re} = \frac{4\dot{m}}{\mu\pi D} = \frac{4\times(70/3600)}{2.83\times 10^{-6}\times\pi\times 0.35} \cong 250$$

As $R_e < 1800$, the assumption of laminar flow is correct.

**Example 6.5** A vertical plate 300 mm wide and 1.2 m high is maintained at 70°C and is exposed to saturated steam at 1 atm pressure. Calculate the heat transfer coefficient and the total mass of steam condensed per hour. What would be the heat transfer coefficient if the plate is inclined at 30° to the vertical?

**Solution** The mean film temperature of the condensate $= \dfrac{70+100}{2} = 85°\text{C}$.

Properties of saturated water at 85°C are:

$$\rho_f = 968 \text{ m}^3/\text{kg}, \mu_f = 3.37 \times 10^{-4} \text{ Pa-s},$$
$$k_f = 0.674 \text{ W/mK}, h_{fg} = 2255 \text{ kJ/kg}.$$

$$h_{av} = 0.943\left[\frac{\rho_f(\rho_f - \rho_v)gh_{fg}k_f^3}{L\mu_f(T_{sat} - T_w)}\right]^{1/4}$$

$$= 0.943\left[\frac{k_f^3\rho_f^2 gh_{fg}}{L\mu_f(T_{sat} - T_w)}\right]^{1/4}, \text{ since } \rho_f >> \rho_v.$$

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$$= 0.943 \left[ \frac{(0.674)^3 \times (968)^2 \times 9.81 \times 2255 \times 10^3}{1.2 \times 3.37 \times 10^{-4} \times 30} \right]^{1/4}$$

$$= 4.51 \times 10^3 \text{ W/m}^2\text{K} \quad Ans.$$

For checking

$$\text{Re} = \frac{4\dot{m}}{P\mu},$$

Also, $hA(T_{sat} - T_w) = \dot{m}\, h_{fg}$

$$\therefore \quad \dot{m} = \frac{hA(T_{sat} - T_w)}{h_{fg}}$$

Substituting, if $W$ is the plate width and $L$ its length,

$$\text{Re} = \frac{4hA(T_{sat} - T_w)}{h_{fg} P\mu}, A = WL, P = W$$

$$= 4hL \frac{T_{sat} - T_w}{h_{fg} \cdot \mu}$$

$$= \frac{4 \times 4.51 \times 10^3 \times 1.2 \times 30}{2255 \times 10^3 \times 3.37 \times 10^{-4}} = 856.6$$

Since Re < 1800, the flow is laminar and the equation used is valid.
Heat transfer through the plate, $Q$

$$= hA\,\Delta T = 4.51 \times 10^3 \times (1.2 \times 0.3) \times 30$$

$$= 48.7 \text{ kW}$$

$\therefore$ Mass flow rate of condensate, $\dot{m}$

$$= Q/h_{fg} = \frac{48.7}{2255} \times 3600$$

$$= 77.76 \text{ kg/h} \quad Ans.$$

If the plate is inclined at 30° to the vertical, $\theta = 60°$,

$$\bar{h} = 0.943 \left[ \frac{k_f^3 \rho_f^2 g h_{fg} \sin\theta}{L\mu\Delta T} \right]^{1/4}$$

$$= 0.943 \left[ \frac{(0.674)^3 \times (968)^2 \times 9.81 \times 2255 \times 10^3 \times \sin 60°}{1.2 \times 3.37 \times 10^{-4} \times 30} \right]^{1/4}$$

$$= 4.35 \times 10^3 \text{ W/m}^2\text{K} \quad Ans.$$

**Example 6.6** A tube of 15 mm outside diameter and 1.5 m long is used for condensing steam at 40 kPa. Calculate the average heat transfer coefficient when the tube is (a) horizontal, (b) vertical and its surface temperature is maintained at 50°C.

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**Solution** Saturation temperature at 40 kPa or 0.4 bar is 76°C. The mean film temperature is $\frac{50+76}{2} = 63°C$ and the properties of saturated water are:

$$\rho_l = 980 \text{ kg/m}^3,\ \mu_l = 0.432 \times 10^{-3} \text{ Pa-s},$$

$$k_l = 0.66 \text{ W/mK},\ h_{fg} = 2320 \text{ kJ/kg}, \quad \rho_l >> \rho_v.$$

(a) Horizontal tube

$$\bar{h} = 0.725\left[\frac{k_l^3 \rho_l^2 g h_{fg}}{\mu_l D_o \theta}\right]^{1/4}$$

$$= 0.725\left[\frac{(0.66)^3 \times (980)^2 \times 9.81 \times 2320 \times 10^3}{0.432 \times 10^{-3} \times 1.5 \times 26}\right]^{1/4}$$

$$\cong 10{,}000 \text{ W/m}^2\text{K} \quad \textit{Ans.}$$

(b) Vertical tube

Equation (6.13) should be used for a vertical tube if the film thickness is very small in comparison with the tube diameter.

Film thickness, $$\delta_L = \left[\frac{4\mu k L \Delta T}{g h_{fg} \rho_l^2}\right]^{1/4}$$

$$\therefore \quad \delta_L = \left[\frac{4 \times 0.432 \times 10^{-3} \times 0.66 \times 1.5 \times 26}{9.81 \times 2320 \times 10^3 \times (980)^2}\right]^{1/4}$$

$$= 0.3413 \times 10^{-3} \text{ m} = 0.3413 \text{ mm}$$

Since $\delta_L << 150$ mm, the tube diameter, Eq. (6.26) is used.

$$\frac{(h_m)_H}{(h_m)_V} = 0.77\left(\frac{L}{D}\right)^{1/4}$$

$$\therefore \quad (h_m)_V = \frac{10.000}{0.77\,(150/15)^{0.25}}$$

$$= 7303 \text{ W/m}^2\text{K} \quad \textit{Ans.}$$

Thus, the performance of horizontal tubes for filmwise laminar condensation is better than vertical tubes, thus horizontal tubes are preferred.

**Example 6.7** A square array of 400 tubes 15 mm outer diameter is used to condense steam at atmospheric pressure. The tube walls are maintained at 88°C by a coolant flowing through the tubes. Calculate the amount of steam condensed per hour per unit length of the tubes.

**Solution** Properties of condensate at mean film temperature of $\frac{88+100}{2} = 94°C$ are:

$$\rho_f = 963 \text{ kg/m}^3,\ \mu_f = 3.06 \times 10^{-4} \text{ kg/m-s},$$
$$k_f = 0.678 \text{ W/mK},\ h_{fg} = 2255 \times 10^3 \text{ J/kg}.$$

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A square array of 400 tubes will have $N = 20$.

$$\therefore \qquad \bar{h} = 0.725\left[\frac{k_f^3 \rho_f^2 g h_{fg}}{N\mu_f D_0 \theta}\right]^{1/4}$$

$$= 0.725\left[\frac{(0.678)^3 \times (963)^2 \times 9.81 \times 2255 \times 10^3}{20 \times 3.06 \times 10^{-4} \times 0.015 \times 12}\right]^{1/4}$$

$$= 6.328 \times 10^3 \text{ W/m}^2\text{K}$$

Surface area of 400 tubes, $A_0 = 400 \times 3.1416 \times 0.015 \times 1$

$$= 18.852 \text{ m}^2 \text{ per metre length of tube}$$

$$\therefore \qquad Q = \bar{h}\, A_0\, \Delta T = 6.328 \times 18.852 \times 12 = 1431.56 \text{ kW}$$

$\therefore$ Condensation rate, $\dot{m} = \dfrac{1431.56}{2255} \times 3600$

$= 2285.4$ kg/h per metre length *Ans.*

**Example 6.8** Estimate the power required to boil water in a copper pan, 0.35 m in diameter. The pan is maintained at 120°C by an electric heater. What is the evaporation rate? Estimate the critical heat flux.

**Solution** From Table A.6, for saturated water at 100°C: $\rho_1 = 1/v_f = 957.9$ kg/m$^3$, $\rho_v = 1/v_g = 0.5955$ kg/m$^3$, $c_{p1} = 4.217$ kJ/kg K, $\mu_1 = 279 \times 10^{-6}$ Ns/m$^2$, $\text{Pr}_1 = 1.76$, $h_{fg} = 2257$ kJ/kg and $\sigma = 58.9 \times 10^{-3}$ N/m.

Excess temperature $\Delta T_e = 120 - 100 = 20$°C. According to the boiling curve of Fig. 6.14, nucleate pool boiling will occur and the recommended correlation is given in Eq. 6.52.

$$q_w = \mu_1 h_{fg}\left[\frac{g(\rho_1 - \rho_v)}{\sigma}\right]^{1/2}\left[\frac{c_{p1}\Delta T_e}{C_{sf}\, h_{fg}\, \text{Pr}_1^n}\right]^3$$

The values of $C_{sf}$ and $n$ corresponding to the polished copper surface water combination are taken from Table 6.1, where $C_{sf} = 0.013$ and $n = 1.0$. Substituting numerical values, the boiling heat flux is

$$q_w = 279 \times 10^{-6} \text{ N s/m}^2 \times 2257 \times 10^3 \text{ J/kg}$$

$$\times\left[\frac{9.8 \text{ m/s}^2 (957.9 - 0.5955)\text{kg/m}^3}{58.9 \times 10^{-3} \text{ N/m}}\right]^{1/2} \times \left(\frac{4.217 \times 10^3 \text{ J/kg K} \times 20°\text{C}}{0.013 \times 2257 \times 10^3 \text{ J/kg} \times 1.76}\right)^3$$

$$= 630 \text{ J/m s} \times 39.1\,\frac{1}{m} \times 4.36$$

$$= 1096.25 \text{ k W/m}^2 = 1.096 \text{ MW/m}^2$$

Boiling heat transfer rate

$$Q = q_w \times A = 1096.25 \text{ kW/m}^2 \times \frac{\pi}{4} \times (0.35)^2 \text{ m}^2$$

$= 105.5$ kW *Ans.*

Evaporation rate of water,

$$\omega = \frac{105.5 \times 10^3 \text{ W}}{2257 \times 10^3 \text{ J/kg}} = 0.0467 \text{ kg/s} = 168.3 \text{ kg/h}$$ *Ans.*

The critical heat flux for nucleate boiling can be estimated from Eq. (6.54).

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$$q_{max} = 0.149\, h_{fg}\, \rho_v \left[\frac{g(\rho_l - \rho_l)}{\rho_v^2}\right]^{1/4}$$

$$= 0.149 \times 2257 \times 10^3 \text{ J/kg} \times 0.5955 \text{ kg/m}^3$$

$$\times \left[\frac{58.9\times10^{-3}\,\text{N/m}\times 9.8\,\text{m/s}^2 \times (957.9-0.5955)\,\text{kg/m}^3}{(0.5955)^2\,(\text{kg/m}^3)^2}\right]^{1/4}$$

$$= 1.26 \text{ MW/m}^2$$

Operation of the heater at $q_w = 1.096$ MW/m$^2$ is below the critical heat flux.

**Example 6.9** A metal-clad heating element of 8 mm diameter and emissivity 0.9 is horizontally immersed in a water bath. The surface temperature of the metal is 260°C under steady-state boiling conditions. Estimate the power dissipation per unit length of heater.

**Solution** Properties of water at 100°C are $\rho_l = 957.9$ kg/m$^3$ and $h_{fg} = 2257$ kJ/kg.

Properties of water vapour at (260 + 100)/2 or 180°C are $\rho_v$ = 4.808 kg/m$^3$, $c_{p,\,v}$ = 2.56 kJ/kg K, $k_v = 0.0331$ W/m K and $\mu_v = 14.85 \times 10^{-6}$ N s/m$^2$.

Excess temperature = 260 – 100 = 160°C.

According to the boiling curve of Fig. 6.14, film pool boiling conditions prevail.

Using Bromley's correlation, Eq. (6.55)

$$\bar{h} = 0.62\left[\frac{g(\rho_l - \rho_v)\rho_v\, k_v^3 (h_{fg} + 0.68\, c_{p_v} \Delta T_e)}{D\mu_v \Delta T_e}\right]^{1/4}$$

$$= 0.62\left[\frac{9.8 \text{ m/s}^2 (957.9 - 4.808)\,\text{kg/m}^3 \times 4.808 \text{ kg/m}^3}{1}\right.$$

$$\left.\times \frac{(0.0331)^3 (\text{W/m K})^3 (2257\times10^3 + 0.68\times2.56\times10^3\,\text{J/kg K}\times160°\text{C})}{14.85\times10^{-6}\,\text{Ns/m}^2 \times 8\times10^{-3}\,\text{m}\times160°\text{C}}\right]^{1/4}$$

$$= 0.62\left[\frac{1.629\,(2535.5\times10^3)}{1.9\times10^{-5}}\right]^{1/4} = 423 \text{ W/m}^2\text{ K}$$

$$h_r = \frac{\varepsilon\sigma(T_w^4 - T_{sat}^4)}{T_w - T_{sat}}$$

$$= \frac{0.9\times5.67\times10^{-8}\text{ W/m}^2\text{K}^4 \times (533^4 - 373^4)\text{K}^4}{(533-373)\text{K}}$$

$$= 0.0319\,(807.066 - 193.569) = 19.57 \text{ W/m}^2\text{ K}$$

$$\bar{h} = h_c + \frac{3}{4} h_r = 423 + 0.75 \times 19.57$$

$$= 437.68 \text{ W/m}^2\text{ K}$$

Heat transfer rate per unit length

$$= \bar{h}A\,(T_w - T_{sat}) = 437.68 \times \pi \times 8 \times 10^{-3} \times 160$$

$$= 1.76 \text{ kW/m} \quad \textit{Ans.}$$

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**Example 6.10** It is desired to generate 100 kg/h of saturated steam at 100°C using a heating element of copper of surface area 5 m$^2$. Calculate the convective heat transfer coefficient, the temperature of the heating surface and the critical heat flux.

**Solution** Mass of steam to be evaporated is 100 kg/h. Enthalpy of vaporization at 100°C,

$$h_{fg} = 2255 \text{ kJ/kg}$$

$$Q = 2255 \times 10^3 \times 100/3600 = 62{,}639 \text{ W}$$

$$\therefore \quad Q/A = q_w = \frac{62639}{5} = 12{,}528 \text{ W/m}^2$$

Properties of saturated water at 100°C are

$$\rho_l = 958.4 \text{ kg/m}^3,\ \rho_v = 0.598 \text{ kg/m}^3,$$

$$\mu = 0.283 \times 10^{-3} \text{ Pa-s},\ c_{pl} = 4217 \text{ J/kgK},\ \text{Pr} = 1.75,\ C_{sf} = 0.013,$$

$$\sigma = 58.8 \times 10^{-3} \text{ N/m}.$$

From Eq. (6.52),

$$\Delta T_c = \frac{C_{sf} h_{fg} P_v}{C_{p,l}} \left[ \frac{q_w}{\mu h_{fg}} \sqrt{\frac{\sigma}{g(\rho_l - \rho_v)}} \right]^{0.33}$$

$$= \frac{0.013 \times 2255 \times 10^3 \times 1.75}{4217} \left[ \frac{12528}{0.283 \times 10^{-3} \times 2255 \times 10^3} \times \sqrt{\frac{58.8 \times 10^{-3}}{9.81 \times 957.8}} \right]^{0.33}$$

$$= 4.5°\text{C}$$

$$\therefore \quad T_w = 104.5°\text{C} \quad \textit{Ans.}$$

$$q_w = h\,\Delta T_c = 12528 \text{ W/m}^2$$

$$\therefore \quad h = \frac{12528}{4.5} = 2784 \text{ W/m}^2\text{K} \quad \textit{Ans.}$$

Critical heat flux is obtained from Eq. (6.54)

$$\therefore \quad q_{\max} = 0.149\, h_{fg}\, \rho_v \left[ \frac{\sigma g(\rho_l - \rho_v)}{\rho_v^2} \right]^{1/4}$$

$$= 0.149 \times 2255 \times 10^3 \times 0.598 \left[ \frac{58.8 \times 10^{-3} \times 9.81 \times 957.8}{(0.598)^2} \right]^{1/4}$$

$$= 1259.7 \times 10^3 \text{ W/m}^2 \quad \textit{Ans.}$$

## Summary

Phase change heat transfer processes such as boiling and condensation which have wide applications in industry are introduced. Vapour can condense on a cooled surface in two ways, i.e., dropwise and filmwise. Attention has been given mainly to film condensation. The classical Nusselt analysis of laminar film condensation on a vertical plate has been presented and the extension of this analysis to account for subcooling of condensate, and superheating of vapour has been discussed. Nusselt's approach to the analysis of laminar film condensation on a single horizontal tube as well as a vertical tier of N horizontal tubes is explained.

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In boiling heat transfer the fundamentals of pool boiling, Nukiyama experiment, typical saturated boiling curve for water at atmospheric pressure and various boiling regimes are explained in detail. Nucleate boiling regime is elaborated, Rohsenow's correlation and Zuber's critical heat flux correlation are presented. Flow regimes for forced convection boiling and two phase flow are explained.

## Important Formulae and Equations

| *Equation number* | *Equation* | *Remarks* |
|---|---|---|
| (6.3) | $\mathrm{Nu}_L = f\left[\dfrac{\rho g(\rho_l - \rho_u)^{L^3}}{\mu^2}, \mathrm{Ja}, \mathrm{Pr}, \mathrm{Bo}\right]$ | In condensation or boiling Nusselt number is a function of Jakob number, $\mathrm{Ja} = \dfrac{c_p \Delta T}{\mu_{fg}}$, Bond number $\mathrm{Bo} = \dfrac{g(\rho_l - p_v)L^3}{\sigma}$, Prandtl number $\mathrm{Pr} = \dfrac{c_p \mu}{k}$ and a dimensionless parameter akin to Grashof number = $\dfrac{\rho g(\rho_l - \rho_v)L^3}{\mu^2}$ |
| (6.10) | $\delta(x) = \left[\dfrac{4k\mu_l\theta_x}{g\rho_l(\rho_l - \rho_v)h_{fg}}\right]^{1/4}$ | Local film thickness of condensate layer on a vertical plate or cylinder |
| (6.11) | $h_x = \left[\dfrac{k_l^3 \rho_l(\rho_l - \rho_v)h_{fg}}{4\mu_l\theta_x}\right]^{1/4}$ | Local heat transfer coefficient of condensate film on a vertical plate |
| (6.13) | $h_m = 0.943\left[\dfrac{k_l^3 \rho_l(\rho_l - \rho_v)gh_{fg}}{\mu_l\theta L}\right]^{1/4}$ | Average heat transfer coefficient of condensate film on a vertical plate |
| (6.14) | $h_m = 0.943\left[\dfrac{k_l^3 \rho_l^2 gh_{fg}}{\mu_l\theta L}\right]^{1/4}$ | Same as above when $\rho_l >> \rho_v$ |
| (6.20) | $T_S - T_B = \dfrac{3}{8}\theta$ | Degree of subcooling of condensate, where $T_S = T_{\text{sat}}$, $T_B$ = bulk temperature of condensate and $\theta = T_s - T_w$, $T_w$ = wall temperature |
| (6.22) | $h'_{fg} = h_{fg} + \dfrac{3}{8}c_{p_e}\theta$ | Average enthalpy change during condensation with subcooling |
| (6.24) | $h_m = 0.943\left[\dfrac{k_l^3 \rho_l(\rho_l - \rho_v)gh_{fg}(1 + 0.68\,\mathrm{Ja})}{\mu_l\theta L}\right]^{1/4}$ | Rohsenow's correlation for condensation on a vertical plate with subcooling where Ja = Jakob number = $\dfrac{c_{pl}(T_s - T_w)}{h_{fg}}$ |

(*Contd*)

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| Equation number | Equation | Remarks |
|---|---|---|
| (6.25) | $h_m = 0.725\left[\dfrac{k_l^3\rho_l(\rho_l-\rho_v)gh_{fg}}{\mu_l D_0\theta}\right]^{1/4}$ | Nusselt's equation for film condensation on a horizontal tube |
| (6.26) | $\dfrac{(h_m)_H}{(h_m)_v} = 0.77\left(\dfrac{L}{D_0}\right)^{1/4}$ | Ratio of average heat transfer coefficients on a horizontal tube and vertical tube. For $L/D_0 = 2.87$, $(h_m)_H = (h_m)_V$ |
| (6.27) | $h_m = 0.725\left[\dfrac{k_l^3\rho_l(\rho_l-\rho_v)gh'_{fg}}{\mu_l ND_0\theta}\right]^{1/4}$ | Average $h$ transfer coefficient of condensate film on a vertical tier of horizontal tubes |
| (6.28) | $h_m = 0.815\left[\dfrac{k_l^3\rho_l(\rho_l-\rho_v)gh'_{fg}}{\mu_l D\theta}\right]^{1/4}$ | Condensation of vapour on a sphere |
| (6.34) | $h_m\left(\dfrac{\mu_l^2}{k_l^3\rho_l^2 g}\right)^{1/3} = 1.47\,\mathrm{Re}^{-1/3}$ | Heat transfer coefficient as a function of Reynolds number, Re $= \dfrac{4\Gamma}{\mu}$, where $\Gamma$ = rate of condensation per unit width |
| (6.35) | $\mathrm{Co} = h_m\left[\dfrac{\mu_l^2}{k_l^3\rho_l(\rho_l-\rho_v)g}\right]^{1/3}$ $= 1.47\,\mathrm{Re}^{-1/3}$ | Condensation number, Co, for a vertical plate |
| (6.37) | $h_m\left(\dfrac{\mu_l^2}{k_l^3\rho_l^2 g}\right)^{1/3} = 1.51\,\mathrm{Re}^{-1/3}$ | Eq. (6.25) expressed in terms of Reynolds number |
| (6.38) | $h = 0.056\left(\dfrac{4\Gamma}{\mu_l}\right)^{0.2}\left(\dfrac{K_l^3\rho_l^2 g}{\mu_{l^2}}\right)^{1/3}(\mathrm{Pr}_l)^{1/2}$ | Colburn's relation for turbulent film condensation (Re > 2000) |
| (6.52) | $q_w = \mu_l h_{fg}\left[\dfrac{g(\rho_l-\rho_v)}{\sigma}\right]^{1/2}\left[\dfrac{c_{p_l}\Delta T_e}{c_{sf}h_{fg}\,\mathrm{Pr}_l^n}\right]^3$ | Rohsenow's correlation to predict the heat flux for nucleate pool boiling |
| (6.54) | $q_{\max} = 0.149 h_{fg}\rho_v\left[\dfrac{\sigma g(\rho_l-\rho_v)}{\rho_v^2}\right]^{1/4}$ | Critical heat flux on the boiling curve |
| (6.55) | $\bar{h}c = 0.62\left[\dfrac{g(\rho_l-\rho_v)\rho_v k_v^3(h_{fg}+0.68cpv\Delta Te)}{D\mu_v\Delta Te}\right]^{1/4}$ | Bromley's correlation for film boiling on tubes |
| (6.56) | $\bar{h}_{\text{total}} = \bar{h}_c + 0.75\bar{h}_r$ where $\bar{h}_r = \dfrac{\sigma\varepsilon_s(T_w^4 - T_{\text{sat}}^4)}{T_w - T_{\text{sat}}}$ | Film boiling including radiation |

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## Review Questions

6.1 Why are heat transfer rates high for a phase change process?

6.2 What are the five significant dimensionless numbers in boiling and condensation?

6.3 Explain the physical significance of Jakob number and Bond number.

6.4 What are the two modes in which condensation can take place on a cooling surface? What is film condensation?

6.5 Explain the conditions under which dropwise condensation can take place. Why is the rate of heat transfer in dropwise condensation many times larger than in filmwise condensation?

6.6 What is a promoter? Why does its effectiveness decay with time?

6.7 State the assumptions made in deriving Nusselt's equation for film condensation.

6.8 Explain how the condensate film thickness on a vertical plate is influenced by different parameters.

6.9 What is bulk temperature of the condensate? Show that it is subcooled and is less than the saturation temperature by $3\theta/8$, where $\theta = T_s - T_B$.

6.10 What is the effect of inclination of the tube or plate on the average condensation heat transfer coefficient?

6.11 How does the Nusselt's equation for condensation on a horizontal tube differ from that on a vertical tube?

6.12 Explain why the condenser tubes are usually horizontal.

6.13 Why does the mean heat transfer coefficient in a condenser decrease if the number of horizontal tubes in a vertical tier ($N$) increases?

6.14 What is condensation number? How is it related with Reynolds number for condensation on (a) a vertical tube and (b) a horizontal tube?

6.15 What is the effect of turbulence on condensation heat transfer coefficient? Why is turbulent flow of condensate hardly ever reached on a horizontal tube?

6.16 Explain the effect of high vapour velocity on the rate of condensation.

6.17 Why is the effect of superheat not significant on condensation heat transfer coefficient?

6.18 Discuss the effect of a noncondensable gas like air on the rate of condensation of steam. How is air removed continuously from the condenser shell in a power plant?

6.19 What do you mean by subcooled boiling? What is saturated boiling?

6.20 What is pool boiling? How is forced convection boiling different from pool boiling?

6.21 State the regimes of pool boiling.

6.22 Draw and explain the Farber–Scorah boiling curve.

6.23 What is nucleate boiling? Why is it important?

6.24 What is excess temperature? What do you mean by ONB and DNB?

6.25 What is critical heat flux? What is its importance?

6.26 Explain partial or unstable film boiling.

6.27 What is Leidenfrost point? What is its significance?

6.28 Explain film boiling. Why is it avoided? What is boiling crisis?

6.29 What are the two separate processes of nucleate boiling?

6.30 State the two conditions which are required to be fulfilled for bubbles to form.

6.31 Why do bubbles form on the heating surface?

6.32 When does a bubble grow or collapse as it moves up through the liquid?

6.33 What are nucleation sites? What are active cavities?

6.34 Why is less liquid superheat required for bubble formation when a noncondensable gas is present in the bubble?

6.35 What is Rohsenow's correlation in nucleate boiling? On what does the coefficient $C_{sf}$ depend?

6.36 Explain the flow regimes in two phase flow through a tube. What is the difference between slug-flow regime and annular flow regime.

6.37 What are dry spots? What is mist flow regime?

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## Objective Type Questions

6.1 The ratio of gravitational body force to the surface tension force is called
(a) Bond number (b) Weber number
(c) Grashof number (d) Jakob number

6.2 The rate of heat transfer is many time larger in this mode of condensation:
(a) Filmwise condensation
(b) Dropwise condensation
(c) Mixed condensation
(d) Forced convection condensation

6.3 Consider the following statements:
1. If a condensing liquid wets a surface, dropwise condensation takes place.
2. Dropwise condensation has a higher heat transfer coefficient than filmwise condensation.
3. Suitable coating is used to promote filmwise condensation.
4. Reynolds number of condensing liquid is calculated on its mass flow rate.

Of these statements.
(a) 1, 2, 3 and 4 are correct
(b) 2 and 4 are correct
(c) 2 and 3 are correct
(d) 1 and 4 are correct

6.4 During filmwise condensation on a vertical surface, as the distance $x$ from the leading edge increases,
(a) the film thickness $\delta$ decreases and the heat transfer coefficient $h$ increases
(b) both $\delta$ and $h$ increase
(c) $\delta$ increases and $h$ decreases
(d) both $\delta$ and $h$ decrease

6.5 The local condensing heat transfer coefficient $h_x$ on a vertical surface varies with the distance $x$ from the leading edge as
(a) $h_x \propto x^{1/4}$ (b) $h_x \propto x^{-1/4}$
(c) $h_x \propto x^{1/3}$ (d) $h_x \propto x^{2/3}$

6.6 The mean condensing heat transfer coefficient on a vertical tier of $N$ horizontal tubes varies with
(a) $N^{-1}$ (b) $N^{-1/2}$
(c) $N^{-1/3}$ (d) $N^{-1/4}$

6.7 It is a measure of the importance of subcooling in condensing heat transfer
(a) Bond number (b) Grashof number
(c) Jakob number (d) Colburn number

6.8 **Assertion (A):** The rate of condensation on a rusty surface is more than that on a polished surface.
**Reasoning (R):** The polished surface promotes dropwise condensation.
**Codes:**
(a) Both A and R are true
(b) A is true, R is false
(c) Both A and R are false
(d) A is false, R is true

6.9 **Assertion (A):** The rate of heat transfer drops significantly when the condensing vapour contains non-condensable gases.
**Reasoning (R):** The non-condensable gases serve as an obstacle to the condensing vapour reaching the surface.
**Codes:**
(a) Both A and R are true
(b) Both A and R are false
(c) A is true, R is false
(d) A is false, R is true

6.10 **Assertion (A):** Surface condensers are designed on the basis that filmwise condensation always exists.
**Reasoning (R):** It is not possible to have dropwise condensation on the surface continuously at all times.
**Codes:**
(a) Both A and R are false
(b) Both A and R are true
(c) A is true, R is false
(d) A is false, R is true

6.11 The mean condensing heat transfer coefficint on a horizontal tube of diameter $D_0$ varies with
(a) $D_0^{-1}$ (b) $D_0$
(c) $D_0^{-1/2}$ (d) $D_0^{-1/4}$

6.12 For filmwise condensation the heat transfer coefficient will be equal whether the tube is horizontal or vertical, when the ratio of length to diameter is
(a) 1.3 (b) 2.87
(c) 0.78 (d) > 5

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6.13 The condensate flow is almost always laminar
(a) on a vertical tube
(b) on a horizontal tube
(c) on a vertical bank of horizontal tubes
(d) on a cluster of vertical tubes.

6.14 When a vapour condenses on a horizontal or vertical surface, the bulk temperature of the condensate is always
(a) equal to the saturation temperature at the vapour pressure
(b) greater than the saturation temperature
(c) less than the saturation temperature
(d) equal to the surface temperature

6.15 Consider the following statements regarding condensation heat transfer:
1. For a single tube, the horizontal position is preferred to the vertical position for better heat transfer.
2. Heat transfer coefficient decreases with increasing vapour velocity.
3. Condensation of steam on an oily surface is dropwise.
4. Condensation of pure benzene vapour is always dropwise.
Of these statements,
(a) 2 and 4 are correct
(b) 1 and 3 are correct
(c) 1 and 2 are correct
(d) 3 and 4 are correct

6.16 In some designs of evaporators, the heating surface is submerged beneath a free surface of liquid. This is known as
(a) pool boiling (b) bulk boiling
(c) convection boiling (d) saturated boiling

6.17 When a liquid flows through a tube with subcooled or saturated boiling, the process is
(a) pool boiling
(b) convection boiling
(c) forced convection boiling
(d) bulk boiling

6.18 In boiling heat transfer, the most significant regime of boiling is
(a) convection boiling
(b) film boiling
(c) nucleate boiling
(d) transition boiling

6.19 Consider the following statements regarding boiling heat transfer:
1. The peak of the boiling curve indicates the critical heat flux and the burnout point.
2. This point also indicates the onset of departure from nucleate boiling (DNB).
3. It is desird to operate the heat transfer surface close to this value, but it is dangerous to exceed it.
4. Nucleate boiling is followed by film boiling when heat transfer is maximum.
Of these statements,
(a) 1, 2 and 4 are correct
(b) 3 and 4 are correct
(c) 1, 2 and 3 are correct
(d) all are correct

6.20 Nucleation or bubble formation in nucleate boiling requires liquid to be
(a) superheated
(b) saturated
(c) subcooled
(d) in the boiling regime

6.21 In nucleate boiling to form the nuclei of bubbles there must be present
(a) superheated vapour
(b) dissolved gases
(c) undissolved gases
(d) superheated liquid

6.22 In nucleate boiling, bubbles always originate
(a) in the bulk of the liquid
(b) on the heating surface
(c) where the liquid superheat is mininum
(d) where the liquid is saturated

6.23 **Assention (A):** In forced convection boiling inside a tube, the heat transfer coefficient increases sharply in the bubble flow regime.
**Reasoning (R):** Because the bubbles appearing at the surface grow and are carried away into the liquid stream.
**Codes:**
(a) A is true, R is false
(b) Both A and R are false
(c) Both A and R are true
(d) A is false, R is true

6.24 Cavities on the heating surface are used most advantageously when the entire surface is exposed to
(a) nucleate boiling (b) film boiling
(c) transition boiling (d) pool boiling

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## Answers

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 6.1 | (a) | 6.2 | (b) | 6.3 | (b) | 6.4 | (c) | 6.5 | (b) |
| 6.6 | (d) | 6.7 | (c) | 6.8 | (d) | 6.9 | (a) | 6.10 | (b) |
| 6.11 | (d) | 6.12 | (b) | 6.13 | (b) | 6.14 | (c) | 6.15 | (b) |
| 6.16 | (a) | 6.17 | (c) | 6.18 | (c) | 6.19 | (c) | 6.20 | (a) |
| 6.21 | (b) | 6.22 | (b) | 6.23 | (c) | 6.24 | (a) | | |

## Open Book Problems

6.1 Air-free saturated steam at $T_s$ = 65ºC ($p$ = 25.03 kPa) condenses on the outer surface of a 2.5 cm OD, 3 m long vertical tube maintained at a uniform temperature $T_w$ = 35ºC by the flow of cooling water through the tube. Assuming film condensation, calculate the average heat transfer coefficient over the entire length of the tube and the rate of condensation.

*Hints:* At the film temperature of condensate (water), $T_f = \dfrac{65+35}{2}$ = 50ºC, find the properties $k_l$, $\mu_l$, $\rho_l$, $h_{fg}$ and use Eq. (6.15).

$$h_m = 1.13\left(\frac{k_l^3 \rho_l^2 g h_{fg}}{\mu_c.\theta.L}\right)^{1/4} \quad \text{where } \theta = T_s - T_w.$$

to find $h_m$. The mass flow rate of condensate

$$\dot{m} = \frac{h_m \pi d L (T_s - T_w)}{h_{fg}}$$

Check the validity of laminar flow of condensate by showing

$$\text{Re} = \frac{4\dot{m}}{\pi d \mu_l} < 1800$$

6.2 Determine the average heat transfer coefficient and the total condensation rate for the above example if the tube is horizontal.

*Hints:* For $\rho_v << \rho_l$, use Eq. (6.25) to find $h_m$.

$$h_m = 0.725\left[\frac{k_l^3 \rho_l^2 g h_{fg}}{\mu_l \theta d}\right]^{1/4}$$

Condensation rate, $\dot{m} = \dfrac{\pi d L\, h_m (T_s - T_w)}{h_{fg}}$

Check $\text{Re}_L < 1800$.

6.3 Air free saturated steam at $T_s$ = 65ºC condenses on the surface of a vertical tube with on OD of 2.5 cm which is maintained at a uniform temperature of 35ºC. Determine the tube length $L$ for a condensate flow rate of $6 \times 10^{-3}$ kg/s per tube.

*Hints:* Properties of water at $T_f$ = 50ºC are the same as in the previous examples. Find $\text{Re} = \dfrac{4\dot{m}}{\pi d \mu_l} < 1800$ to prove that the condensate flow is laminar. Use Eq. (6.34) to find $h_m$ increased by 20% as recommended by McAdams.

$$h_m \left(\frac{\mu_l^2}{k_l^3 \rho_l^2 g}\right)^{1/3} = 1.47(\text{Re})^{-1/3} \times 1.20$$
$$= 1.76\,\text{Re}^{-1/3}$$

$L$ is calculated from the equation $\dot{m} = \dfrac{h_m \pi d L (T_s - T_w)}{h_{fg}}$

6.4 Air-free saturated steam at $T_s$ = 85ºC ($p$ = 57.83 kPa) condenses on the outer surface of 225 horizontal tubes of 1.27 cm OD arranged in a 15 × 15 square array. Tube surfaces are maintained at $T_w$ = 75ºC. Calculate the condensation rate per metre length of the tube bundle.

*Hints:* Physical properties of water are obtained at 80ºC from the Appendix. Use Eq. (6.27) to find $h_m$ for a vertical tier of 15 horizontal tubes.

$$h_m = 0.725\left[\frac{k_l^3 \rho_l^2 g h_{fg}}{N \mu_l.\theta.d}\right]^{1/4}, \text{ where } N = 15.$$

Surface area of 225 tubes, $A_o = N\pi dL$, where $L$ = 1 m, $N$ = 225.

$$Q = h_m A_0 (T_s - T_w) = \dot{m} \times h_{fg}$$

∴ $m$ can be found out from above.

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## Problems for Practice

6.1 Saturated steam at 1.46 bar and 110°C condenses on a 25 mm outer diameter vertical tube which is 50 cm long. The tube wall is maintained at 100°C. Calculate the average heat transfer coefficient and the rate of condensation. Check that the condensate flow is laminar. The properties of the condensate at 150°C are $k = 0.68$ W/m K, $\rho = 954.7$ kg/m$^3$, $\mu = 0.271 \times 10^{-3}$ kg/ms and $h_{fg} = 2243.7$ kJ/kg.
(*Ans.* 7789.33 W/m$^2$ K, $1.36 \times 10^{-3}$ kg/s)

6.2 Air-free saturated steam at 65°C condenses on the outer surface of a 25 mm outer diameter 3 m long vertical tube maintained at a uniform temperature of 35°C by flow of cooling water through the tube. Assuming film condensation and 20% in excess of Nusselt's value calculate the average heat transfer coefficient over the entire length of the tube and the rate of condensate flow at the bottom of the tube. Confirm that the flow is laminar. (*Ans.* 3729 W/m$^2$ K, $11.23 \times 10^{-3}$ kg/s, Re = 1018)

6.3 Determine the average heat transfer coefficient and the total condensation rate for problem 6.2 when the tube is horizontal.
(*Ans.* 7918 W/m$^2$ K, $23.86 \times 10^{-3}$ kg/s)

6.4 Saturated air-free steam at 75°C condenses on a 0.5 m × 0.5 m vertical plate maintained at a uniform temperature of 45°C. Calculate (a) the average heat transfer coefficient over the entire length of the plate, (b) the total rate of condensation and (c) the thickness of the condensate at the bottom of the plate.
(*Ans.* (a) 6142 W/m$^2$ K, (b) $1.953 \times 10^{-2}$ kg/s, (c) 17 mm)

6.5 Saturated air-free steam at 85°C condenses on the outer surface of 225 horizontal tubes of 12.7 mm outer diameter arranged in a 15 × 15 array. Tube surfaces are maintained at 75°C. Calculate the total condensation rate per metre length of the tube bundle.
(*Ans.* $h_m = 7150$ W/m$^2$ K, $Q/L = 697.84$ kW/m, W/L = 0.3 kg/sm)

6.6 Saturated steam at atmospheric pressure condenses on a horizontal copper tube of 25 mm inner diameter and 29 mm outer diameter through which water flows at the rate of 25 kg/min entering at 30°C and leaving at 70°C. Making necessary assumptions, calculate (a) the condensing heat transfer coefficient, (b) the inside heat transfer coefficient and (c) the length of the tube. (*Ans.* (a) 11,489.0 W/m$^2$ K, (b) 4618.2 W/m$^2$ K, (c) 5.47 m)

6.7 Saturated water at 100°C is boiled with a copper heating element having a heating surface of area 0.04 m$^2$ which is maintained at 15°C. Calculate the surface heat flux and the rate of evaporation.
(*Ans.* 484 kW/m$^2$, 30.9 kg/h)

6.8 In the above problem, if the heating element were made of brass instead of copper, what would be the heat flux at the surface of the heater? (*Ans.* 4930 kW/m$^2$)

6.9 Water at atmospheric pressure is boiling on a mechanically polished stainless steel surface that is heated electrically from below. Determine the heat flux from the surface to the water when the surface temperature is 106°C, and compare it with the critical heat flux for nucleate boiling. Repeat for the case of water boiling on a Teflon-coated stainless steel surface. (*Ans.* 29.29 kW/m$^2$, $q_{max} = 1.10^7$ MW/m$^2$, 345.3 kW/m$^2$)

6.10 During the boiling of saturated water at 100°C with an electric heating element, a heat flux of 500 kW/m$^2$ is achieved with a temperature difference of 9.3°C. What is the value of the coefficient $C_{sf}$ for the heater surface? (*Ans.* 0.008)

6.11 Repeat Problem 6.9 using a surface temperature of 400°C for the mechanically polished stainless steel surface.
(*Ans.* 44.74 kW/m$^2$)

6.12 Water at saturation temperature and atmospheric pressure is boiled with an electrically heated, horizontal platinum wire

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of 1.27 mm diameter. Determine the boiling heat transfer coefficient and the heat flux for a temperature difference $\Delta T_e$ = 650°C.

(*Ans.* 368.2 W/m$^2$ K, 239.33 kW/m$^2$)

6.13 An electrically heated copper kettle with a flat bottom of diameter 25 cm is to boil water at atmospheric pressure at a rate of 2.5 kg/h. What is the temperature of the bottom surface of the kettle? (*Ans.* 106.1°C)

6.14 An electrically heated, copper, spherical heating element of diameter 10 cm is immersed in water at atmospheric pressure and saturation temperature. The surface of the element is maintained at a uniform temperature at 115°C. Calculate (a) the surface heat flux, (b) the rate of evaporation and (c) the peak heat flux.

(*Ans.* 484 kW/m$^2$, (b) 24.3 kg/h, (c) 0.933 MW/m$^2$)

6.15 Water at saturation temperature and atmospheric pressure is boiled in the stable film boiling regime with an electrically heated, horizontal platinum wire of diameter 1.27 mm. Calculate the surface temperature necessary to produce a heat flux of 150 kW/m$^2$.

6.16 Calculate the heat transfer coefficient during stable film boiling of water from 9 mm diameter horizontal carbon tube. The water is saturated at 100°C and the tube surface is at 1000°C. Assume the emissivity of carbon surface to be 0.8. Properties of steam are $\rho_v$ = 0.266 kg/m$^3$, $\mu_v$ = 28.7 × 10$^{-6}$ kg/ms and $k_v$ = 0.0616 W/mk.

(*Ans.* 283.8 W/m$^2$ K)

6.17 Saturated water at 100∞C flows through a 20 mm diameter copper tube with an average velocity of 2 m/s. The tube well is maintained at 111°C. Calculate the heat flux, assuming nucleate boiling. Take $\rho_v$ = 0.6 kg/m$^3$

(*Ans.* 326 kW/m$^2$)

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# Radiation Heat Transfer

# 7

Thermal radiation refers to the radiant energy emitted by bodies by virtue of their own temperatures, resulting from the thermal excitation of the molecules. Radiation is assumed to propagate in the form of electromagnetic waves. The assumption of wave nature of radiation can explain the phenomena of diffraction, interference and polarisation of light. Radiation is also assumed to be emitted in discrete quanta called photons, each quantum of energy being $h\nu$, where $h$ is Planck's constant, $6.625 \times 10^{-34}$ Js, and $\nu$ is the frequency. The particle nature of radiation can explain the Compton effect in which a photon striking an electron changes its trajectory. Thermal radiation thus exhibits the wave–particle duality [1], behaving both like particles (photons which have no rest mass, but possess momenta) and like waves.

## 7.1 THERMAL RADIATION

Thermal radiation is generally described in terms of electromagnetic waves, all of which travel at the velocity of light. The wavelength and frequency of radiation propagating in a medium are related by

$$c = \nu\lambda$$

where $c$ is the velocity of light in the medium,
$\nu$ is the frequency and,
$\lambda$ is the wavelength.

When the medium in which radiation travels is vacuum, the velocity of propagation is $c = 2.9979 \times 10^8$ m/s.

The wavelength is expressed in micron $\mu$ or angstrom Å:

1 micron = 1 micrometer = 1 $\mu$m = $10^{-6}$ m

1Å = $10^{-10}$ m

Electromagnetic radiation of all kinds is similar in nature, and is differentiated only by wavelength as shown in Table 7.1.

**Table 7.1** *Classification of electromagnetic radiation by wavelength*

| *Kind of radiation* | *Wavelength* |
|---|---|
| Cosmic rays | upto $4 \times 10^{-7}$ $\mu$m |
| Gamma rays | $4 \times 10^{-7}$ to $1.4 \times 10^{-4}$ $\mu$m |
| X-rays | $1 \times 10^{-5}$ to $2 \times 10^{-2}$ $\mu$m |
| Ultraviolet rays | 0.02 to 0.4 $\mu$m |
| Visible radiation | 0.4 to 0.8 $\mu$m |
| Thermal (infrared) radiation | 0.8 $\mu$m to 800 $\mu$m or 0.8 mm |
| Heat rays | 0.4 $\mu$m to 800 $\mu$m |
| Radio waves | 0.2 mm to $2 \times 10^{10}$ $\mu$m or 20 km |

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The electromagnetic spectrum (Fig. 7.1) covering a wide range of wavelengths from less than $10^{-10}$ µm for cosmic rays to more than $10^{10}$ µm for electrical power waves includes gamma rays, X-rays, ultraviolet radiation, visible light, infrared radiation, thermal radiation, microwaves and radio waves. The type of electromagnetic radiation that is pertinent to heat transfer is the thermal radiation emitted as a result of vibration and rotational motions of molecules, atoms and electrons of a substance, and higher the temperature, higher is the rate of emission. It is the portion of electromagnetic spectrum that extends from 0.1 µm to 100 $\mu$m wavelength range. Thus, it includes the entire visible and infrared radiation as well as a portion of the ultraviolet radiation.

The visible portion of the electromagnetic spectrum lying between 0.4 µm and 0.76 µm is what we call light, which triggers the sensation of vision in the human eye. Light consists of narrow bands of colour from violet (0.40 – 0.44 µm) to red (0.63 – 0.76 µm), as shown in Table 7.2. A surface that reflects radiation in the wavelength range 0.63 – 0.76 µm and absorbs the rest of visible radiation appears red to the eye. A surface that reflects all of the light appears white, while a surface that absorbs all the incident radiation appears black.

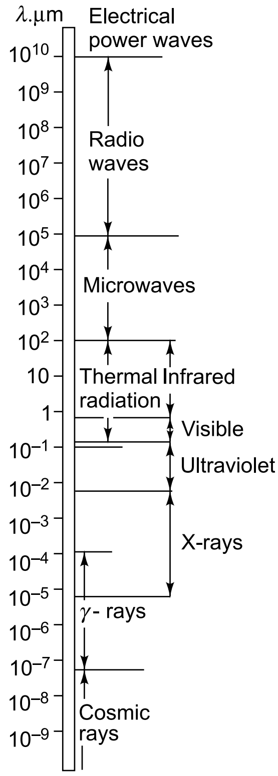


**Fig. 7.1** *Electromagnetic wave spectrum*

**Table 7.2** *The wavelength ranges of different colours*

| *Colour* | *Wavelength band* |
|---|---|
| Voilet | 0.40–0.44 $\mu$m |
| Blue | 0.44–0.49 $\mu$m |
| Green | 0.49–0.54 $\mu$m |
| Yellow | 0.54–0.60 $\mu$m |
| Orange | 0.60–0.63 $\mu$m |
| Red | 0.63–0.76 $\mu$m |

A body that emits some radiation in the visible range is called a light source. The sun is our primary light source. The electromagnetic radiation emitted by the sun is called the solar radiation which falls in the wavelength band 0.1 – 3 µm. About half of solar radiation is light (visible range), while the rest is ultraviolet and infrared.

Radiation emitted by bodies at room temperature falls into the infrared region extending from 0.76 to 100 µm. Bodies start emitting visible radiation at temperatures above 800 K. The tungsten filament in the light bulb heated to temperatures above 2000 K emits radiation significantly in the visible range.

The ultraviolet radiation in wavelengths varying from 0.01 to 0.40 µm forms about 12% of solar radiation and is highly pernicious, killing microorganisms and causing serious damage to humans and other living organisms. Fortunately, the ozone layer in the atmosphere absorbs most of this ultraviolet radiation.

Microwave ovens utilise electromagnetic radiation in the microwave region of the spectrum, $10^2 – 10^5$ $\mu$m, generated by microwave tubes called magnetrons. These microwaves are very suitable for use in cooking, since they are reflected by metals, transmitted by glass and plastics, and absorbed by food (especially water) molecules. The electric energy converted to radiation in a microwave oven eventually becomes part of the internal energy of the food. This hastens cooking which becomes more efficient and less power-consuming.

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Radars and cordless telephones also use electromagnetic radiation in the microwave region. The electromagnetic waves used in radio and TV broadcasting varying in wavelengths between 1 m and 1000 m fall in the radio wave region of the spectrum.

Radiation emitted from heated bodies comprises rays with wavelengths 0.4 – 800 $\mu$m whereas the visible range is from 0.4 – 0.8 $\mu$m. The large part of thermal radiation between 0.8 and 800 $\mu$m which has waves outside the visible range is called the *infrared* radiation.

Unlike conduction and convection, radiation does not require any intervening medium for energy propagation. In vacuum, it is the only mode of heat transfer. Life on this planet is supported and sustained by the radiant energy streaming out from the sun, $150 \times 10^6$ km away.

If the intervening space between two bodies is vacuum, the radiative energy transfer between them occurs with the speed of light. If an absorbing medium intervenes the bodies, the net radiative transfer is reduced, since the medium itself absorbs some energy. The study of radiation energy transfer is thus separated into two distinct parts:

1. Radiation energy transfer in a *nonparticipating medium* where the medium is either vacuum or does not interfere with the energy propagation.
2. Radiation energy transfer in a *participating medium*, which intervenes with its propagation by absorbing, emitting or scattering energy. Layers of certain gases and vapours which absorb and emit radiation at certain ranges of wavelengths would be considered as the participating medium.

## 7.2 PREVOST'S THEORY

All bodies emit thermal radiation, unless the body is at absolute zero temperature. If a body is placed in a surrounding at the same temperature as itself, its temperature does not change, since the rate at which energy is radiated is equal to the rate at which energy is received from the surroundings. When a hot body is placed in cooler surroundings, the rate at which it radiates is faster than the rate at which it absorbs the incident radiation, as a result of which its temperature decreases till the state of thermal equilibrium is reached. This is known as *Prevost's theory of heat exchange*.

## 7.3 ABSORPTIVITY, REFLECTIVITY AND TRANSMISSIVITY

Matter can emit, absorb, reflect and transmit radiant energy. If $Q$ is the total radiant energy incident upon the surface of a body, some part of it ($Q_A$) will be absorbed, some part ($Q_R$) reflected and some part ($Q_{Tr}$) transmitted through the body (Fig. 7.2). By energy balance,

$$Q_A + Q_R + Q_{Tr} = Q$$

or,

$$\frac{Q_A}{Q} + \frac{Q_R}{Q} + \frac{Q_{Tr}}{Q} = 1$$

$$\alpha + \rho + \tau = 1 \tag{7.1}$$

where $\alpha$ is the fraction of incident radiation which is absorbed, called *absorptivity*, $\rho$ is the fraction which is reflected, called *reflectivity*, and $\tau$ is the fraction which is transmitted through the body, called *transmissivity* or *transmittance*.

A body is said to be opaque if $\tau = 0$ and $\alpha + \rho = 1$. Most solids do not transmit any radiation and are opaque. If $\rho$ is reduced, $\alpha$ increases. The reflectivity depends on the character of the surface. Therefore, the absorptivity of an opaque body can be increased or decreased by appropriate surface treatment. Roughening a solid surface enhances its absorptivity of incident radiation.

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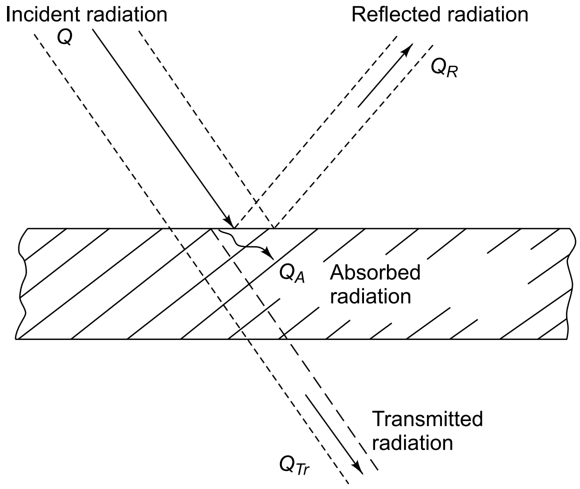


**Fig. 7.2** *Radiation incident on a surface*

When the surface is highly polished, the angle of incidence $\theta_i$ is equal to the angle of reflection $\theta_r$, and the reflection is said to be *specular*. When the surface is rough, the incident radiation is distributed in all directions, and the reflection is said to be *diffuse* (Fig. 7.3).

Most gases have high values of $\tau$ and low values of $\alpha$ and $\rho$. Air at atmospheric pressure and temperature is transparent to thermal radiation for which $\tau = 1$ and $\alpha = \rho = 0$. Gases like $CO_2$ and $H_2O$ vapour are highly absorptive at certain ranges of wavelengths.

## 7.4 BLACK BODY

A body is said to be *black* if it absorbs all incident radiation. So, for a black body $\alpha = 1$ and $\rho = \tau = 0$. There is no such perfectly black body in nature. The term black is used, since most black coloured surfaces normally show high values of absorptivity, and they also absorb all visible light rays, because of which they appear black to our eyes. There are some surfaces which absorb nearly all incident radiation, yet do not appear black. Ice, snow, white-washed walls have absorptivities greater than 0.95.

## 7.5 EMISSIVE POWER

If the radiation from a heated body is dispersed into a spectrum by a prism, it is found that the radiant energy is distributed among various wavelengths.

The *total emissive power of a body*, $E$, is defined as the total radiant energy emitted by the body at a certain temperature per unit time and per unit surface area at all wavelengths.

The *monochromatic emissive power of a body*, $E_\lambda$, is defined as the radiant energy emitted by the body per unit time and per unit surface area at a particular wavelength and temperature.

The radiation energy emitted by a black body per unit time and per unit surface area is given by

$$\mathbf{E_b = \sigma T^4 \ (W/m^2)} \tag{7.2}$$

where $\sigma = 5.67 \times 10^{-8}$ W/m$^2$ K$^4$ is called the *Stefan–Boltzmann constant* and $T$ is the absolute temperature of the surface in kelvin. Equation (7.2) is called the **Stefan–Boltzmann law** and $E_b$ is called the total emissive power of a black body.

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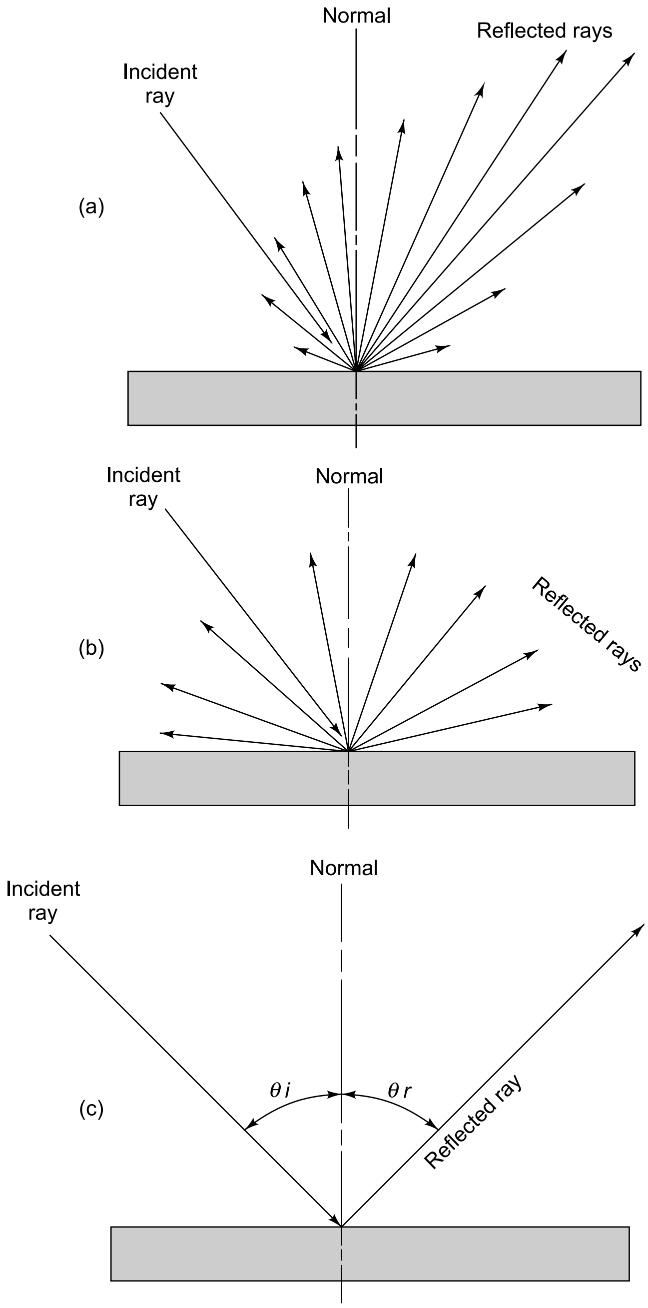


**Fig. 7.3** *Types of reflection from a surface: (a) actual or irregular (b) diffuse and (c) specular or mirrorlike*

The variation of $E_{b\lambda}$ with $\lambda$ and $T$ is shown in Fig. 7.4. The total radiation, $E_b$, is distributed among wavelengths varying from 0 to $\infty$. Therefore, at a particular temperature,

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$$E_b = \int_0^\infty E_{b\lambda}\, d\lambda$$
$$= \text{Area under the curve at that temperature}$$

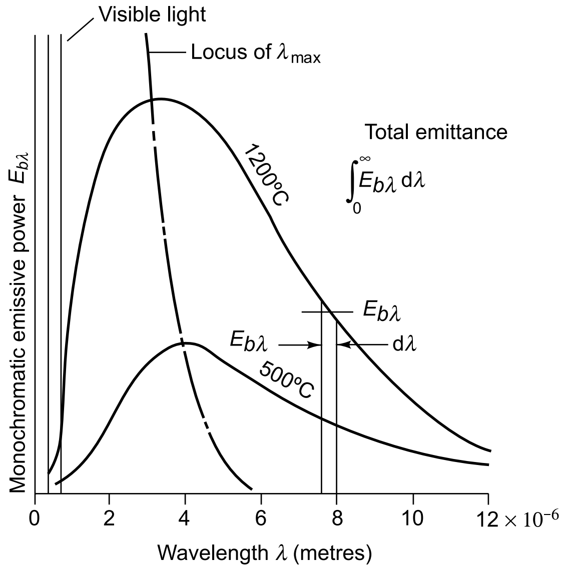


**Fig. 7.4** *Variation of monochromatic emissive power of a black body with $\lambda$ and $T$*

For wavelengths ranging from $\lambda$ to $\lambda + d\lambda$, $dE_b$ is the change of total emissive power.

$$dE_b = E_{b\lambda}\, d\lambda$$

or
$$\mathbf{E_{b\lambda}} = \frac{\mathbf{dE_b}}{\mathbf{d\lambda}} \ \mathbf{(W/m^3)} \tag{7.3}$$

which is also called *spectral or radiation intensity* of a black body. Since the area under each curve for a certain temperature represents the total emissive power $E_b$, as the temperature increases, the area increases and therefore $E_b$ increases.

## 7.6 EMISSIVITY

The *emissivity* of a surface is defined as the ratio of radiation emitted by the surface to the radiation emitted by a black body at the same temperature. It is denoted by $\varepsilon$, and it varies between 0 and 1. Emissivity is a measure of how closely a surface approximates a black body for which $\varepsilon = 1$.

The emissivity of a real surface varies with the temperature of the surface as well as the wavelength and direction of emitted radiation. The emissivity of a surface at a certain wavelength is called spectral emissivity $\varepsilon_\lambda$. The emissivity in a certain direction is called directional emissivity $\varepsilon_\theta$, where $\theta$ is the angle between the direction of radiation and the normal to the surface. The emissivity of a surface averaged over all directions is called the *hemispherical emissivity*, and the emissivity averaged over all wavelengths is called the *total emissivity*. Thus, the total hemispherical emissivity of a surface is simply the average emissivity over all directions and wavelengths, and can be expressed as

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$$\varepsilon(T) = \frac{E(T)}{E_b(T)} = \frac{E(T)}{\sigma T^4}$$

where $E(T)$ is the total emissive power of the real surface at temperature $T$ and $E_b(T)$ is the same for a black body. Thus,

$$E(T) = \varepsilon(T)\, \sigma\, T^4 \text{ (W/m}^2\text{)} \tag{7.4}$$

Spectral emissivity or monochromatic emissivity is defined in a similar manner

$$\varepsilon_\lambda(T) = \frac{E_\lambda(T)}{E_b\lambda(T)}$$

where $E_\lambda(T)$ is the spectral or monochromatic emissive power of the real surface.

When the emissivity of the material does not change with temperature, it is called a *gray body*. A gray body has a constant monochromatic emissivity, $\varepsilon_\lambda$, with respect to wavelength (Fig. 7.5). Thus, a gray body has a characteristic emissivity value (< 1) which does not vary with temperature.

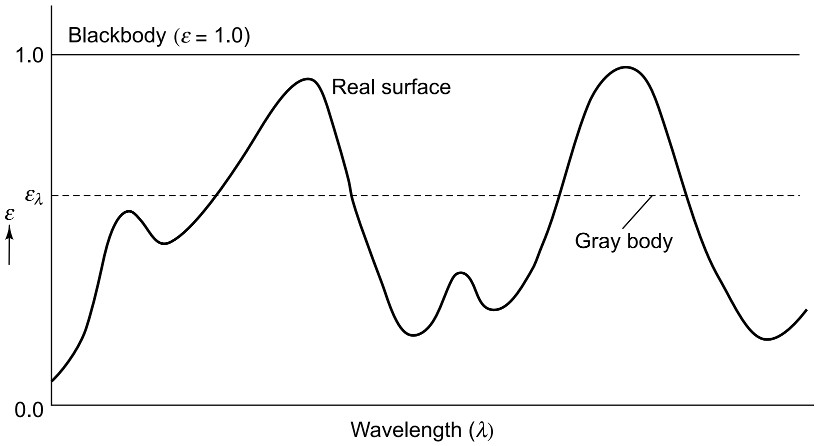


**Fig. 7.5** *Emissivities of real surface, gray body and black body*

The monochromatic emissive power, $\varepsilon_\lambda$, of a real surface can vary significantly from black body emissive power, as illustrated in Fig. 7.6. This is due to the fact that $E_\lambda = \varepsilon_\lambda E_{b\lambda}$. Since $\varepsilon_\lambda$ for the real surface is a characteristic of the surface and not the temperature of the surface, the area under the real surface $E_{b\lambda}$ curve will not be a constant fraction of the area under the $E_{b\lambda}$ curve as the temperature varies. Thus, the effective emissivity of a real surface varies with temperature.

## 7.7 KIRCHHOFF'S LAW

A small body of surface area $A_1$ is placed in a hollow evacuated space kept at a constant uniform temperature $T$ (Fig. 7.7). After some time, the body will attain at steady state the same temperature as that of the interior of the space and thereafter will radiate as much energy as it receives.

Let $I$ be the radiant energy falling upon the body per unit time per unit surface area,

$E_1$ be the total emissive power of the body and,

$\alpha_1$ be the absorptivity of the body.

At steady state,

Energy absorbed = Energy emitted

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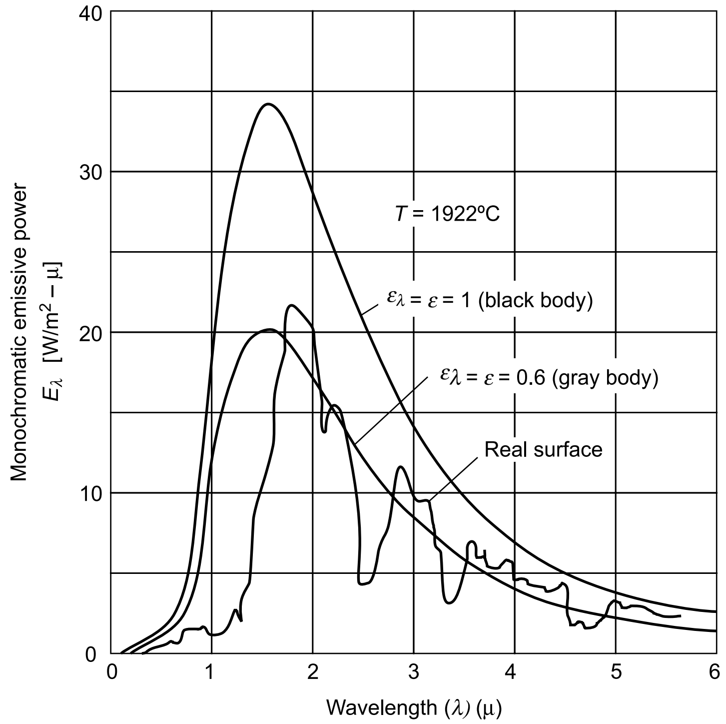


**Fig. 7.6** *Comparison of emissive power of a black body, gray body and a real surface varying with* $\lambda$

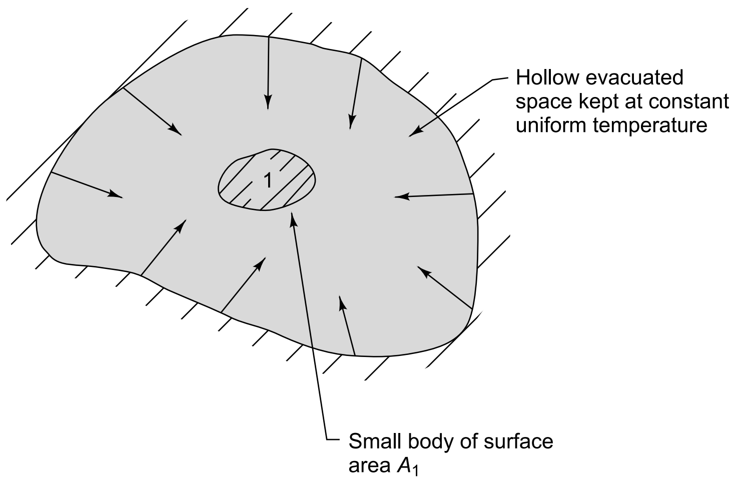


**Fig. 7.7** *Energy exchange between a body and the enclosing wall*

$$I A_1 \alpha_1 = E_1 A_1$$

$$I = \frac{E_1}{\alpha_1}$$

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If this body is replaced by a second body of the same shape and surface area, but of different material at exactly the same location, at steady state

$$I A_1 \alpha_2 = E_2 A_1$$

or, $$I = E_2/\alpha_2$$

Similarly, if the second body is replaced at the same location by a black body of the same shape and surface area,

$$I A_1 \alpha_B = E_b A_1$$

$$I = E_b/\alpha_B$$

Since $I$ is the same on each body,

$$E_1/\alpha_1 = E_2/\alpha_2 = E_b/\alpha_B = E_b \text{ (since } \alpha_B = 1)$$

$$\therefore \quad \alpha_1 = E_1/E_b,\ \alpha_2 = E_2/E_b$$

Therefore, for any body

$$\alpha = E/E_b \tag{7.5}$$

But the ratio $E/E_b$ has been defined as emissivity. Hence,

$$\alpha = \varepsilon \tag{7.6}$$

This is *Kirchhoff's law* which states that the emissivity of the surface of a body is equal to its absorptivity when the body is in thermal equilibrium with its surroundings.

A black body in addition to being a perfect absorber ($\alpha_B = 1$) is also a perfect emitter ($\varepsilon_B = 1$) of radiant energy.

Kirchhoff's law also holds for monochromatic radiation, for which

$$\frac{E_{\lambda_1}}{\alpha_{\lambda_1}} = \frac{E_{\lambda_2}}{\alpha_{\lambda_2}} = \frac{E_{b\lambda}}{\alpha_{\lambda B}} = \frac{E_{b\lambda}}{1}$$

Therefore, $$E_\lambda/\alpha_\lambda = E_{b\lambda}$$

or $$\alpha_\lambda = E_\lambda/E_{b\lambda} \tag{7.7}$$

From Eqs (7.4) and (7.7),

$$\alpha_\lambda = \varepsilon_\lambda \tag{7.8}$$

Therefore, the monochromatic emissivity of a body is equal to the monochromatic absorptivity at the same wavelength.

## 7.8 LABORATORY BLACK BODY

A black body may be made in the laboratory from a hollow enclosure or cavity with a small hole, the walls being kept at a uniform temperature (Fig. 7.8). A ray of radiant energy entering through the hole will be partially absorbed and reflected many times before it is almost fully absorbed.

The energy emitted from a portion of the surface in the cavity is $\varepsilon E_b$. After one reflection, this becomes $\rho\varepsilon E_b$, after two reflections, $\rho^2\varepsilon E_b$, etc. Let us imagine the radiation leaving the hole of this cavity to be composed of rays which have been directly radiated, reflected once, twice, etc. Then the energy emitted from the hole is

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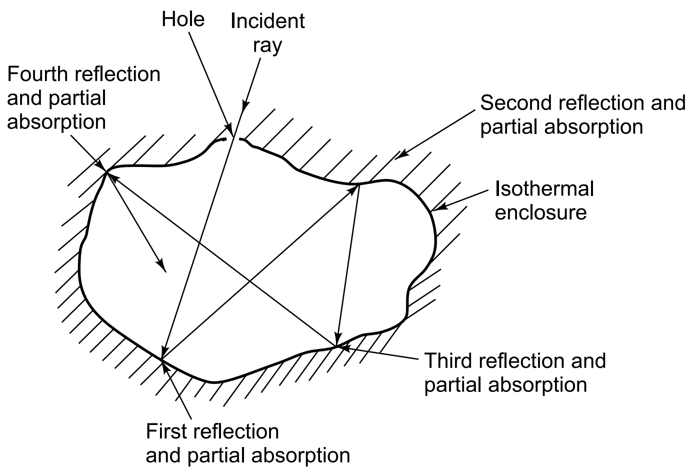


**Fig. 7.8** *Schematic diagram of a black body cavity*

$$
\begin{aligned}
E &= \varepsilon E_b + \rho\varepsilon E_b + \rho^2\varepsilon E_b + \rho^3\varepsilon E_b + \cdots \\
&= \varepsilon E_b(1 + \rho + \rho^2 + \rho^3 + \cdots) \\
&= \varepsilon E_b \frac{1}{1-\rho} = \varepsilon E_b \frac{1}{\alpha} = E_b
\end{aligned}
$$

The energy streaming out from the hole is *black body radiation*.

## 7.9 SPECTRAL ENERGY DISTRIBUTION OF A BLACK BODY

It is the characteristic of thermal radiation from a solid body that when it is dispersed by being passed through a prism, a continuous spectrum is formed with an energy distribution as shown below for a black body. The area under each constant temperature curve is the total rate of energy emission per unit area given by

$$\mathbf{E_b = \int_0^\infty E_{b\lambda}\, d\lambda} \tag{7.9}$$

where $E_{b\lambda}$ is the monochromatic emissive power of a black body.

### 7.9.1 Planck's Law

Planck deduced by his quantum theory the expression for $E_{b\lambda}$ as a function of temperature and wavelength, as given below

$$\mathbf{E_{b\lambda} = \frac{C_1\lambda^{-5}}{e^{C_2/\lambda T} - 1}} \tag{7.10}$$

where $C_1 = 3.74 \times 10^{-16}$ W m$^2$, $C_2 = 1.438 \times 10^{-2}$ m K, $\lambda$ is the wavelength, *m*, and $T$ is the absolute temperature, K.

**Planck's law** is the basic law of thermal radiation. From this law it follows that the emissive power $E_b$ characterised by individual isotherms passes through a maximum, as shown in Fig. 7.9. With wavelengths $\lambda = 0$ and $\lambda = \infty$ it vanishes.

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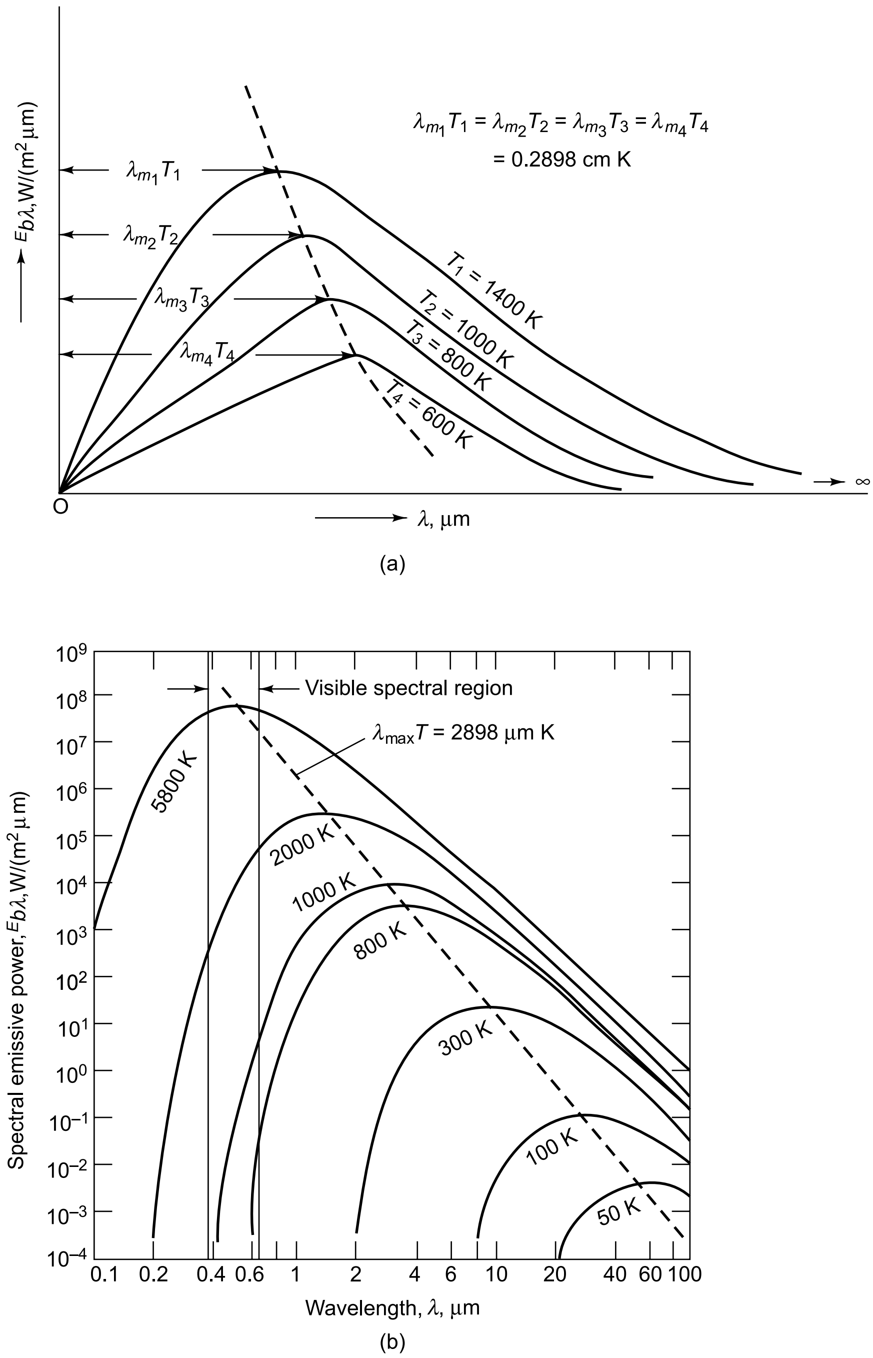


**Fig. 7.9** *Energy distribution of a black body varying with λ and T. (a) Maxima of isotherms lying on a hyperbola and (b) maxima of isotherms lying on a straight line (log-log scale)*

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### 7.9.2 Rayleigh–Jeans' Law

Planck's law has two extreme cases. One of the extreme cases is the condition

$$\lambda T >> C_2$$

or $$C_2/\lambda T << 1$$

Now, $$e^{C_2/\lambda T} = 1 + \frac{1}{\lfloor 1}\left(\frac{C_2}{\lambda T}\right) + \frac{1}{\lfloor 2}\left(\frac{C_2}{\lambda T}\right)^2 + \frac{1}{\lfloor 3}\left(\frac{C_3}{\lambda T}\right)^3 + \cdots \quad (7.11)$$

Taking only the first two terms of Eq. (7.11), Eq. (7.10) becomes

$$\mathbf{E_{b\lambda} = \frac{C_1\lambda^{-5}\lambda T}{C_2} = \frac{C_1 T}{C_2\lambda^4}} \quad (7.12)$$

This equation is known as *Rayleigh–Jeans law* which is found to be *valid for large wavelengths*.

### 7.9.3 Wien's Law

The second extreme case corresponds to the condition

$$\lambda T << C_2$$

or $$C_2/\lambda T >> 1$$

The unity present in the denominator of Eq. (7.10) can be neglected, and the relationship becomes

$$\mathbf{E_{b\lambda} = \frac{C_1\lambda^{-5}}{e^{C_2/\lambda T}} = \frac{C_1}{\lambda^5}\, e^{-C_2/\lambda T}} \quad (7.13)$$

This equation is known as *Wien's law*, which is found to be *valid for short wavelengths*.

### 7.9.4 Wien's Displacement Law

The maximum values of the emissive power $E_b$ (Fig. 7.9) can be obtained by differentiating Planck's Eq. (7.10) with respect to $\lambda$ and equating it to zero. Therefore,

$$\frac{\mathrm{d}E_{b\lambda}}{\mathrm{d}\lambda} = \frac{d}{\mathrm{d}\lambda}\left[C_1\lambda^{-5}\left(e^{C_2/\lambda T} - 1\right)^{-1}\right]$$

$$= C_1\lambda^{-5}(-1)(e^{C_2/\lambda T} - 1)^{-2}(-C_2/\lambda^2 T)\, e^{C_2/\lambda T} + (e^{C_2/\lambda T} - 1)^{-1} C_1 (-5)\lambda^{-6} = 0$$

or $$\frac{e^{C_2/\lambda T}}{e^{C_2/\lambda T} - 1} = \frac{5\lambda T}{C_2}$$

Putting $C_2/\lambda T = x$ and re-arranging,

$$e^{-x} + \frac{x}{5} - 1 = 0$$

Solution of this equation gives

$$x = \frac{C_2}{\lambda_{max}\, T} = 4.965$$

$$\lambda_{max}\, T = \frac{C_2}{4.965} = \frac{1.438 \times 10^{-2}\,\text{m K}}{4.965}$$

$$= \mathbf{2.898 \times 10^{-3}\ m\ K} \quad (7.14)$$

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This is known as *Wien's displacement law*. Here, $\lambda_{max}$ is the wavelength at which $E_{b\lambda}$ is the maximum at a particular temperature. The value of $(E_{b\lambda})_{max}$ shifts towards the shorter wavelengths with increasing temperature.

The maximum emissive power of a black body can be found from Planck's law; $\lambda = \lambda_{max}$ is replaced from Eq. (7.14).

$$(E_{b\lambda})_{max} = \frac{C_1 \lambda^{-5}}{e^{C_2/\lambda T} - 1} = \frac{3.74 \times 10^{-16} \left(2.898 \times 10^{-3}/T\right)^{-5}}{e^{4.965} - 1}$$

or $$\mathbf{(E_{b\lambda})_{max} = C_3 T^5 \ W/m^3} \tag{7.15}$$

where the constant $C_3 = 1.287 \times 10^{-5}$ W/m$^3$ K$^5$. From Eq. (7.15) it follows that the magnitude of $(E_b)_{max}$ is proportional to the fifth power of the absolute temperature of the body.

### 7.9.5 Planck's Law in Dimensionless Form

Planck's law can be expressed in dimensionless form by using Eq. (7.15). Therefore,

$$\frac{E_{b\lambda}}{(E_{b\lambda})_{max}} = \frac{C_1}{C_3 (\lambda T)^5} \frac{1}{e^{C_2/\lambda T} - 1}$$

$$= f(\lambda T) \tag{7.16}$$

If we substitute the value of $T$ from Eq. (7.14),

$$\frac{E_{b\lambda}}{(E_{b\lambda})_{max}} = \phi\left(\frac{\lambda}{\lambda_{max}}\right) \tag{7.17}$$

Figure 7.10 is obtained by plotting Eqs (7.16) and (7.17). The maximum of this relationship corresponds to the values of

$$\frac{E_{b\lambda}}{(E_{b\lambda})_{max}} = 1 \text{ and } \frac{\lambda}{\lambda_{max}} = 1$$

$$f = \frac{\int_0^{\lambda} E_{b\lambda}\, d\lambda}{\int_0^{\infty} E_{b\lambda}\, d\lambda} = \frac{\int_0^{\lambda} E_{b\lambda}\, d\lambda}{\sigma T^4} \tag{7.18}$$

This is also equal to the ratio of area $A_1$ to the total area under the curve at $T$ (Fig. 7.11). Similarly, for the wavelength range between $\lambda_1$ and $\lambda_2$, the fraction of radiation at temperature $T$, as shown in Fig. 7.11, would be

$$f = \frac{\int_{\lambda_1}^{\lambda_2} E_{b\lambda}\, d\lambda}{\sigma T^4} = \frac{1}{\sigma T^4}\left(\int_0^{\lambda_2} E_{b\lambda}\, d\lambda - \int_0^{\lambda_1} E_{b\lambda}\, d\lambda\right)$$

$$= \frac{\textbf{Area } \mathbf{A_2}}{\textbf{Total area}} \tag{7.19}$$

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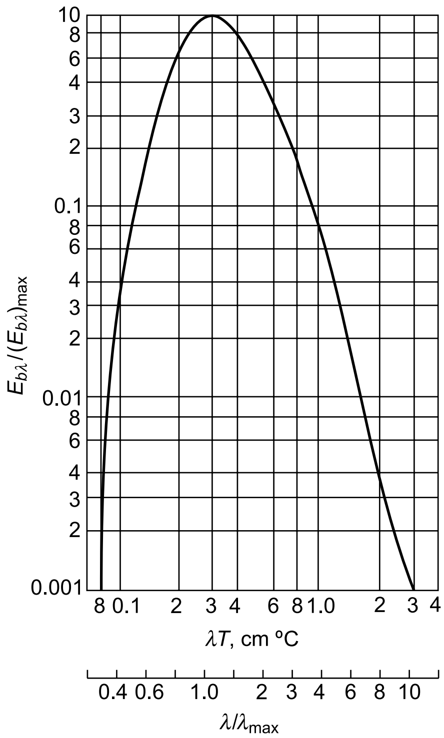


**Fig. 7.10** *Graphical presentation of Planck's law in dimensionless form*

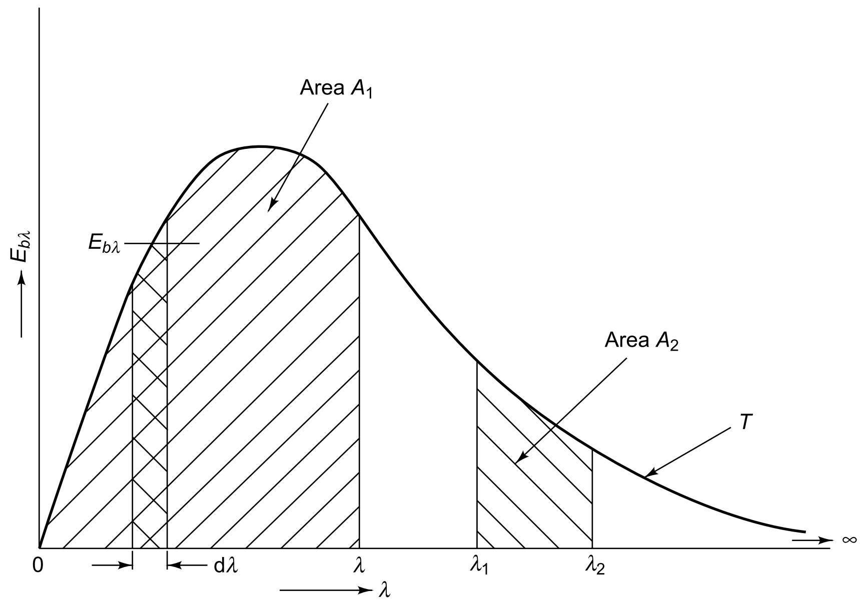


**Fig. 7.11** *Fraction of radiation in a range of wavelength expressed as area ratio*

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Dunkle [3] has evaluated the values of fractional areas representing $[E_b\,(0 \to \lambda T)]/\sigma T^4$ for various values of $T$ and tabulated them. Table 7.3 gives these values for convenient use in computing the fraction of radiant energy falling in a certain wavelength range. The fraction of the total black body emission between 0 and a given $\lambda$ is presented in Fig. 7.12 and also Table 7.3 as a universal function of $\lambda T$.

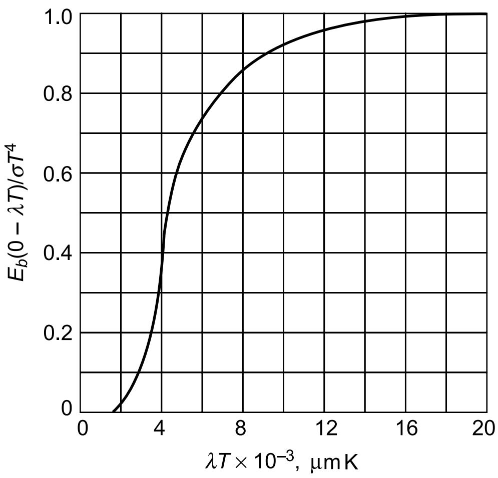


**Fig. 7.12** *Ratio of black body emission between 0 and l to the total emission, $E_b\,(0 - \lambda T)/\sigma T^4$ versus $\lambda T$*

**Table 7.3** *Black body radiation function*

| $\lambda T$ μm · K | $E_{b\lambda}/T^5$ W/$m^2 \cdot K^5 \cdot \mu m \times 10^{11}$ | $\frac{E_{b0-\lambda T}}{\sigma T^4}$ |
|---|---|---|
| 555.6 | $0.400 \times 10^{-5}$ | $0.170 \times 10^{-7}$ |
| 666.7 | $0.120 \times 10^{-3}$ | $0.756 \times 10^{-6}$ |
| 777.8 | 0.00122 | $0.106 \times 10^{-4}$ |
| 888.9 | 0.00630 | $0.738 \times 10^{-4}$ |
| 1,000.0 | 0.02111 | $0.321 \times 10^{-3}$ |
| 1,111.1 | 0.05254 | 0.00101 |
| 1,222.2 | 0.10587 | 0.00252 |
| 1,333.3 | 0.18275 | 0.00531 |
| 1,444.4 | 0.28091 | 0.00983 |
| 1,555.6 | 0.39505 | 0.01643 |
| 1,666.7 | 0.51841 | 0.02537 |
| 1,777.8 | 0.64404 | 0.03677 |
| 1,888.9 | 0.76578 | 0.05059 |
| 2,000.0 | 0.87878 | 0.06672 |
| 2,111.1 | 0.97963 | 0.08496 |
| 2,222.2 | 1.0663 | 0.10503 |
| 2,333.3 | 1.1378 | 0.12665 |

(*Contd*)

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**Table 7.3** (*Contd*)

| | | |
|---|---|---|
| 2,444.4 | 1.1942 | 0.14953 |
| 2,555.6 | 1.2361 | 0.17337 |
| 2,666.7 | 1.2645 | 0.19789 |
| 2,777.8 | 1.2808 | 0.22285 |
| 2,888.9 | 1.2864 | 0.24803 |
| 3,000.0 | 1.2827 | 0.27322 |
| 3,111.1 | 1.2713 | 0.29825 |
| 3,222.2 | 1.2532 | 0.32300 |
| 3,333.3 | 1.2299 | 0.34734 |
| 3,444.4 | 1.2023 | 0.37118 |
| 3,555.6 | 1.1714 | 0.39445 |
| 3,666.7 | 1.1380 | 0.41708 |
| 3,777.8 | 1.1029 | 0.43905 |
| 3,888.9 | 1.0665 | 0.46031 |
| 4,000.0 | 1.0295 | 0.48085 |
| 4,111.1 | 0.99221 | 0.50066 |
| 4,222.2 | 0.95499 | 0.51974 |
| 4,333.3 | 0.91813 | 0.53809 |
| 4,444.4 | 0.88184 | 0.55573 |
| 4,555.6 | 0.84629 | 0.57267 |
| 4,666.7 | 0.81163 | 0.58891 |
| 4,777.8 | 0.77796 | 0.60449 |
| 4.888.9 | 0.74534 | 0.61941 |
| 5,000.0 | 0.71383 | 0.63371 |
| 5,111.1 | 0.68346 | 0.64740 |
| 5,222.2 | 0.65423 | 0.66051 |
| 5,333.3 | 0.62617 | 0.67305 |
| 5,444.4 | 0.59925 | 0.68506 |
| 5,555.6 | 0.57346 | 0.69655 |
| 5,666.7 | 0.54877 | 0.70754 |
| 5,777.8 | 0.52517 | 0.71806 |
| 5,888.9 | 0.50261 | 0.72813 |
| 6,000.0 | 0.48107 | 0.73777 |
| 6,111.1 | 0.46051 | 0.74700 |
| 6,222.2 | 0.44089 | 0.75583 |
| 6,333.3 | 0.42218 | 0.76429 |
| 6,444.4 | 0.40434 | 0.77238 |
| 6,555.6 | 0.38732 | 0.78014 |
| 6,666.7 | 0.37111 | 0.78757 |
| 6,777.8 | 0.35565 | 0.79469 |
| 6,888.9 | 0.34091 | 0.80152 |
| 7,000.0 | 0.32687 | 0.80806 |

(*Contd*)

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**Table 7.3** (*Contd*)

| | | |
|---|---|---|
| 7,111.1 | 0.31348 | 0.81433 |
| 7,222.2 | 0.30071 | 0.82035 |
| 7,333.3 | 0.28855 | 0.82612 |
| 7,444.4 | 0.27695 | 0.83166 |
| 7,555.6 | 0.26589 | 0.83698 |
| 7,666.7 | 0.25534 | 0.84209 |
| 7,777.8 | 0.24527 | 0.84699 |
| 7,888.9 | 0.23567 | 0.85171 |
| 8,000.0 | 0.22651 | 0.85624 |
| 8,111.1 | 0.21777 | 0.86059 |
| 8,222.2 | 0.20942 | 0.86477 |
| 8,333.3 | 0.20145 | 0.86880 |
| 8,888.9 | 0.16662 | 0.88677 |
| 9,444.4 | 0.13877 | 0.90168 |
| 10,000.0 | 0.11635 | 0.91414 |
| 10,555.6 | 0.09817 | 0.92462 |
| 11,111.1 | 0.08334 | 0.93349 |
| 11,666.7 | 0.07116 | 0.94104 |
| 12,222.2 | 0.06109 | 0.94751 |
| 12,777.8 | 0.05272 | 0.95307 |
| 13,333.3 | 0.04572 | 0.95788 |
| 13,888.9 | 0.03982 | 0.96207 |
| 14,444.4 | 0.03484 | 0.96572 |
| 15,000.0 | 0.03061 | 0.96892 |
| 15,555.6 | 0.02699 | 0.97174 |
| 16,111.1 | 0.02389 | 0.97423 |
| 16,666.7 | 0.02122 | 0.97644 |
| 22,222.2 | 0.00758 | 0.98915 |
| 27,777.8 | 0.00333 | 0.99414 |
| 33,333.3 | 0.00168 | 0.99649 |
| 38,888.9 | $0.940 \times 10^{-3}$ | 0.99773 |
| 44,444.4 | $0.564 \times 10^{-3}$ | 0.99845 |
| 50,000.0 | $0.359 \times 10^{-3}$ | 0.99889 |
| 55,555.6 | $0.239 \times 10^{-3}$ | 0.99918 |

### 7.9.6 Stefan–Boltzmann Law

Planck's law also permits to derive Stefan–Boltzmann law, which establishes the dependence of total hemispherical radiation on temperature.

The total emissive power of a black body is given by

$$E_b = \int_0^\infty E_{b\lambda} \mathrm{d}y = \int_0^\infty \frac{C_1 \lambda^{-5}}{e^{C_2/\lambda T} - 1} \mathrm{d}\lambda$$

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Substituting $\lambda^{-1} = x$, or $-\lambda^{-2}\, d\lambda = dx$, or $d\lambda = -1/x^2\, dx$, and when $\lambda = \infty$, $x = 0$ and $\lambda = 0$, $x = \infty$, we have

$$E_b = \int_{\infty}^{0} \frac{C_1 x^5}{e^{C_2 x/T} - 1}\left(-\frac{1}{x^2}\right) dx$$

$$= C_1 \int_0^{\infty} x^3 \left(e^{C_2 x/T} - 1\right)^{-1} dx$$

$$= \int_0^{\infty} x^3 \left(e^{-C_2 x/T} + e^{-2C_2 x/T} + e^{-3C_2 x/T} + \cdots\right) dx$$

$$= C_1 \int_0^{\infty} x^3 e^{-C_2 i x/T}\, dx,\ i = 1, 2, 3, \cdots$$

Since $$\int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}},$$

$$E_b = C_1 \frac{3!}{\left(C_2 i/T\right)^{3+1}} = \frac{6 C_1 T^4}{C_2^4 i^4}$$

$$= \frac{6 C_1 T^4}{C_2^4}\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots\right)$$

$$= \mathbf{\frac{6 C_1\ T^4}{C_2^4}\ \frac{\pi^4}{90}}$$

Therefore, $$E_b = \sigma T^4 \tag{7.20}$$

where $\sigma$ is the Stefan-Boltzmann constant given by

$$\sigma = \frac{6 C_1 \pi^4}{C_2^4 \times 90} = \frac{6 \times 3.74 \times 10^{-16} \times \pi^4}{\left(1.438 \times 10^{-2}\right)^4 \times 90}$$

$$\mathbf{= 5.67 \times 10^{-8}\ W/m^2\ K^4}$$

Equation (7.20) is known as *Stefan–Boltzmann law*, which establishes that the emissive power of a black body depends only on temperature and is *proportional to the fourth power of its absolute temperature*. The values of all radiation constants have been put together by Snyder [3].

Historically, it is worth noting that Stefan–Boltzmann law as well as Rayleigh-Jeans' law and Wien's law were independently developed by different methods much earlier than Planck's law. By his quantum hypothesis Planck developed the most general equation of thermal radiation from which all other equations can be derived, as illustrated above.

### 7.9.7 Black Body Radiation in a Certain Range of Wavelength

The fraction of total radiation $f$ from a black body in the wavelength range of $0 - \lambda$ at a certain temperature $T$ is given by Eq. (7.18).

From Planck's law,

$$E_{b\lambda} = \frac{C_1 \lambda^{-5}}{e^{C_2/\lambda T} - 1}$$

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or
$$\frac{E_{b\lambda}}{T^5} = \frac{C_1 (\lambda T)^{-5}}{e^{C_2/\lambda T} - 1} \tag{7.21}$$

Therefore, $E_{b\lambda}/T^5$ is a function of $\lambda T$.

Again,
$$E_b = \int_0^{\infty} E_{b\lambda}\, \mathrm{d}y = \sigma T^4$$

$\therefore$
$$\sigma = \int_0^{\infty} \frac{E_{b\lambda}}{T^4} \mathrm{d}y = \int_0^{\infty} \frac{E_{b\lambda}}{T^5} \mathrm{d}(\lambda T) \tag{7.22}$$

From Eq. (7.21),

$$\sigma = \int_0^{\infty} \frac{\mathbf{C_1 (\lambda T)^{-5}}}{\mathbf{e^{C_2/\lambda T} - 1}} \mathbf{d(\lambda T)} \tag{7.23}$$

Figure 7.13 shows the plot of $E_{b\lambda}/T^5$ vs $\lambda T$.

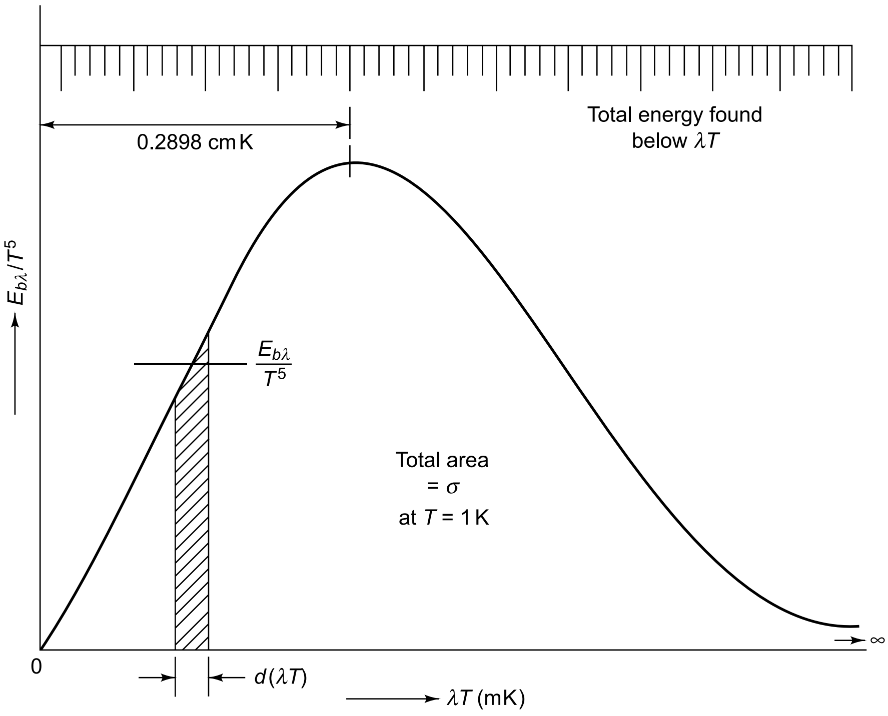


**Fig. 7.13** *Plot of* $\frac{E_{b\lambda}}{T^5}$ *vs* $\lambda T$

At $T = 1$ K, the curve represents $E_{b\lambda}$ vs $\lambda$. The area under the curve is equal to the Stefan–Boltzmann constant. The fractional energy in the range of $\lambda_1 T$ and $\lambda_2 T$ is given by

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$$f = \frac{\int_{\lambda_1}^{\lambda_2} E_{b\lambda}\, d\lambda}{\int_0^{\infty} E_{b\lambda}\, d\lambda} = \frac{\int_{\lambda_1}^{\lambda_2} E_{b\lambda}\, d\lambda}{\sigma T^4}$$

or
$$\mathbf{f = \frac{1}{\sigma} \int_{\lambda_1 T}^{\lambda_2 T} \frac{E_{b\lambda}}{T^5}\, d(\lambda T)} \tag{7.24}$$

which also represents the area ratio which can be read from the scale at the top of Fig. 7.13.

## 7.10 RADIATION FROM REAL SURFACES

A gray body is one the monochromatic emissivity of which has the same value at all wavelengths. For a gray body, $\varepsilon_\lambda$ is independent of $\lambda$ so that

$$\varepsilon_\lambda = \alpha_\lambda = \varepsilon = \alpha$$

even though the temperatures of the incident radiation and of the receiving surface are not the same.

For the purpose of heat transfer calculations, most real surfaces are considered to be gray. In general, however, the emissivity of real surfaces varies with wavelength. If the variation of monochromatic emissivity $\varepsilon_\lambda$ with $\lambda$ is known, the emissive power of the real body can be found by plotting the product $\varepsilon_\lambda$ $E_b$ vs $\lambda$ (Fig. 7.14). An average emissivity of a real body can then be obtained from since $\varepsilon_l = E_\lambda / E_{b\lambda}$. If the real body is gray body, then the ratio $E_{g\lambda}/E_{b\lambda}$ is constant for all wavelengths.

$$\varepsilon = \frac{E}{E_b} = \frac{\int_0^\infty E_\lambda\, d\lambda}{\int_0^\infty E_{b\lambda}\, d\lambda}$$

$$= \frac{\int_0^\infty \varepsilon_\lambda E_{b\lambda}\, d\lambda}{\int_0^\infty E_{b\lambda}\, d\lambda} \tag{7.25}$$

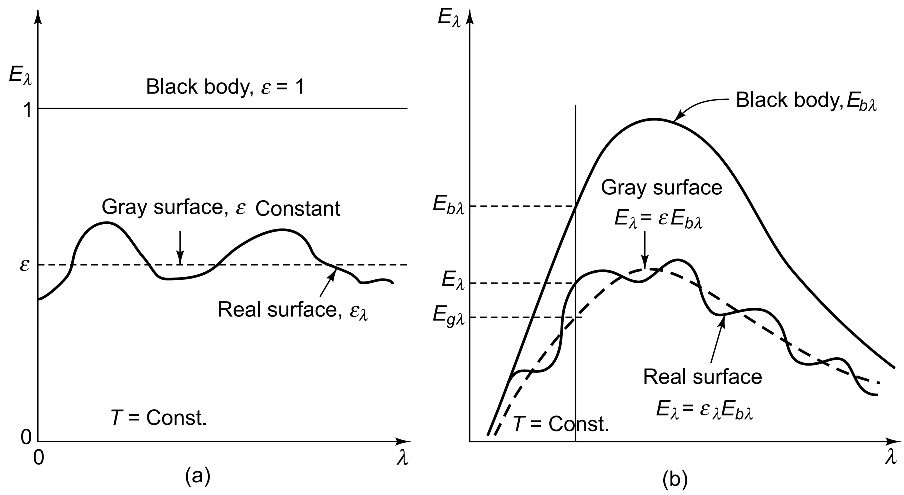


**Fig. 7.14** *Plot of $E_\lambda$ vs $\lambda$ for ideal and real body*

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The total average absorptivity can be similarly obtained from the distribution of monochromatic absorptivity $\alpha_\lambda$. Thus

$$\alpha = \frac{\int_0^\infty a_\lambda \, E_{b\lambda} \, \mathrm{d}\lambda}{\int_0^\infty E_{b\lambda} \, \mathrm{d}\lambda}$$

$$= \int_0^\infty \frac{E_{b\lambda}}{\sigma T^5} \, d(\lambda T) \quad (7.26)$$

If a plot of $\alpha_\lambda$ with $\lambda$ is known, $\alpha$ can be obtained from the above equation.

The total hemispherical emissivity $\varepsilon$ varies with temperature, degree of roughness, and if a metal, degree of oxidation, Table A.9 in the Appendix A enlists average values of emissivities of some materials at specified temperatures. It is seen that clean and polished surfaces have low values of emissivity, whereas most other surfaces of engineering importance have their emissivities higher than 0.85. The oxidised metal surfaces have higher values of $\varepsilon$ than unoxidised ones.

## 7.11 INTENSITY OF RADIATION

The *intensity of radiation*, $I$ is defined as the rate of heat radiation in a given direction from a surface per unit solid angle per unit area of the projection of the surface on a plane normal to the direction of radiation (Fig. 7.15).

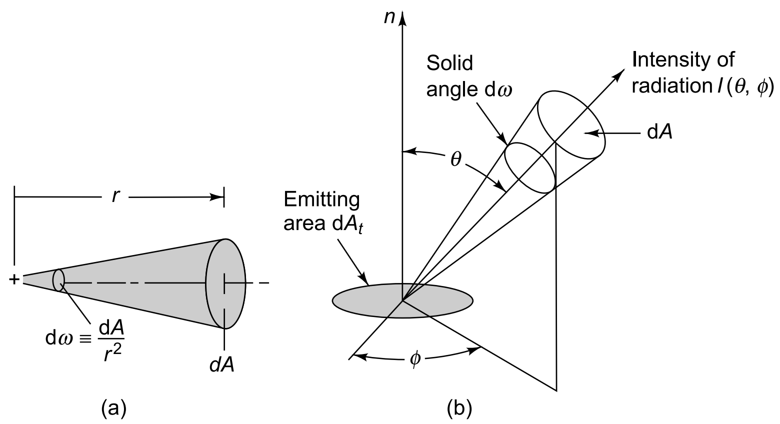


**Fig. 7.15** *(a) Differential solid angle and (b) intensity of radiation*

A *solid angle* is defined as the ratio of the spherical surface enclosed by a cone, with its vertex at the centre of the sphere, to the square of the radius of the sphere (Fig. 7.15). The solid angle subtended by the spherical surface d$A$ at the centre would be d$A/r^2$, where $r$ is the radius of the sphere. If d$A = r^2$, the solid angle subtended is 1 steradian. The whole spherical surface subtends at the centre an angle of $4\pi$ steradian or 1 steragon.

Let us consider the radiation emanating from an elemental black surface of area d$A_1$ at the temperature $T_1$ (Fig. 7.16). Energy will be radiated in all directions in the entire hemisphere. Let us consider the spherical strip area d$A$ of radius $r \sin \phi_1$ and thickness $r \, \mathrm{d}\phi_1$, which subtends at the centre a solid angle of d$A/r^2$. The amount of radiation d$Q_1$ directed towards this area is

$$dQ_1 = I_1\, dA_1 \cos\phi_1 \frac{dA}{r^2}$$

$$= I_1 dA_1 \cos\phi_1 \frac{(2\pi r \sin\phi_1)(r\, d\phi_1)}{r^2}$$

$$= I_1 dA_1 \sin 2\phi_1\, d\phi_1$$

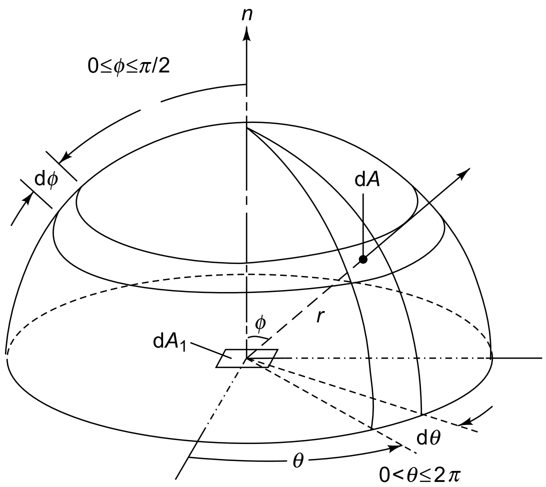


**Fig. 7.16** *Radiation from a differential area $dA_1$ into surrounding hemisphere centred at $dA_1$*

Total radiation in the entire hemisphere

$$Q_1 = I_1\, dA_1 \pi \int_0^{\pi/2} \sin 2\phi_1\, d\phi_1$$

$$= I_1\, dA_1\; \pi$$

$$\mathbf{E_{b_1} = \frac{Q_1}{dA_1} = \pi I_1} \qquad (7.27)$$

The emissive power of a black body is $\pi$-*times the intensity of the emitted radiation*.

$$\mathbf{I_1 = \frac{E_{b_1}}{\pi} = \frac{\sigma T_1^4}{\pi}} \qquad (7.28)$$

Thus, the intensity $I_1$ depends only on temperature $T_1$.

## 7.12 RADIANT HEAT EXCHANGE BETWEEN TWO BLACK BODIES SEPARATED BY A NON-ABSORBING MEDIUM

The radiant heat exchange between two bodies depends upon

1. The views the surfaces have of each other i.e., how they “see” each other.
2. Their emitting and absorbing characteristics and
3. The medium that intervenes the two bodies.

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Let us assume that the two bodies are black and the medium is nonparticipating in the energy exchange. Let us consider the area elements $dA_1$ and $dA_2$ on the two surfaces (Fig. 7.17). The distance between them is $r$ and the angles made by the normals to the two area elements with the line joining them are $\phi_1$ and $\phi_2$ respectively. The projected area of $dA_1$ in the direction of radiation is $dA_1 \cos \phi_1$.

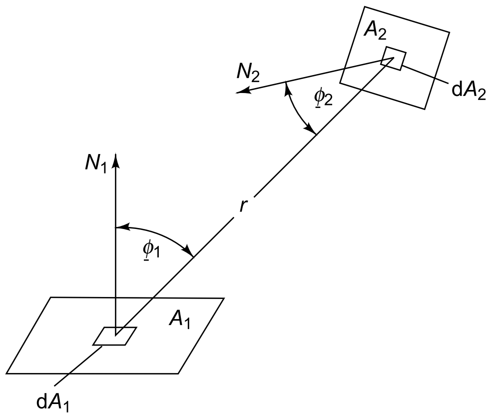


**Fig. 7.17** *Radiant heat exchange between two black surfaces*

Energy leaving $dA_1$ and intercepted (and absorbed, since the surface is black) by $dA_2$ is

$$dQ_{1-2} = I_1 \, dA_1 \cos \phi_1 \, \frac{dA_1 \cos \phi_2}{r^2} \tag{7.29}$$

where the solid angle subtended by the element $dA_2$ at the centre of $dA_1$ is $(dA_2 \cos \phi_2)/r^2$. Since $I_1$ is independent of the direction $\phi_1$, the energy emitted per unit of area $dA_1$ per unit solid angle is proportional to the cosine of the angle $\phi_1$. This is known as *Lambert's cosine law*. From Eq. (7.28)

$$dA_2 \, dQ_{1-2} = \frac{\sigma T_1^4}{\pi} \, dA_1 \cos \phi_1 \, \frac{dA_2 \cos \phi_2}{r^2}$$

The total radiation leaving $A_1$ and being absorbed by $A_2$ is

$$Q_{1-2} = \frac{\sigma T_1^4}{\pi} \int_{A_1}\int_{A_2} \frac{\cos \phi_1 \cos \phi_2}{r^2} \, dA_1 \, dA_2$$

$$= A_1 F_{12} \, \sigma T_1^4 \tag{7.30}$$

where $$A_1 F_{12} = \frac{1}{\pi} \int_{A_1}\int_{A_2} \frac{\cos \phi_1 \cos \phi_2}{r^2} \, dA_1 \, dA_2 \tag{7.31}$$

Similarly, energy leaving $dA_2$ and intercepted (and absorbed) by $dA_1$ is

$$dQ_{2-1} = I_2 \, dA_2 \cos \phi_2 \, \frac{dA_1 \cos \phi_1}{r^2}$$

$$= \frac{\sigma T_2^4}{\pi} \, dA_2 \cos \phi_2 \, \frac{dA_1 \cos \phi_1}{r^2} \tag{7.32}$$

Therefore, the total radiation leaving $A_2$ and being absorbed by $A_1$ is

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$$Q_{2-1} = \frac{\sigma T_2^4}{\pi} \int_{A_1}\int_{A_2} \frac{\cos\phi_1 \cos\phi_2}{r^2} \, dA_1 \, dA_2$$

$$= A_2 F_{21} \sigma T_2^4 \tag{7.33}$$

where
$$\mathbf{A_2 F_{21}} = \frac{1}{\pi} \int_{A_1}\int_{A_2} \frac{\cos\phi_1 \cos\phi_2}{r^2} \, dA_1 \, dA_2 \tag{7.34}$$

Therefore, the net energy exchange between $A_1$ and $A_2$ using Eqs. (7.30) and (7.33),

$$(Q_{12})_{net} = Q_{1-2} - Q_{2-1}$$

$$= \sigma A_1 F_{12} (T_1^4 - T_2^4) \tag{7.35}$$

where from Eqs (7.30) and (7.34),

$$\mathbf{A_1 F_{12} = A_2 F_{21}} = \frac{1}{\pi} \int_{A_1}\int_{A_2} \frac{\cos\phi_1 \cos\phi_2}{r^2} \, dA_1 \, dA_2$$

Here, $F_{12}$ is called the *shape factor* of $A_1$ with respect to $A_2$. This is the fraction of energy leaving $A_1$ that strikes $A_2$ (and is absorbed, because the surface is black). Similarly, $F_{21}$ is the shape factor of $A_2$ with respect to $A_1$, which is also the fraction of energy leaving $A_2$ that strikes $A_1$ (and is absorbed). Since the values of $F_{12}$ and $F_{21}$ depend on how the two surfaces are exposed to each other or "see" each other, these are also called *view factors*, *geometry factors* or *configuration factors*. Unless $A_1$ is equal to $A_2$, $F_{12} \neq F_{21}$. But

$$\mathbf{A_1 F_{12} = A_2 F_{21}} \tag{7.36}$$

This is known as the *reciprocity theorem*.

## 7.13 SHAPE FACTOR

If the interior surface of a completely enclosed space, such as a room or a furnace is subdivided into parts having areas $A_1, A_2, A_3, \cdots, A_n$ (Fig. 7.18) then

$$F_{11} + F_{12} + F_{13} + \cdots + F_{1n} = 1$$
$$F_{21} + F_{22} + F_{23} + \cdots + F_{2n} = 1$$
$$\vdots$$
$$F_{n1} + F_{n2} + F_{n3} + \cdots + F_{nn} = 1 \tag{7.37}$$

In addition, the reciprocity theorem [Eq. (7.36)] holds true for any two surfaces of the enclosure, e.g.

$$A_1 F_{12} = A_2 F_{21}, A_1 F_{13} = A_3 F_{31}$$
$$A_2 F_{23} = A_3 F_{32} \text{ and so on}$$

The decomposition of one or both the surfaces into subdivisions produces a combination of geometrical configurations for which the shape factor is easily determined. Of the two surfaces $A_1$ and $A_2$, if $A_1$ is subdivided into two parts $A_3$ and $A_4$, then the radiant heat exchange between $A_1$ and $A_2$ is

$$Q_{1-2} = Q_{3-2} + Q_{4-2}$$

If the surfaces are black,

$$\sigma A_1 F_{12}(T_1^4 - T_2^4) = \sigma A_3 F_{32}(T_3^4 - T_2^4) + \sigma A_4 F_{42}(T_4^4 - T_2^4)$$

Since, $T_1 = T_3 = T_4$,

$$\mathbf{A_1 F_{12} = A_3 F_{32} + A_4 F_{42}}$$

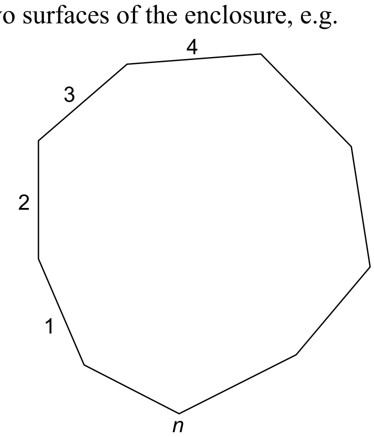


**Fig. 7.18** *Enclosure of black surfaces*

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For the radiant exchange from $A_2$ to $A_1$, $A_2 F_{21} = A_2 F_{23} + A_2 F_{24}$

or $$F_{21} = F_{23} + F_{24} \tag{7.38}$$

For two large parallel plates, $F_{12} = F_{21} = 1.0$.

A surface has a shape factor with respect to itself if it is concave, because some of the emitting radiation will be intercepted by the surface itself. If the surface is flat or convex, the shape factor with respect to itself is zero. Let us consider a hemispherical black cavity with a flat black plate over it (Fig. 7.19). The surface of the cavity is denoted by 1 and that of the plate by 2. Then

$$F_{11} + F_{12} = 1$$
$$F_{21} + F_{22} = 1$$

But $F_{22} = 0$, therefore, $F_{21} = 1$

$$A_1 F_{12} = A_2 F_{21} = A_2$$

or, $$F_{12} = \frac{A_2}{A_1}$$

$$\therefore \quad F_{11} = 1 - \frac{A_2}{A_1} = 1 - \frac{\pi r^2}{2\pi r^2} = \frac{1}{2}$$

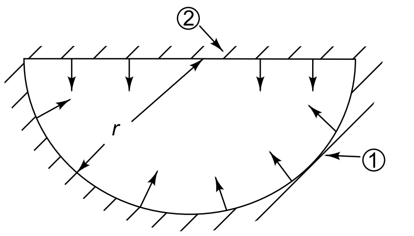


**Fig. 7.19** *Radiation in a hemispherical cavity*

Therefore, 50% of radiation emitted from the hemispherical surface is striking the surface itself, and is absorbed.

In two concentric black cylinders, if 1 represents the outer surface of the inner cylinder and 2 represents the inner surface of outer cylinder (Fig. 7.20), then

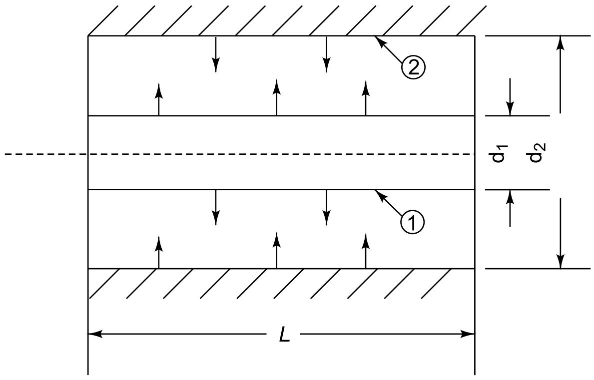


**Fig. 7.20** *Radiant heat exchange between concentric cylinders*

$$F_{11} + F_{12} = 1,$$
$$F_{11} = 0.$$
$$\therefore \quad F_{12} = 1,$$
$$A_1 F_{12} = A_2 F_{21}, F_{21} = A_1/A_2$$
$$F_{21} + F_{22} = 1$$
$$F_{22} = 1 - \frac{A_1}{A_2} = 1 - \frac{\pi d_1 L}{\pi d_2 L} = 1 - \frac{d_1}{d_2}$$

If $d_2 = 2d_1$, 50% of radiation from surface 2 falls on itself.

## 7.14 ELECTRICAL ANALOGY

Oppenheim [4] showed that the black enclosure may be represented by an analogous electrical network (Fig. 7.21). The rate of radiant flux between two black surfaces may be written as

$$Q_{1-2} = \sigma A_1 F_{12} (T_1^4 - T_2^4)$$
$$= \frac{\mathbf{E_{b_1} - E_{b_2}}}{\mathbf{1/(A_1 F_{12})}} \tag{7.39}$$

where $(E_{b_1} - E_{b_2})$ is the driving force or potential between the two nodes 1 and 2 for radiative energy transfer and $1/A_1 F_{12}$ is the resistance. The corresponding network is given in Fig. 7.21.

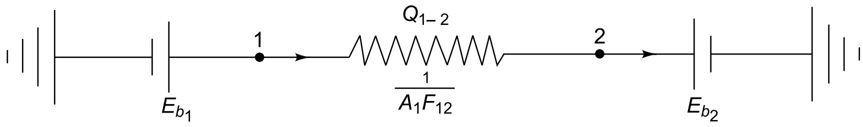


**Fig. 7.21** *Electrical analogy of radiative energy transfer between two black surfaces 1 and 2*

For radiative flux between the four walls of a black enclosure (Fig. 7.22),

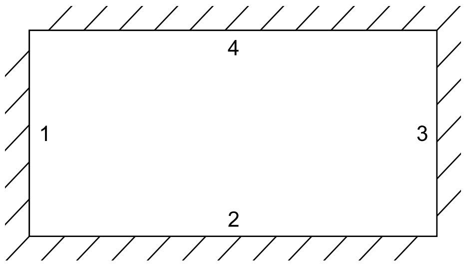


**Fig. 7.22** *Radiative flux between the walls of a four-wall black enclosure*

$$(Q_1)_{net} = \sigma A_1 F_{12} (T_1^4 - T_2^4) + \sigma A_1 F_{13} (T_1^4 - T_3^4) + \sigma A_1 F_{14} (T_1^4 - T_4^4) \tag{7.40}$$

or

$$(\boldsymbol{Q_1})_{\textbf{net}} = \frac{\mathbf{E_{b_1} - E_{b_2}}}{\mathbf{1/(A_1\, F_{12})}} + \frac{\mathbf{E_{b_1} - E_{b_3}}}{\mathbf{1/(A_1\, F_{13})}} + \frac{\mathbf{E_{b_1} - E_{b_4}}}{\mathbf{1/(A_1\, F_{14})}} \tag{7.41}$$

The equivalent network is shown in Fig. 7.23.

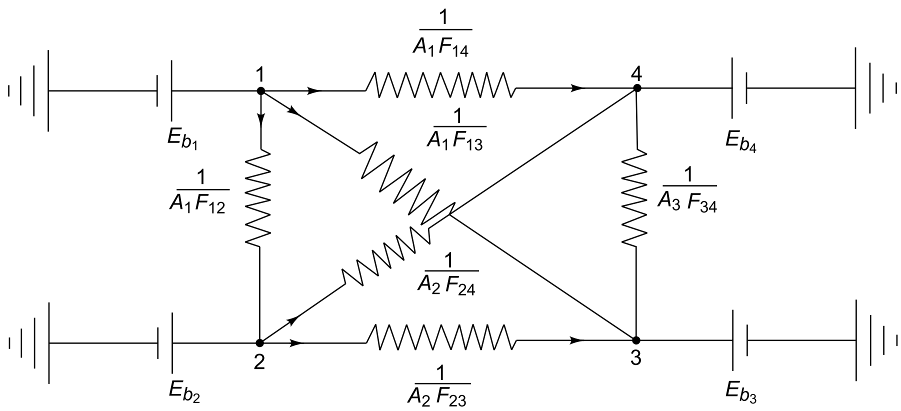


**Fig. 7.23** *Equivalent electrical analogy of radiant heat flux between the walls of the four-wall black enclosure*

From an analogous Kirchhoff's law, the net heat flux leaving node point 1 is

$$(Q_1)_{net} = \sum_{k=2}^{n} A_1 F_{1k}\, \sigma (T_1^4 - T_k^4) \tag{7.42}$$

which is identical with Eq. (7.40).

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## 7.15 RADIANT HEAT TRANSFER BETWEEN TWO BLACK SURFACES CONNECTED BY NONCONDUCTING AND RERADIATING WALLS

An industrial furnace may be assumed to be consisting of a *heat source* (fuel bed, electrical resistor bank, etc.), a *heat sink* (water tube bank of a boiler, surface of billets, etc.) and intermediate refractory surfaces which are essentially adiabatic surfaces except for a very small heat loss by conduction to the furnace exterior. The radiant heat fluxes are usually so large that they predominate in furnace heat transfer calculations. The refractory furnaces are considered to be "no-net-flux" surfaces for radiant transfer.

Let us consider an enclosure having a single source surface 1, a single sink surface 2 and refractory surface $R$, each of the surfaces maintained at uniform temperatures $T_1$, $T_2$ and $T_R$ respectively (Fig. 7.24).

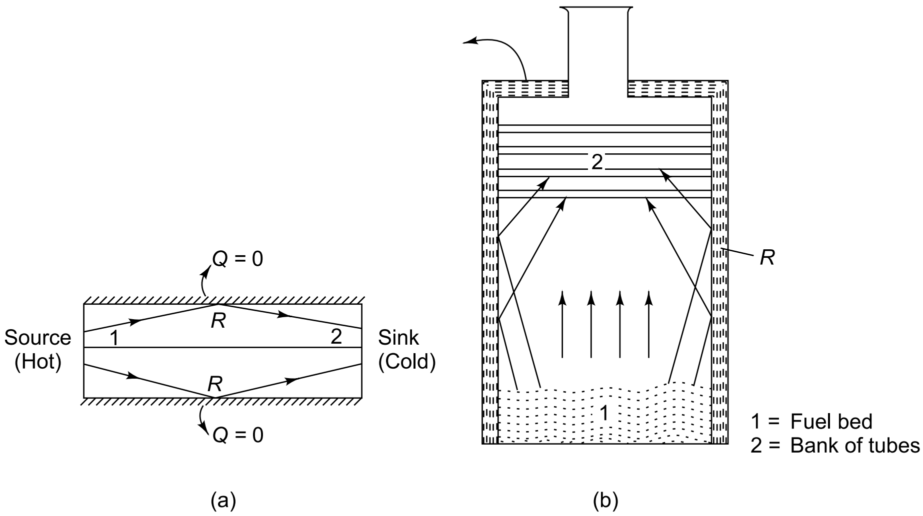


**Fig. 7.24** *Enclosure of black surfaces with a no-net-flux refractory surface (R)*

The net radiant heat transfer from hot surface 1

$$(Q_1)_{net} = \sigma A_1 F_{12} (T_1^4 - T_2^4) + \sigma A_1 F_{1R} (T_1^4 - T_R^4) \quad (7.43)$$

Since refractory walls store no energy,

$$\sigma A_1 F_{1R}(T_1^4 - T_R^4) = \sigma A_R F_{R2} (T_R^4 - T_2^4)$$

$$= \sigma A_2 F_{2R}(T_R^4 - T_2^4)$$

$$T_R^4 = \frac{A_1 F_{1R} T_1^4 + A_2 F_{2R} T_2^4}{A_1 F_{1R} + A_2 F_{2R}} \quad (7.44)$$

Substituting $T_R^4$ in Eq. (7.43)

$$(Q_1)_{net} = \sigma A_1 F_{12} (T_1^4 - T_2^4) + \sigma A_1 F_{1R} \left( T_1^4 - \frac{A_1 F_{1R} T_1^4 + A_2 F_{2R} T_2^4}{A_1 F_{1R} + A_2 F_{2R}} \right)$$

$$= \sigma \left[ A_1 F_{12} + \frac{1}{1/(A_1 F_{1R}) + 1/(A_2 F_{2R})} \right] (T_1^4 - T_2^4)$$

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or $$(Q_1)_{net} = \sigma A_1 \bar{F}_{12} (T_1^4 - T_2^4) \tag{7.45}$$

where $$\mathbf{A_1 \bar{F}_{12} = A_1 F_{12} + \frac{1}{1/(A_1F_{1R}) + 1/(A_2F_{2R})}} \tag{7.46}$$

$F_{1R} = 1 - F_{12}$ and $F_{2R} = 1 - F_{21}$. Figure 7.25 shows the electrical network equivalent to the furnace enclosure of Fig. 7.24. The node $R$ has no potential of its own.

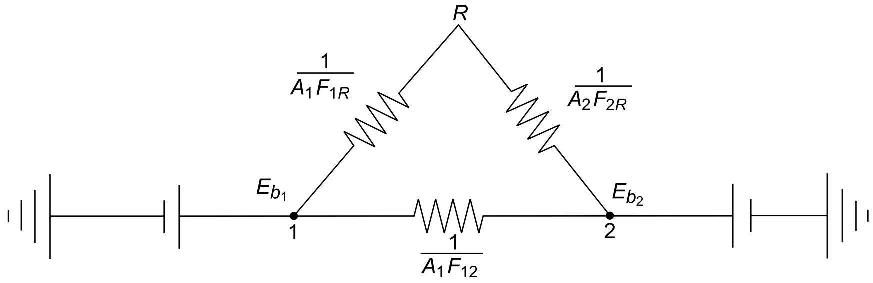


**Fig. 7.25** *Electrical analog of enclosure of black surfaces 1 and 2 with a refractory (re-radiating) surface R*

Now, $$A_2 F_{21} = A_1 F_{12}$$

$$F_{21} = \frac{A_1}{A_2} F_{12}$$

$$F_{2R} = 1 - \frac{A_1 F_{12}}{A_2} \text{ and } F_{1R} = 1 - F_{12}$$

From Eq. (7.46),

$$A_1 \bar{F}_{12} = A_1 F_{12} + \frac{1}{\dfrac{1}{A_1(1 - F_{12})} + \dfrac{1}{A_2\left[1 - (A_1 F_{12}/A_2)\right]}}$$

$$= \frac{A_1(A_2 - A_1 F_{12}^2)}{A_1 + A_2 - 2A_1 F_{12}}$$

$$\mathbf{\bar{F}_{12} = \frac{A_2 - A_1 F_{12}^2}{A_1 + A_2 - 2A_1 F_{12}}} \tag{7.47}$$

If $A_1 = A_2$,

$$\bar{F}_{12} = \frac{1 + F_{12}}{2} \tag{7.48}$$

If the surface 2 does not "see" the surface 1, there is no direct incident radiation on surface 2 from surface 1, then $F_{12} = 0$.

$$\bar{F}_{12} = \frac{A_2}{A_1 + A_2} \tag{7.49}$$

If $A_1 = A_2$ for $F_{12} = 0$,

$$\bar{F}_{12} = \frac{1}{2}$$

Equations (7.45) and (7.46) solve many furnace problems. But these are in error because the temperatures of the surfaces are not uniform in practice. Hottel [5, 6] extended this type of analysis to multisurface cases, each surface being at a uniform temperature.

## 7.16 EVALUATION OF SHAPE FACTOR

The shape factor is defined by the equation

$$\mathbf{F_{12}} = \frac{\mathbf{1}}{\mathbf{A_1 \pi}} \int_{A_1} \int_{A_2} \frac{\cos \varphi_1 \cos \varphi_2}{r^2} \, dA_1 \, dA_2 \tag{7.50}$$

The evaluation of $F_{12}$ by solving the double integrals, each taken over a surface, even for simple geometrical shapes or configurations of the surfaces, is quite involved entailing a lot of mathematical intricacies.

Hamilton and Morgan [7] evaluated shape factors for simple configurations involving rectangles, cylinders, etc. and presented the results in tables and charts. For an extensive study of shape factors, the reader is referred to Ozisik [8], Siegel and Howell [9], Sparrow and Cess [10] and Jakob [11].

For convenience of practical computation, shape factors are presented in the form of charts as shown in Figs 7.26 – 7.32. For the geometrical arrangements for which the charts are not available, the shape factor can be computed by the method of shape-factor algebra. Let us consider the determination of the shape factor $F_{dA_1-A_2}$ from an elemental surface $dA_1$ to a finite rectangular parallel surface $A_2$ (Fig. 7.33). Expressing the area $A_2$ as the algebraic sum of the four areas ($A_2 = A_3 - A_4 - A_5 + A_6$), the shape factor $F_{dA_1-A_2}$ can be written as

$$F_{dA_1-A_2} = F_{dA_1-A_3} - F_{dA_1-A_4} - F_{dA_1-A_5} + F_{dA_1-A_6}$$

The shape factors on the right-hand side of this equation can be obtained from Fig. 7.26.

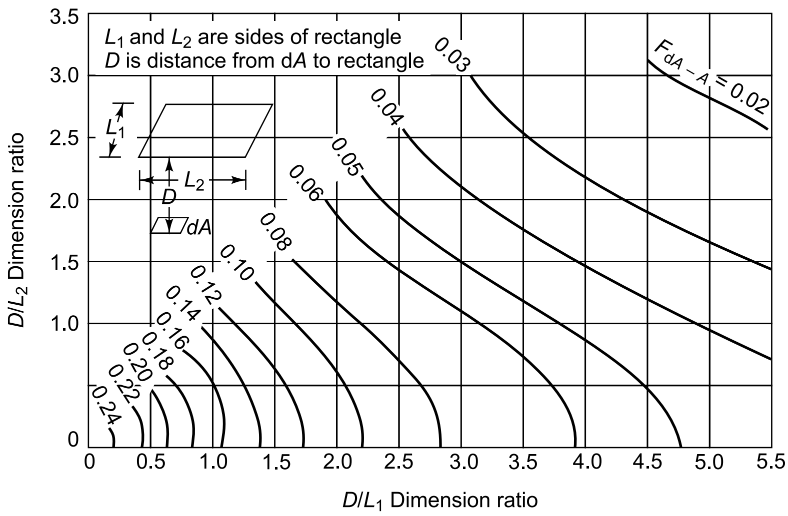


**Fig. 7.26** *Shape factor for a surface element dA and a rectangular surface A parallel to it*

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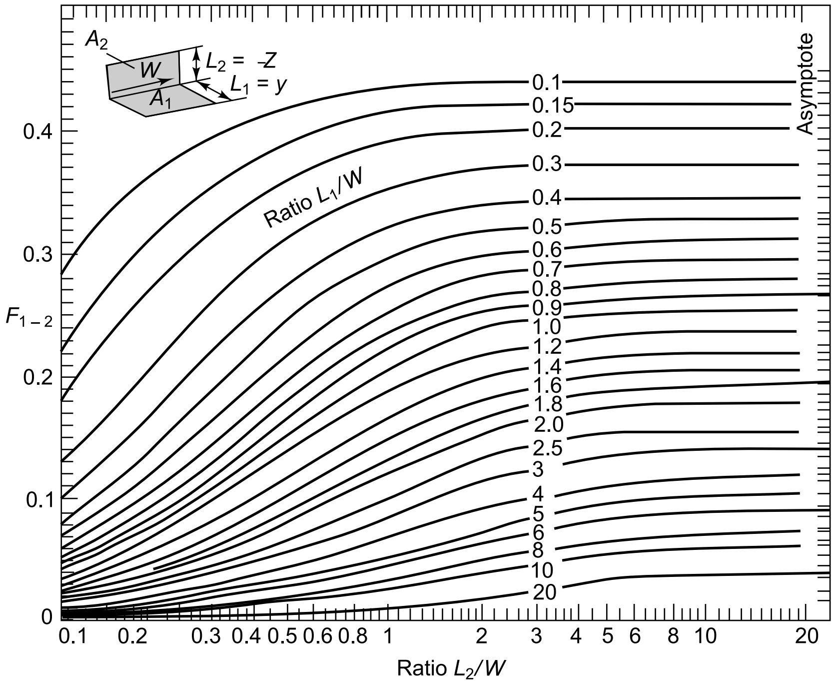


**Fig. 7.27** *Shape factor for adjacent rectangles in perpendicular planes sharing a common edge*

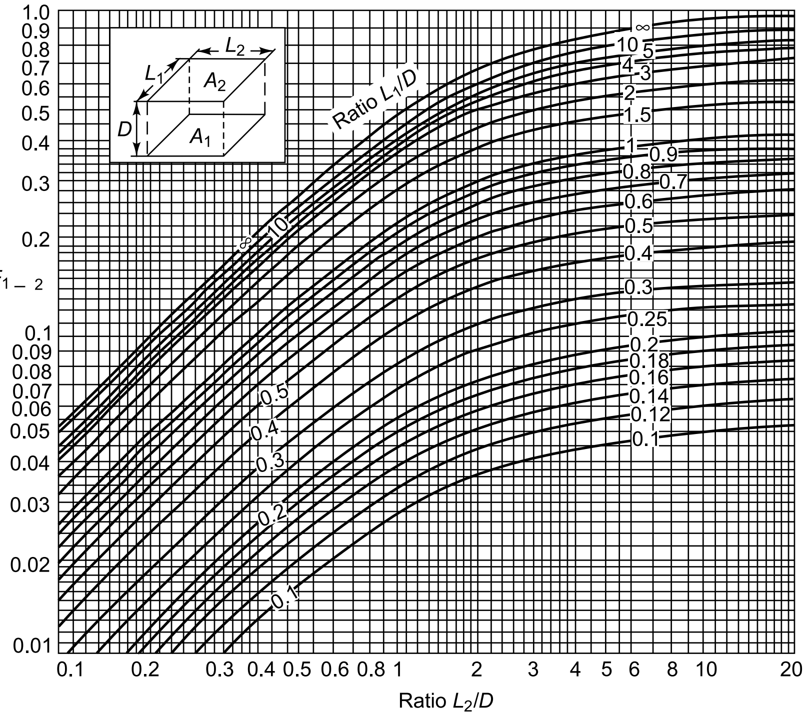


**Fig. 7.28** *Shape factor for directly opposed rectangles*

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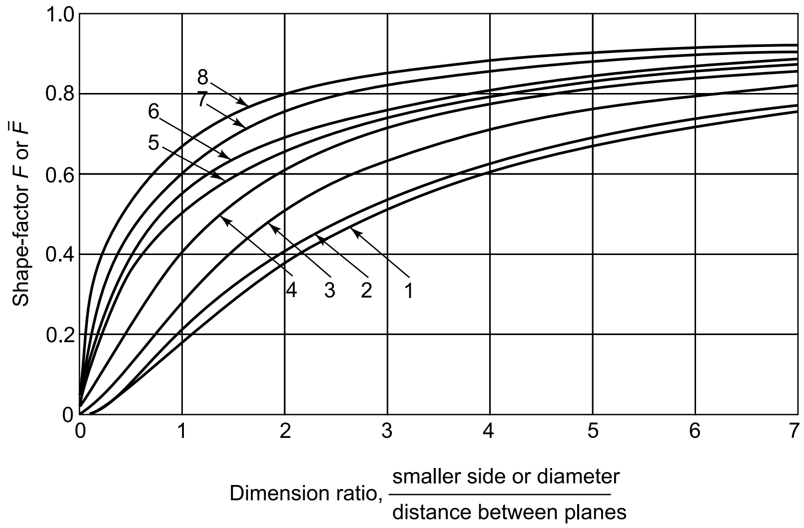


**Radiation between parallel planes, directly opposed:**

- 1,2,3 and 4: Direct radiation between the planes, $F$
- 5,6,7 and 8: Planes connected by nonconducting but reradiating walls, $\bar{F}$
- 1 and 5: Disks
- 2 and 6: Squares
- 3 and 7: 2:1 Rectangles
- 4 and 8: Long, narrow rectangles

**Fig. 7.29** *Shape factors for equal and parallel squares, rectangles, and disks*

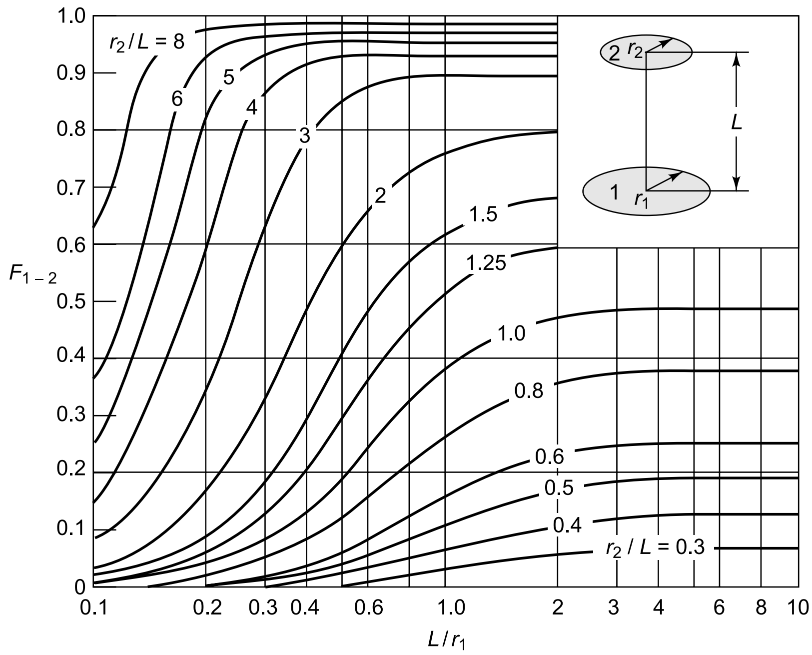


**Fig. 7.30** *Shape factor between two coaxial parallel disks*

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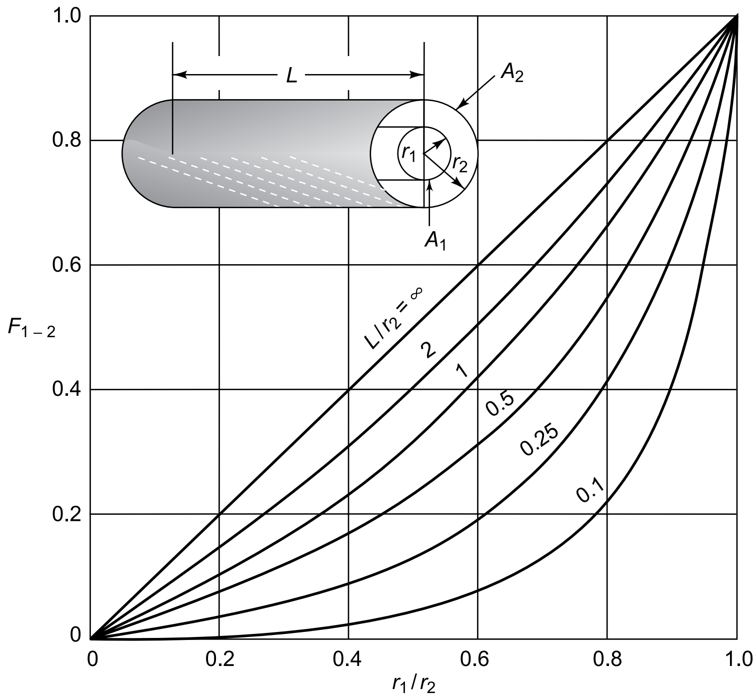
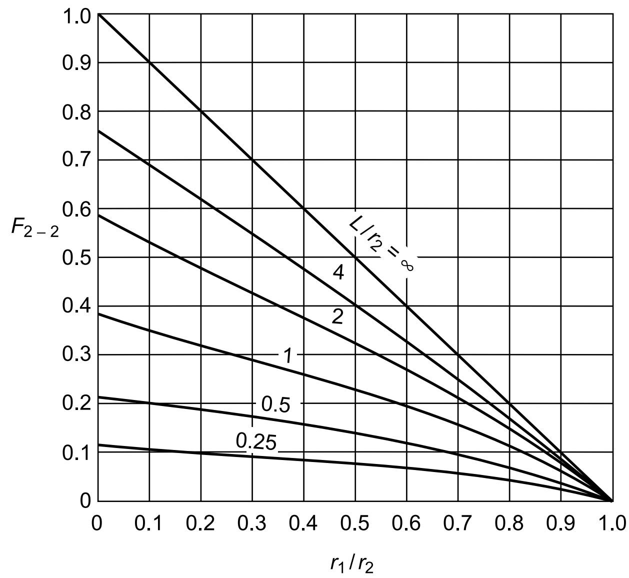



(a)




(b)

**Fig. 7.31** *Shape factor for two concentric cylinders of finite length: (a) outer cylinder to inner cylinder and (b) outer cylinder to itself*

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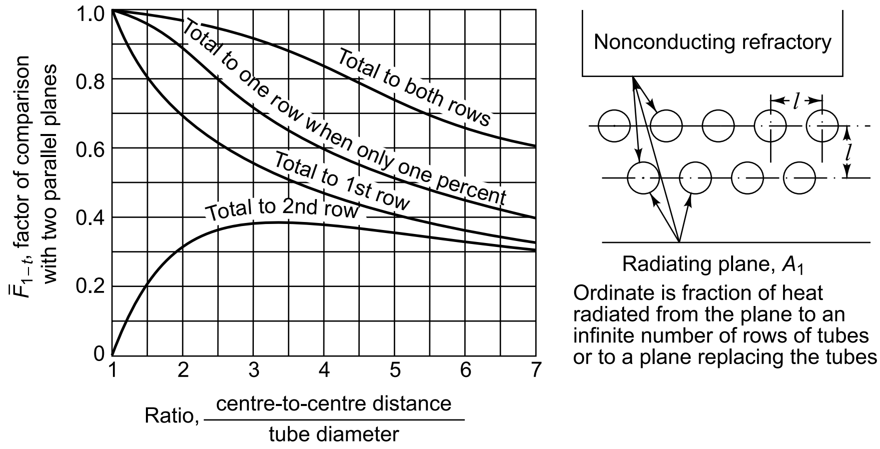


**Fig. 7.32** *Shape factor for a plane and one or two rows of tubes parallel to it*

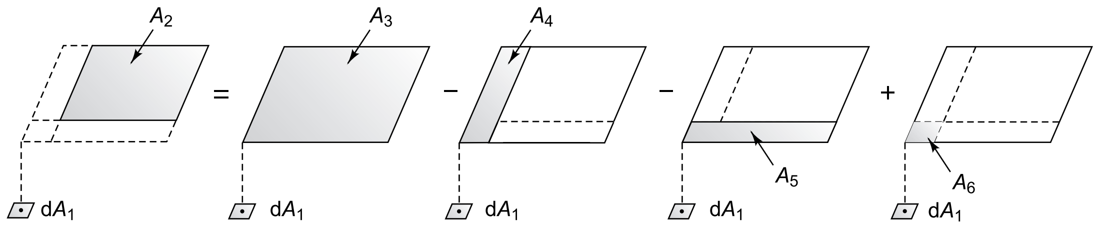


**Fig. 7.33** *Determination of shape factor by shape factor algebra*

## 7.17 RADIATION HEAT TRANSFER BETWEEN GRAY BODIES

Calculation of radiant heat exchange between black surfaces is relatively easy, since all the radiant energy which strikes a surface is absorbed. Once the geometrical shape factor has been determined, the calculation of heat transfer is quite simple. This is, however, not the case with nonblack bodies, since all the radiation striking a surface is not absorbed, some part is reflected. Also the emissivities and absorptivities are not uniform in all directions and for all wavelengths. The problem is somewhat simplified if the bodies are considered gray in which case $\varepsilon_\lambda$ and $\alpha_\lambda$ are constant over the entire spectrum of wavelength and so their average values are equal irrespective of temperatures, i.e.

$$\alpha = \varepsilon$$

Since the transmissivity $\tau$ is zero for solid bodies, $\alpha + \rho = 1$ and for a gray body, the reflectivity of the surface is considered as an additional parameter.

### 7.7.1 Radiant Exchange between Two Small Gray Bodies

Let us consider two gray bodies, represented by suffices 1 and 2, having emissivities $\varepsilon_1$ and $\varepsilon_2$. Let us suppose that the bodies are small compared with the distance between them. It may thus be assumed that of the radiation unabsorbed and reflected diffusely at each surface, a negligible proportion returns to the original emitting body.

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The energy emitted by the body 1 is $\sigma\,\varepsilon_1\,A_1\,T_1^4$, of which $F_{12}\,(\sigma\,\varepsilon_1\,A_1\,T_1^4)$ is incident on the second body, of which $\alpha_2\,(F_{12}\,\sigma\,\varepsilon_1\,A_1\,T_1^4)$ is absorbed. Since the bodies are gray, $\alpha_2 = \varepsilon_2$ and energy transfer from 1 to 2 is

$$Q_{1-2} = \sigma A_1\,F_{12}\,\varepsilon_1\,\varepsilon_2\,T_1^4$$

Similarly, $$Q_{2-1} = \sigma A_2\,F_{21}\,\varepsilon_1\,\varepsilon_2\,T_2^4$$

Therefore, the net radiant heat transfer between the two bodies is

$$(Q_{1-2})_{\text{net}} = \sigma A_1\,F_{12}\,\varepsilon_1\,\varepsilon_2\,(T_1^4 - T_2^4) \tag{7.51}$$

The equivalent emissivity of two small gray bodies is

$$\bar{\varepsilon} = \varepsilon_1\ \varepsilon_2 \tag{7.52}$$

This is also called the view factor $\mathcal{F}_{12}$ for gray bodies

or $$\mathcal{F}_{12} = \varepsilon_1\,\varepsilon_2 \tag{7.53}$$

In practice, of course, a portion of the reflected radiation from each is returned to the other body and reradiated so that the true equivalent emissivity $\bar{\varepsilon}$ or $\mathcal{F}_{12}$ is greater than that given by Eq. (7.52), which gives, in fact, the least possible value.

### 7.17.2 Radiant Exchange between Two Infinite Parallel Gray Planes

The radiant interchange between two infinite parallel gray planes involves no geometry factor, since $F_{12} = F_{21} = 1.0$. Let us consider two gray planes, as shown in Fig. 7.34. For gray surfaces, $\alpha = \varepsilon$ and $\rho = 1 - \varepsilon$. Surface 1 emits $\varepsilon_1\,E_{b_1}$ per unit time and area. Surface 2 absorbs the fraction $\alpha_2\,\varepsilon_1\,E_{b_1}$ or $\varepsilon_2\varepsilon_1 E_{b_1}$

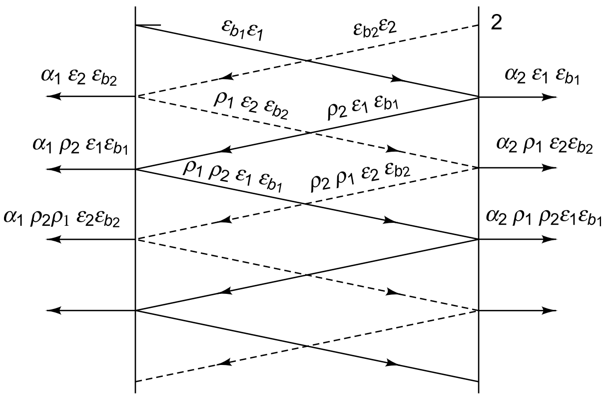


**Fig. 7.34** *Radiant heat exchange between two infinite parallel gray planes*

and reflects $\rho_2\,\varepsilon_1\,E_{b_1}$ or $(1-\varepsilon_2)\varepsilon_1\,E_{b_1}$ back towards $A_1$. The net heat transferred per unit of surface 1 to 2 is the emission $\varepsilon_1\,E_{b_1}$ minus the fraction of $\varepsilon_1\,E_{b_1}$ and $\varepsilon_2\,E_{b_2}$ which is ultimately absorbed by surface 1 after successive reflections. Therefore,

$$\begin{aligned}(Q_{1-2})_{\text{net}} &= A_1\,\varepsilon_1\,E_{b_1}\,[1 - \varepsilon_1\,(1-\varepsilon_2) - \varepsilon_1\,(1-\varepsilon_1)\,(1-\varepsilon_2)^2 - \varepsilon_1\,(1-\varepsilon_1)^2\,(1-\varepsilon_2)^3 - \cdots] \\ &\quad - A_2\,\varepsilon_2\,E_{b_2}\,[\varepsilon_1 + \varepsilon_1\,(1-\varepsilon_1)\,(1-\varepsilon_2) + \varepsilon_1\,(1-\varepsilon_1)^2\,(1-\varepsilon_2)^2 + \cdots]\end{aligned}$$

$$= A_1\, \varepsilon_1\, E_{b_1}\left[1 - \frac{\varepsilon_1\,(1-\varepsilon_2)}{1-(1-\varepsilon_1)\,(1-\varepsilon_2)}\right] - A_2\, \varepsilon_2\, E_{b_2}\, \frac{\varepsilon_1}{1-(1-\varepsilon_1)\,(1-\varepsilon_2)}$$

$$= A\frac{\varepsilon_1\varepsilon_2}{\varepsilon_1 + \varepsilon_2 - \varepsilon_1\varepsilon_2}(E_{b_1} - E_{b_2})$$

Since $A_1 = A_2 = A$.

$$(Q_{1-2})_{\text{net}} = A\sigma\frac{1}{1/\varepsilon_1 + 1/\varepsilon_2 - 1}(T_1^4 - T_2^4)$$

or $$\mathbf{(Q_{1-2})_{net} = \sigma A\, \mathcal{F}_{12}\, (T_1^4 - T_2^4)} \tag{7.54}$$

where $$\mathcal{F}_{12} = \frac{\mathbf{1}}{\mathbf{1/\varepsilon_1 + 1/\varepsilon_2 - 1}} \tag{7.55}$$

where $\mathcal{F}_{12}$ is the view factor for gray bodies.

## 7.18 RADIOSITY AND IRRADIATION

In calculating radiant heat transfer from gray surfaces, two terms will now be introduced.

*Radiosity* is the term used to indicate the total radiant energy leaving a surface per unit time and per unit surface area. It is denoted by the symbol $J$. This quantity differs from the emissive power in that the radiosity includes reflected energy as well as the original emission, regardless of any directional dependence or spectral preference.

*Irradiation* is the term used to denote the total radiation incident on a surface per unit time and per unit surface area. It is denoted by the symbol $G$.

The radiosity is the sum of the energy emitted and the energy reflected, when no energy is transmitted (Fig. 7.35), so that

$$J = \varepsilon E_b + \rho G$$

$$= \boldsymbol{\varepsilon}\mathbf{E_b + (1 - \boldsymbol{\varepsilon})G} \tag{7.56}$$

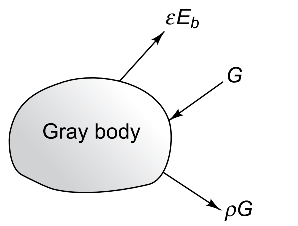


**Fig. 7.35** *Radiosity and irradiation in a gray body*

The net energy leaving the surface is the difference between the radiosity and irradiation

$$\frac{Q}{A} = J - G = \varepsilon E_b + (1-\varepsilon)G - G$$

$$= \boldsymbol{\varepsilon}\mathbf{(E_b - G)} \tag{7.57}$$

From Eq. (7.56),

$$\mathbf{G} = \frac{\mathbf{J - \boldsymbol{\varepsilon} E_b}}{\mathbf{1 - \boldsymbol{\varepsilon}}}$$

Substituting in Eq. (7.57),

$$\frac{Q}{A} = \varepsilon\left(E_b - \frac{J - \varepsilon E_b}{1-\varepsilon}\right) = \varepsilon\,\frac{E_b - J}{1-\varepsilon}$$

$$\mathbf{Q_{net}} = \frac{\mathbf{E_b - J}}{\mathbf{(1-\boldsymbol{\varepsilon})/A\boldsymbol{\varepsilon}}} \tag{7.58}$$

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Equation (7.58) provides a basis for network representation of the gray enclosure. If the numerator on the right side is considered as the potential difference, the denominator as the "*surface resistance*" to radiation heat transfer and the heat flow as current, then a network element could be drawn, as shown in Fig. 7.36.

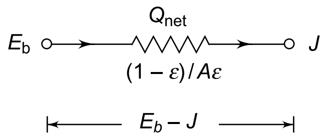


**Fig. 7.36** *Element representing "surface resistance" in radiation network method*

Let us now consider the exchange of radiant energy by two surfaces $A_1$ and $A_2$ (Fig. 7.37). Of the total radiation which leaves the surface 1, the amount that reaches surface 2 is $J_1A_1F_{12}$, and of the total energy leaving surface 2, the amount that reaches surface 1 is $J_2A_2F_{21}$.

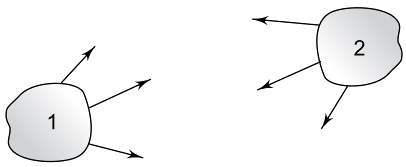


**Fig. 7.37** *Radiant interchange between two gray surfaces*

The net energy interchange between two surfaces is

$$Q_{1-2} = J_1 A_1 F_{12} - J_2 A_2 F_{21}$$
$$= (J_1 - J_2) A_1 F_{12}$$

Since $\qquad A_1 F_{12} = A_2 F_{21}$

$$\mathbf{Q_{1-2} = \frac{J_1 - J_2}{1/(A_1F_{12})}} \tag{7.59}$$

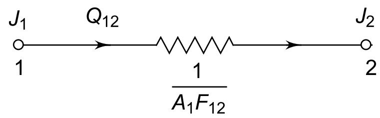


**Fig. 7.38** *Element representing "space resistance" in radiation network method*

We may thus construct a network element (Fig. 7.38) which represents Eq. (7.59). The denominator $1/(A_1\ F_{12})$ is called the "*space resistance*" and the numerator $(J_1 - J_2)$ is the potential difference.

## 7.19 RADIATION NETWORK FOR GRAY SURFACES EXCHANGING ENERGY

To construct a network for a particular radiation heat transfer problem, we only have to connect a "*surface resistance*", $(1 - \varepsilon)/A\varepsilon$, to each surface and a "*space resistance*", $1/(A_x\ F_{xy})$, between the radiosity potentials.

Figure 7.39 shows a network which represents two surfaces exchanging radiative energy with each other.

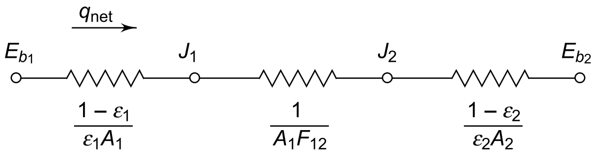


**Fig. 7.39** *Radiation network for two surfaces which see each other and nothing else*

The net heat transfer would be given by dividing the overall potential difference with the sum of the resistances.

$$(Q_{1-2})_{net} = \frac{E_{b_1} - E_{b_2}}{(1-\varepsilon_1)/\varepsilon_1 A_1 + 1/A_1F_{12} + (1-\varepsilon_2)/\varepsilon_2 A_2}$$

$$= \mathbf{\frac{\sigma(T_1^4 - T_2^4)}{(1-\varepsilon_1)/\varepsilon_1 A_1 + 1/A_1F_{12} + (1-\varepsilon_2)/\varepsilon_2 A_2}} \tag{7.60}$$

If $1/A_1\ \mathcal{F}_{12}$ is used to represent the overall resistance or $A_1\ \mathcal{F}_{12}$ as the overall conductance, then

$$(Q_{1-2})_{net} = \sigma A_1\ \mathcal{F}_{12}\ (T_1^4 - T_2^4) \tag{7.61}$$

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where $$\frac{1}{A_1\mathcal{F}_{12}} = \frac{1-\varepsilon_1}{\varepsilon_1 A_1} + \frac{1}{A_1 F_{12}} + \frac{1-\varepsilon_2}{\varepsilon_2 A_2} \tag{7.62}$$

or, $$\mathcal{F}_{12} = \frac{\mathbf{1}}{\left[(1/\varepsilon_1)-1\right] + 1/F_{12} + A_1/A_2\left[(1/\varepsilon_2)-1\right]} \tag{7.63}$$

For a three-body problem, the radiation network is shown in Fig. 7.40.

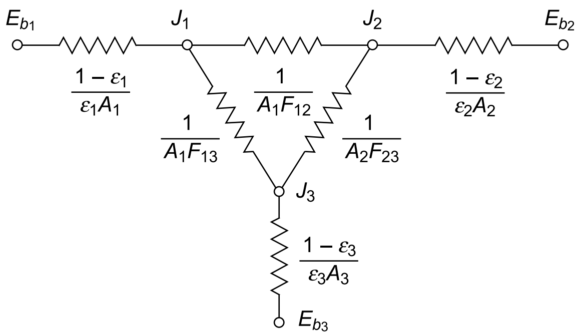


**Fig. 7.40** *Radiation network for three surfaces which see each other and nothing else*

The radiation heat exchange between body 1 and body 2 would be

$$Q_{1-2} = \frac{J_1 - J_2}{1/A_1F_{12}} \tag{7.64}$$

and that between body 1 and body 3,

$$Q_{1-3} = \frac{J_1 - J_3}{1/A_1F_{13}} \tag{7.65}$$

The values of radiosities have to be calculated for determining the heat flows in a problem of this type. Kirchhoff's law on electrical dc network theory which states that the algebraic sum of the currents entering a node is zero is applied.

The network method can be conveniently used to solve the problem of energy exchange between two gray surfaces connected by a nonconducting and reradiating surface to form an enclosure (Fig. 7.24). Figure 7.25 will get modified, since unlike black surfaces, gray surfaces have surface resistance. The corresponding radiation network for the system is shown in Fig. 7.41.

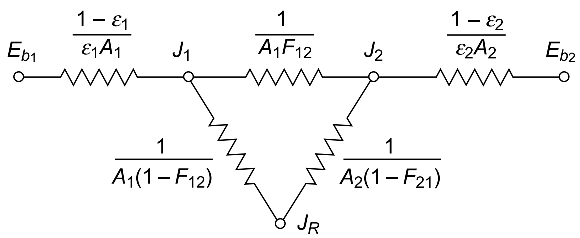


**Fig. 7.41** *Radiation network for two gray surfaces enclosed by a third surface which is nonconducting and reradiating*

Node $J_R$ is not connected to any surface resistance, since the surface $R$ has no source of its own and it only reflects energy. It is called the *floating node*. Now

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$$\overset{=0}{F_{11}} + F_{12} + F_{1R} = 1$$

$$F_{1R} = 1 - F_{12}$$

Similarly, $$F_{2R} = 1 - F_{21}$$

$$A_2 F_{2R} = A_2(1 - F_{21}) = A_R F_{R2}$$

The network above (Fig. 7.41) is a simple series parallel system and may be solved to compute the heat flow. Let $R'$ is the sum of the resistances $1/[A_1(1 - F_{12})]$ and $[1/A_2(1 - F_{12})]$, which are in series,

$$R' = \frac{1}{A_1(1 - F_{12})} + \frac{1}{A_2(1 - F_{21})}$$

$$= \frac{A_1 + A_2 - 2A_1F_{12}}{A_1A_2 - A_1^2F_{12} - A_1A_2F_{12} + A_1^2F_{12}^2}$$

Let $R_{eq}$ be the equivalent resistance of the parallel resistances of $R'$ and $1/(A_1 F_{12})$, so that

$$\frac{1}{R_{eq}} = \frac{1}{R'} + A_1F_{12}$$

$$= \frac{A_1A_2 - A_1^2F_{12} - A_1A_2F_{12} + A_1^2F_{12}^2}{A_1 + A_2 - 2A_1F_{12}} + A_1F_{12}$$

$$= \frac{A_1\left(A_2 - A_1F_{12}^2\right)}{A_1 + A_2 - 2A_1F_{12}}$$

$$\mathbf{R_{eq}} = \frac{\mathbf{A_1 + A_2 - 2A_1F_{12}}}{\mathbf{A_1\left(A_2 - A_1F_{12}^2\right)}}$$

The total resistance offered to heat flow is

$$\overline{R} = \frac{1 - \varepsilon_1}{\varepsilon_1 A_1} + \frac{A_1 + A_2 - 2A_1F_{12}}{A_1\left(A_2 - A_1F_{12}^2\right)} + \frac{1 - \varepsilon_2}{\varepsilon_2 A_2} \tag{7.66}$$

Therefore, the heat flow

$$(Q_{12})_{net} = \frac{E_{B_1} - E_{B_2}}{\dfrac{1 - \varepsilon_1}{\varepsilon_1 A_1} + \dfrac{A_1 + A_2 - 2A_1F_{12}}{A_1\left(A_2 - A_1F_{12}^2\right)} + \dfrac{1 - \varepsilon_2}{\varepsilon_2 A_2}}$$

or $$(Q_{12})_{net} = \frac{\sigma A_1\left(T_1^4 - T_2^4\right)}{\left[\dfrac{1}{\varepsilon_1} - 1\right] + \dfrac{A_1 + A_2 - 2A_1F_{12}}{A_2 - A_1F_{12}^2} + \dfrac{A_1}{A_2}\left[\dfrac{1}{\varepsilon_2} - 1\right]} \tag{7.67}$$

$$= \frac{\boldsymbol{\sigma}\mathbf{A_1\left(T_1^4 - T_2^4\right)}}{\left[\dfrac{\mathbf{1}}{\boldsymbol{\varepsilon}_1} - \mathbf{1}\right] + \dfrac{\mathbf{1}}{\overline{\mathbf{F}}_{12}} + \dfrac{\mathbf{A_1}}{\mathbf{A_2}}\left[\dfrac{\mathbf{1}}{\boldsymbol{\varepsilon}_2} - \mathbf{1}\right]} \tag{7.68}$$

where $$\mathbf{1/\overline{F}_{12}} = \frac{\mathbf{A_1 + A_2 - 2A_1F_{12}}}{\mathbf{A_2 - A_1F_{12}^2}} \tag{7.68a}$$

which is the same as Eq. (7.47).

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(i) If the surfaces are black, $\varepsilon_1 = 1$, $\varepsilon_2 = 1$, for which

$$(Q_{12})_{\text{net}} = \sigma A_1 \bar{F}_{12} (T_1^4 - T_2^4)$$

(ii) For infinite parallel planes, $A_1$ and $A_2$ are equal and radiation shape factor $F_{12} = F_{21} = 1$, since all the radiation leaving one plane reaches the other. Equation (7.67) reduces to

$$(Q_{1-2})_{\text{net}} = \frac{\sigma A_1 \left(T_1^4 - T_2^4\right)}{\left(\dfrac{1}{\varepsilon_1} - 1\right) + \dfrac{2A_1(1 - F_{12})}{A_1\left(1 - F_{12}^2\right)} + \left(\dfrac{1}{\varepsilon_2} - 1\right)}$$

$$= \frac{\sigma A_1 \left(T_1^4 - T_2^4\right)}{1/\varepsilon_1 + 1/\varepsilon_2 - 1}$$

$$= \boldsymbol{\sigma A_1 \mathcal{F}_{12} (T_1^4 - T_2^4)}$$

where $$\mathcal{F}_{12} = \frac{1}{1/\varepsilon_1 + 1/\varepsilon_2 - 1} \qquad (7.54)$$

This can also be obtained from Eq. (7.63).

(iii) For two long concentric cylinders as shown in Fig. 7.42, $F_{12} = 1$, Eq. (7.67) becomes

$$(Q_{12})_{\text{net}} = \frac{\sigma A_1 \left(T_1^4 - T_2^4\right)}{\left[\dfrac{1}{\varepsilon_1} - 1\right] + \dfrac{A_1 + A_2 - 2A_1}{A_2 - A_1} + \dfrac{A_1}{A_2}\left[\dfrac{1}{\varepsilon_2} - 1\right]}$$

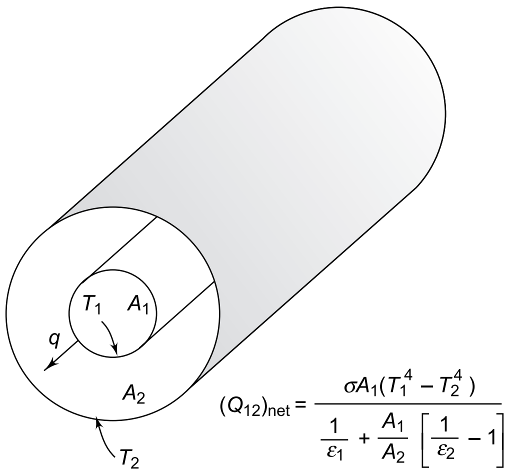


**Fig. 7.42** *Radiation interchange between two concentric cylindrical gray surfaces*

or $$(Q_{12})_{\text{net}} = \frac{\sigma A_1 \left(T_1^4 - T_2^4\right)}{\dfrac{1}{\varepsilon_1} + \dfrac{A_1}{A_2}\left[\dfrac{1}{\varepsilon_2} - 1\right]} \qquad (7.69)$$

$$= A_1 \mathcal{F}_{12} (T_1^4 - T_2^4)$$

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where $$\mathcal{F}_{12} = \frac{1}{\dfrac{1}{\varepsilon_1} + \dfrac{A_1}{A_2}\left[\dfrac{1}{\varepsilon_2} - 1\right]} \qquad (7.70)$$

This can also be obtained directly from Eq. (7.63).

If $A_1 = \pi d_1 L$ and $A_2 = \pi d_2 L$,

$$\mathcal{F}_{12} = \frac{1}{\dfrac{1}{\varepsilon_1} + \dfrac{d_1}{d_2}\left[\dfrac{1}{\varepsilon_2} - 1\right]} \qquad (7.71)$$

where $d_1$ and $d_2$ are the diameters of the inner and outer cylinder respectively.

(iv) Equations (7.69) and (7.70) would also hold good for concentric spheres, for which

$$\mathcal{F}_{12} = \frac{1}{\dfrac{1}{\varepsilon_1} + \dfrac{4\pi r_1^2}{4\pi r_2^2}\left[\dfrac{1}{\varepsilon_2} - 1\right]}$$

$$= \frac{1}{\dfrac{1}{\varepsilon_1} + \dfrac{r_1^2}{r_2^2}\left[\dfrac{1}{\varepsilon_2} - 1\right]} \qquad (7.72)$$

where $r_1$ and $r_2$ are the radii of the inner and outer spheres.

(v) When a small body is enclosed by a large body, $A_1/A_2 \sim 0$, and $F_{12} = 1$, where suffix 1 stands for the small body, Eq. (7.63) reduces to

$$Q_{12} = \sigma A_1 \varepsilon_1 (T_1^4 - T_2^4) \qquad (7.73)$$

which can be used to estimate the radiation energy loss from a hot object in a large room.

For gray enclosures, any of the three network circuits may represent them (Fig 7.43).

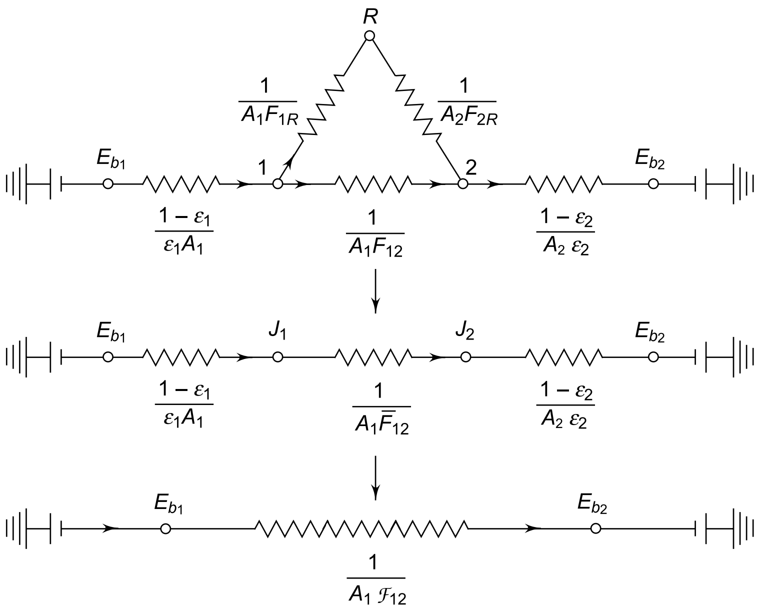


**Fig. 7.43** *Electrical analog of enclosure of gray surfaces with a refractory surface*

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The overall conductance is $A_1 \mathcal{F}_{12}$ and the overall resistance is

$$\overline{R} = \frac{1}{A_1\mathcal{F}_{12}} = \frac{1-\varepsilon_1}{\varepsilon_1 A_1} + \frac{1}{A_1\overline{F}_{12}} + \frac{1-\varepsilon_2}{\varepsilon_2 A_2}$$

so that

$$\mathcal{F}_{12} = \frac{1}{\left[\frac{1}{\varepsilon_1} - 1\right] + \frac{1}{\overline{F}_{12}} + \frac{A_1}{A_2}\left[\frac{1}{\varepsilon_2} - 1\right]} \tag{7.74}$$

If no reradiating surfaces exist, then $\overline{F}_{12} = F_{12}$.

For the network (Fig. 7.43), it is clear that the following equation is an alternative form of the equation

$$Q_{12} = \sigma A_1 \mathcal{F}_{12} (T_1^4 - T_2^4)$$

$$(Q_1)_{net} = A_1 \overline{F}_{12} (J_1 - J_2) \tag{7.75}$$

where
$$A_1 \overline{F}_{12} = \frac{(Q_{12})_{net}}{J_1 - J_2}$$

If $A_1$ and $A_2$ do not see each othe, $F_{12} = 0$. Therefore, from Eq (7.68a)

$$\frac{1}{\overline{F}_{12}} = \frac{A_1 + A_2 - 2A_1F_{12}}{A_2 - A_1F_{12}^2} = 1 + \frac{A_1}{A_2}$$

From Eq. (7.75),

$$\mathcal{F}_{12} = \frac{1}{\left(\frac{1}{\varepsilon_1}\right) - 1 + 1 + \frac{A_1}{A_2} + \left(\frac{A_1}{A_2}\right)\left(\frac{1}{\varepsilon_2}\right) - \frac{A_1}{A_2}}$$

or

$$\mathcal{F}_{12} = \frac{1}{\frac{1}{\varepsilon_1} + \left(\frac{A_1}{A_2}\right)\left(\frac{1}{\varepsilon_2}\right)} \tag{7.76}$$

Even if the two surfaces do not see each other, still they receive energy via the reradiating surface connected between them.

If the temperature of each type of enclosure is not uniform, the surfaces are then to be divided into a number of parts, each of which may be considered to have a uniform temperature. This would result in a number of source and sink surfaces and of no-net-flux or reradiating surfaces. Hottel treated such a complex problem extensively with an energy analysis and Oppenheim [4] attempted to solve it with a network analysis.

## 7.20 HOTTEL'S CROSSED STRING METHOD FOR ESTIMATING SHAPE FACTOR FOR INFINITELY LONG SURFACES

Many problems encountered in practice involve geometries of constant cross-section such as channels and ducts that are very long in one direction and can be considered to be two-dimensional. The shape factor between their surfaces can be determined by the very simple *crossed-strings method* developed by H.C. Hottel in the 1950s. The surfaces of the geometry do not need to be flat; they can be convex, concave or of any irregular shape.

Let us consider the geometry shown in Fig. 7.44 and find the shape factor $F_{1-2}$ between surfaces 1 and 2. Let us connect the end points with tightly stretched strings indicated by dashed lines.

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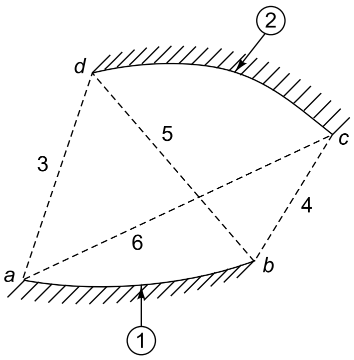


**Fig. 7.44** *Determination of shape factor $F_{12}$ by using the crossed-string method*

If we consider the triangle *adb*,

$$F_{13} + F_{15} = 1.0$$
$$F_{31} + F_{35} = 1.0$$
$$F_{51} + F_{53} = 1.0 \tag{7.77}$$

By reciprocity theorem,

$$A_1F_{13} = A_3F_{31}$$
$$A_1F_{15} = A_5F_{51} \tag{7.78}$$
$$A_3F_{35} = A_5F_{53}$$

From Eqs (7.77) and (7.78),

$$A_1F_{13} + A_1F_{15} = A_1 \qquad A_3F_{31} + {}_3F_{35} = A_3$$
$$A_1F_{13} + A_3F_{35} = A_3 \qquad A_5F_{51} + A_5F_{53} = A_5$$
$$A_1F_{15} + A_3F_{35} = A_5 \qquad A_3F_{35} = A_3 - A_1F_{13}$$
$$A_1F_{15} - A_3F_{35} = A_1 - A_3$$
$$A_1F_{15} = A_1 - A_3 + A_3F_{35}$$
$$= A_1 - A_3 + A_5 - A_1F_{15}$$
$$A_1F_{15} = \frac{A_1 - A_3 + A_5}{2}$$
$$A_1F_{13} = A_1 - F_{15} = A_1 - \frac{A_1}{2} + \frac{A_3}{2} - \frac{A_5}{2}$$
$$F_{13} = \frac{A_1 + A_3 - A_5}{2A_1}$$

Similarly, $$F_{14} = \frac{A_1 + A_4 - A_6}{2A_1}$$

Now, $$F_{12} + F_{13} + F_{14} = 1.0$$

$$F_{12} = 1 - F_{13} - F_{14} = 1 - \frac{A_1 + A_3 - A_5}{2A_1} - \frac{A_1 + A_4 - A_6}{2A_1}$$

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$$= \frac{2A_1 - A_1 - A_3 + A_5 - A_1 - A_4 + A_6}{2A_1}$$

$$= \frac{(A_5 + A_6) - (A_3 + A_4)}{2A_1}$$

In terms of lengths of strings instead of areas,

$$F_{12} = \frac{(L_5 + L_6) - (L_3 + L_4)}{2L_1}$$

Hottel's cross-string method can thus be expressed as

$$\mathbf{F_{i-j}} = \frac{\sum \textbf{(crossed strings)} - \sum \textbf{(uncrossed strings)}}{\textbf{2} \times \textbf{(string on surface i)}} \tag{7.79}$$

## 7.21 RADIATION SHIELDS

Radiation heat transfer between two surfaces may be reduced either by using the materials which are highly reflective or by introducing radiation shields between them.

Figure 7.45 shows two infinite parallel gray planes interchanging radiative energy between them with and without a radiation shield. For the case (a) without a radiation shield.

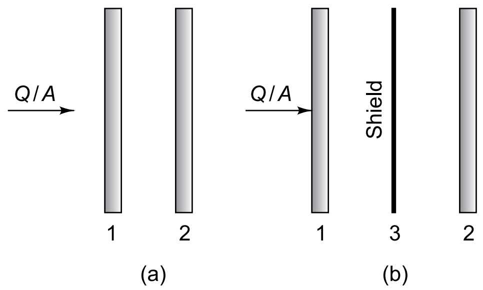


**Fig. 7.45** *Radiation between parallel infinite planes with and without radiation shield*

$$\frac{Q_{12}}{A} = \frac{\sigma\,(T_1^4 - T_2^4)}{1/\varepsilon_1 + 1/\varepsilon_2 - 1}$$

For the case (b), with a radiation shield between the surfaces, at equilibrium

$$\frac{\mathbf{Q_{13}}}{\mathbf{A}} = \frac{\mathbf{Q_{32}}}{\mathbf{A}} = \frac{\boldsymbol{\sigma}\,(\mathbf{T_1^4 - T_3^4})}{\mathbf{1/\boldsymbol{\varepsilon}_1 + 1/\boldsymbol{\varepsilon}_3 - 1}} = \frac{\boldsymbol{\sigma}(\mathbf{T_3^4 - T_2^4})}{\mathbf{1/\boldsymbol{\varepsilon}_3 + 1/\boldsymbol{\varepsilon}_2 - 1}}$$

where $T_3$ is the equilibrium temperature of the shield. If $T_3$ is known, the heat transfer rate can easily be calculated. The radiation network with one shield is shown in Fig. 7.46.

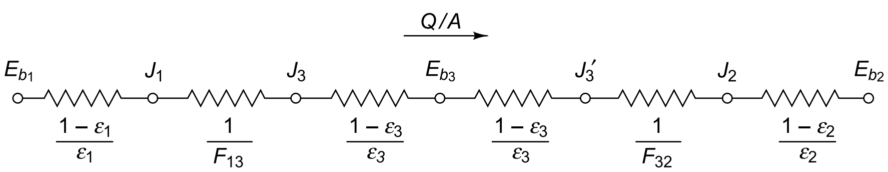


**Fig. 7.46** *Radiation network for two parallel planes separated by one radiation shield*

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If the two parallel planes are of equal emissivity $\varepsilon$,

$$\frac{Q_{12}}{A} = \frac{\sigma(T_1^4 - T_2^4)}{2/\varepsilon - 1} \tag{7.80}$$

If the third plane placed between them also has the same emissivity, at equilibrium

$$\frac{Q_{13}}{A} = \frac{\sigma(T_1^4 - T_3^4)}{2/\varepsilon - 1} = \frac{Q_{32}}{A} = \frac{\sigma(T_3^4 - T_2^4)}{2/\varepsilon - 1} \tag{7.81}$$

At thermal equilibrium $Q_{13}/A = Q_{32}/A$, therefore

$$T_1^4 - T_3^4 = T_3^4 - T_2^4$$

or

$$T_3^4 = \frac{T_1^4 + T_2^4}{2} \tag{7.82}$$

Substituting in Eq. (7.81),

$$\frac{Q_{13}}{A} = \frac{Q_{32}}{A} = \frac{\sigma\left[T_1^4 - (T_1^4/2) - (T_2^4/2)\right]}{2/\varepsilon - 1}$$

$$= \frac{1}{2}\frac{(T_1^4 - T_2^4)}{2/\varepsilon - 1} \tag{7.83}$$

From Eqs (7.80) and (7.83)

$$\left(\frac{Q_{12}}{A}\right)_{\text{with 1 shield}} = \frac{1}{2}\left(\frac{Q_{12}}{A}\right)_{\text{without shield}}$$

By the use of one radiation shield, the net radiant heat transfer is reduced by 50%. The position of the shield so long as it does not touch either of the planes does not alter its effectiveness.

If $N$ shields are placed between the two planes 1 and 2, there would be $(2N + 2)$ "surface resistances", two for each shield and one for each heat transfer surface, and $(N + 1)$ "space resistances" (which would all be unity). The total resistance would thus be

$$R\text{ (with } N \text{ shields)} = (2N + 2)\ \frac{1-\varepsilon}{\varepsilon} + (N+1)(1)$$

$$= (N + 1)\left(\frac{2}{\varepsilon} - 1\right)$$

The resistance when no shield is present is

$$R(\text{no shield}) = \frac{2}{\varepsilon} - 1$$

The resistance with the shields in place is $(N + 1)$ times as large as when the shields are absent. Thus

$$\left(\frac{Q}{A}\right)_{\text{with N shields}} = \frac{1}{N+1}\left(\frac{Q}{A}\right)_{\text{without shields}} \tag{7.84}$$

From Fig. 7.46,

$$\frac{Q_{12}}{A} = \frac{\sigma A(T_1^4 - T_2^4)}{\dfrac{1-\varepsilon_1}{\varepsilon_1} + \dfrac{1}{F_{13}} + \dfrac{1-\varepsilon_3}{\varepsilon_3} + \dfrac{1-\varepsilon_3}{\varepsilon_3} + \dfrac{1}{F_{23}} + \dfrac{1-\varepsilon_2}{\varepsilon_2}}$$

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$$= \frac{\sigma A(T_1^4 - T_2^4)}{\frac{1}{\varepsilon_1} - 1 + 1 + \frac{1}{\varepsilon_3} - 1 + \frac{1}{\varepsilon_3} - 1 + 1 + \frac{1}{\varepsilon_2} - 1}, \quad (\because F_{13} = F_{23} = 1)$$

$$= \frac{\sigma A(T_1^4 - T_2^4)}{\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_3} - 1\right) + \left(\frac{1}{\varepsilon_2} + \frac{1}{\varepsilon_3} - 1\right)}$$

$$= \frac{\boldsymbol{\sigma A(T_1^4 - T_2^4)}}{\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} + 2\left(\frac{1}{\varepsilon_3}\right) - 2}$$

This equation can be generalized for a system of two parallel plates separated by $N$ shields of emissivities $\varepsilon_{s_1}, \varepsilon_{s_2}, \ldots \varepsilon_{s_N}$ as

$$\frac{\mathbf{Q_{12}}}{\mathbf{A}} = \frac{\boldsymbol{\sigma A(T_1^4 - T_2^4)}}{\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} + 2\sum_{i=1}^{N} \frac{1}{\varepsilon_{3i}} - (N + 1)} \tag{7.84a}$$

### 7.21.1 Radiation Error in High Temperature Measurement

If the temperature of a high temperature gas stream is measured by the insertion of a thermometer or thermocouple, the effects of the radiant exchange between the pipe walls and the temperature sensing element introduce considerable error.

If $T_g$ is the gas temperature to be measured and $T_c$ is the measured temperature, at steady state, the heat transfer by convection from gas to the thermocouple is equal to the heat transfer by radiation from thermocouple to the wall [Fig. 7.47(a)], so that

$$Q = hA_c (T_g - T_c)$$

$$= \sigma A_c \mathcal{F}_{cw} (T_c^4 - T_w^4) \tag{7.85}$$

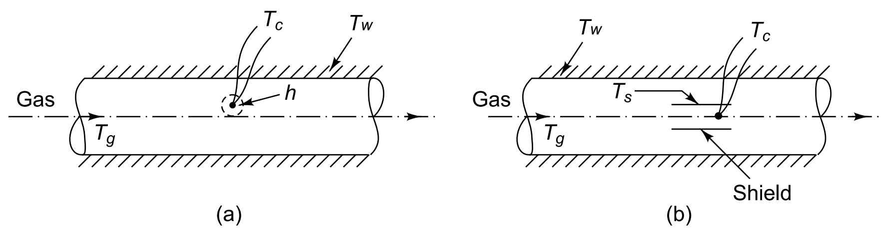


**Fig. 7.47** *Gas temprature measurement with (a) a bare thermocouple and (b) a shielded thermocouple*

where $\mathcal{F}_{cw} = \varepsilon_c$, the emissivity of the thermocouple which is very small compared to the enclosing wall, and $A_c$ is the area of the thermocouple. Here, $(T_g - T_c)$ is the *thermocouple error*.

When the couple is shielded, this error is considerably reduced. At steady state, (i) the heat transfer by convection from gas to couple is equal to that by radiation from couple to shield and (ii) the heat transfer by convection from gas to shield and that by radiation from couple to shield are equal to heat transfer by radiation from shield to wall [Fig. 7.47(b)].

(i) $hA_c (T_g - T_c) = \sigma A_c \mathcal{F}_{cs} (T_c^4 - T_s^4)$ (7.86)

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(ii) $h2A_s(T_g - T_s) + \sigma A_c \mathcal{F}_{cs}(T_c^4 - T_s^4) = \sigma A_s \mathcal{F}_{sw}(T_s^4 - T_w^4)$ (7.87)

where $\mathcal{F}_{cs} = \dfrac{1}{(1/\varepsilon_c) + (A_c/A_s)\left[(1/\varepsilon_s) - 1\right]}$ and $\mathcal{F}_{sw} = \varepsilon_s$

$T_s$ is the equilibrium temperature of the shield, $A_s$ is the shield surface area and $\varepsilon_s$ is the emissivity of the shield. Since heat is transferred by convection from gas to both sides of the shield, the total surface area of the shield is $2A_s$, as provided in the first term of Eq. (7.87). Assuming a value of $T_s$ and solving for $T_g$ from Eqs (7.86) and (7.87) by trial and error, the true gas temperature can be computed.

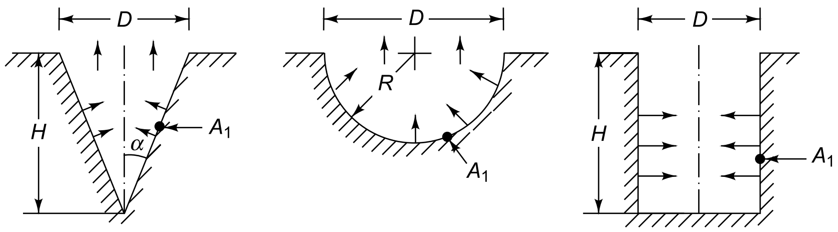
## 7.22 RADIATION FROM CAVITIES

The emission of radiant energy from cavities of regular geometrical shapes (Fig. 7.48) can be estimated.

Let us consider the conical cavity, as shown in Fig. 7.48(a), which is of diameter $D$, height $H$, lateral length $L$, semi-vertex angle $\alpha$ and surface area $A_1$. The temperature $T_1$ of the surface is uniform. Part of the radiation from the surface falls on itself, of which a portion is absorbed and the remainder is reflected.

Rate of emission from the surface $= A_1\, \varepsilon_1\, \sigma T_1^4$

Of this, the amount falling on $A_1$ and absorbed by it $= A_1\, \varepsilon_1\, \sigma T_1^4 F_{11}\, \varepsilon_1$, where $F_{11}$ is the shape factor of the conical surface with respect to itself.




(a) Conical cavity (b) Hemispherical cavity (c) Cylindrical cavity

**Fig. 7.48** *Radiant emission from cavities*

The amount reflected $= (1 - \varepsilon_1)\, A_1 \varepsilon_1\, \sigma T_1^4\, F_{11}$. Of this reflected energy, quantity falling on $A_1$ and absorbed $= (1 - \varepsilon_1)\, A_1\, \varepsilon_1\, \sigma T_1^4 F_{12}^2\, \varepsilon_1$.

Reflected $= (1 - \varepsilon_1)\, A_1\, \varepsilon_1\, \sigma T_1^4 F_{11}^2\, (1 - \varepsilon_1)$

Absorbed $= (1 - \varepsilon_1)\, A_1\, \varepsilon_1\, \sigma T_1^4 F_{11}^3\, (1 - \varepsilon_1)\, \varepsilon_1$

Reflected $= (1 - \varepsilon_1)\, A_1\, \varepsilon_1\, \sigma T_1^4 F_{11}^3\, (1 - \varepsilon_1)^2$ and so on

Net rate of emission from the surface

$$Q = \text{Total emission rate} - \text{Total absorption rate}$$

$$= A_1\, \varepsilon_1\, \sigma T_1^4\, [A_1\, \varepsilon_1\, \sigma T_1^4 F_{11}\, \varepsilon_1 + (1 - \varepsilon_1)\, A_1 \varepsilon_1\, \sigma T_1^4 F_{11}^2\, \varepsilon_1$$

$$+ (1 - \varepsilon_1)\, A_1 \varepsilon_1\, \sigma T_1^4 F_{11}^3\, (1 - \varepsilon_1)\, \varepsilon_1 + \cdots]$$

$$= A_1\, \varepsilon_1\, \sigma T_1^4\, [1 - \varepsilon_1 F_{11} - \varepsilon_1 (1 - \varepsilon_1) F_{11}^2 - \varepsilon_1 (1 - \varepsilon_1)^2 F_{11}^3 - \cdots]$$

$$= A_1\, \varepsilon_1\, \sigma T_1^4\, [1 - \varepsilon F_{11}\, (1 + (1 - \varepsilon_1) F_{11} + (1 - \varepsilon_1)^2 F_{11}^2 + \cdots)]$$

$$= A_1\, \varepsilon_1\, \sigma T_1^4 \left[1 - \frac{\varepsilon_1 F_{11}}{1 - (1 - \varepsilon_1) F_{11}}\right]$$

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$$= \mathbf{A_1 \varepsilon_1 \sigma T_1^4 \frac{1 - F_{11}}{1 - F_{11}(1-\varepsilon_1)}} \tag{7.88}$$

Let us consider an imaginary flat surface $A_2$ closing the cavity (Fig. 7.49). Since $A_1$ and $A_2$ together form an enclosure,

$$F_{11} + F_{12} = 1$$

$$F_{22} + F_{21} = 1$$

Since $F_{22} = 0, F_{21} = 1$

$$A_1 F_{12} = A_2 F_{21}$$

$$F_{12} = A_2/A_1$$

$\therefore$ $$F_{11} = 1 - F_{12} = 1 - \frac{A_2}{A_1}$$

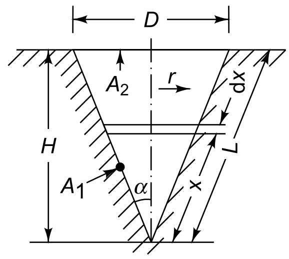


**Fig. 7.49** *Conical cavity with a flat plate on top*

$$A_2 = \frac{\pi}{4} D^2$$

$$\frac{r}{x} = \frac{D}{2L}, \quad \tan\alpha = \frac{D}{2H}, \quad \sin\alpha = \frac{D}{2L}$$

$$D = 2H \tan\alpha, \quad L = \frac{D}{2\sin\alpha}, \quad L = \frac{H}{\cos\alpha}$$

$$A_1 = \int_0^L 2\pi r\, dx = \int_0^L 2\pi \frac{D}{2L} x\, dx$$

$$= \frac{\pi D L}{2} = \text{surface area of the cone}$$

$$F_{11} = 1 - \frac{(\pi/4)D^2}{(\pi DL/2)} = 1 - \frac{1}{2}\frac{D}{L}$$

or $$\mathbf{F_{11} = 1 - \sin\alpha} \tag{7.89}$$

Substituting in Eq. (7.88),

$$Q = \sigma A_1 \varepsilon_1 T_1^4 \frac{\sin\alpha}{1 - (1-\varepsilon_1)(1-\sin\alpha)} \tag{7.90}$$

Putting for $$A_1 = \frac{\pi DL}{2} = \frac{\pi}{2} 2H \tan\alpha \frac{H}{\cos\alpha}$$

$$\mathbf{Q = \varepsilon_1 \sigma T_1^4 \pi H^2 \tan^2\alpha \frac{1}{1-(1-\varepsilon_1)(1-\sin\alpha)}} \tag{7.91}$$

Similarly, expressions for radiant emissions from cylindrical and hemispherical cavities can be obtained.

## 7.23 RADIATION FROM GASES AND VAPOURS

Gases such as oxygen, nitrogen, hydrogen, dry air, etc. are transparent to thermal radiation i.e., these gases neither emit nor absorb radiant energy at the temperature of interest. On the other hand, some gases and vapours such as $CO_2$, CO, $H_2O$, $SO_2$, $NH_3$, hydrocarbons, etc. absorb and emit radiant energy significantly. Of these gases, $CO_2$ and $H_2O$ vapour are the most important so far as the atmosphere and the industrial furnaces are concerned.

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There are, however, certain differences between radiation from gases and that from solids.

1. Solids radiate and absorb energy at all wavelengths over the entire spectrum (from $\lambda = 0$ to $\lambda = \infty$). Gases like $CO_2$ and $H_2O$ emit and absorb radiation only between narrow ranges of wavelengths called *bands* (Fig. 7.50). Gases are, therefore, called *selective radiators*. Table 7.4 gives the emission bands of $CO_2$ and $H_2O$.

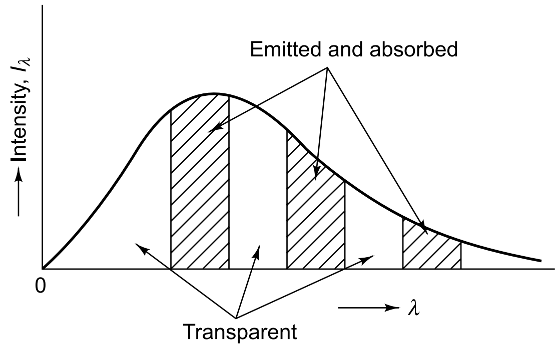


**Fig. 7.50** *Emission or absorption bands of gas radiation*

**Table 7.4** *Emission Bands of $CO_2$ and $H_2O$*

| *Band number* | *$CO_2$* | | *$H_2O$* | |
|---|---|---|---|---|
| | *Wavelength range (μm)* | *Band width (μm)* | *Wavelength range (μm)* | *Band width (μm)* |
| First | 2.64 – 2.84 | 0.20 | 2.24 – 3.37 | 1.13 |
| Second | 4.01 – 4.80 | 0.79 | 4.8 – 8.5 | 3.70 |
| Third | 12.5 – 16.5 | 4.00 | 12 – 25 | 13.00 |

(2) For a solid body, the emission and absorption of radiation are essentially surface phenomena. But in calculating the radiation emitted or absorbed by a gas layer, its thickness, pressure and shape as well as surface area must be taken into account. Incident radiation is slowly absorbed by the gas layer, the intensity decreasing with the thickness of the layer.

### 7.23.1 Absorptivity of Gases

When monochromatic radiation at an intensity $I_{\lambda 0}$ passes through a gas layer of thickness $L$ (Fig. 7.51), the radiant energy absorption in a differential distance d$x$ is governed by the equation

$$dI_{\lambda_x} = -k_\lambda I_{\lambda_x} dx \tag{7.92}$$

where $I_{\lambda_x}$ is the radiation intensity at a distance $x$ from the wall where the intensity is $I_{\lambda_0}$ and $k_\lambda$ is the monochromatic absorption coefficient. The decrease in intensity is proportional to $I_{\lambda_x}$ as well as d$x$, and the constant of proportionality is $k_\lambda$.

The integration of Eq. (7.92) gives

$$\int_{I_{\lambda_0}}^{I_{\lambda_L}} \frac{dI_{\lambda_x}}{I_{\lambda_x}} = \int_0^L k_\lambda \, dx$$

or

$$\mathbf{I_{\lambda_L} = I_{\lambda_0} e^{-k\lambda_L}} \tag{7.93}$$

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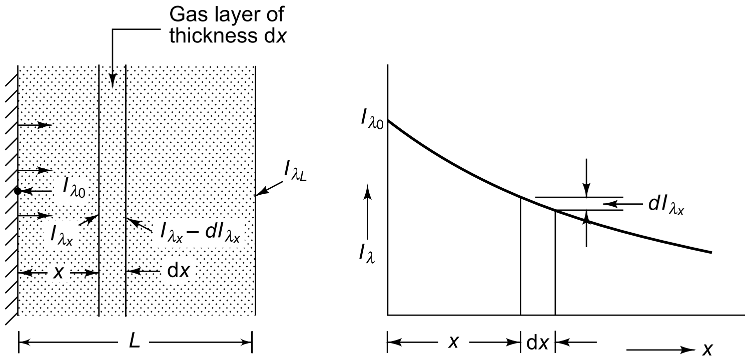


**Fig. 7.51** *Absorption of radiant energy by a gas layer*

The radiation intensity $I_{\lambda_L}$ decreases exponentially with the thickness of gas layer $L$. This is known as *Beer's law*.

Amount of energy absorbed in the gas layer

$$= I_{\lambda_0} - I_{\lambda_L} = I_{\lambda_0}(1 - e^{-k_\lambda L})$$

$$= \boldsymbol{\alpha}_{\mathbf{G}_\lambda} \mathbf{I}_{\lambda_0} \tag{7.94}$$

where $\alpha_{G_\lambda} = 1 - e^{-k_\lambda L}$ is called the *monochromatic absorptivity* of the gas. By Kirchhoff's law,

$$\boldsymbol{\alpha}_{\mathbf{G}_\lambda} = \boldsymbol{\varepsilon}_{\mathbf{G}_\lambda}.$$

For large values of $L$, $\lambda_{G_\lambda} = 1$. Thus, for thick gas layers, gas radiation approaches black body radiation within the wavelength bands.

To find the *effective absorptivity* $\alpha_G$ or emissivity $\varepsilon_G$ of a gas volume over all wavelength bands, the total intensity of radiation at a distance $x$ is

$$I_x = I_0 \, e^{-kx}$$

and at length $L$,

$$I_L = I_0 \, e^{-kL}$$

$$I_0 - I_L = \text{Energy absorbed by the gas layer in all bands}$$

$$= I_0 (1 - e^{-kL}) = I_0 \, \alpha_G \tag{7.95}$$

where $\alpha_G = 1 - e^{-kL}$ and $k$ is the total absorption coefficient. It may be noted that $k$ is not generally equal to $k_\lambda$. The accurate determination of $k$ is quite difficult. For this reason, the effective absorptivity of gases is usually calculated by employing an approximate method developed by Hottel [5], as explained below.

### 7.23.2 Hottel's Curves

Hottel evaluated the emissivities of various gases at different pressures and temperatures and plotted his results graphically, as shown in Figs 7.53 to 7.57. Hottel's curves are strictly valid for hemispherical gas volumes of radius $L$, radiating to an elemental surface d$A$ at the centre of the base (Fig. 7.52). However, for other gas shapes a *mean beam length* can be determined from Table 7.5. For approximate calculations,

$$\mathbf{L} = \mathbf{3.4} \times \frac{\textbf{Volume of gas}}{\textbf{Surface area of gas}} \tag{7.96}$$

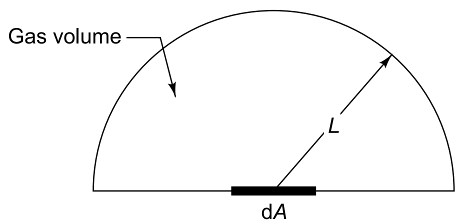


**Fig. 7.52** *Gas volume for Hottel's graphs*

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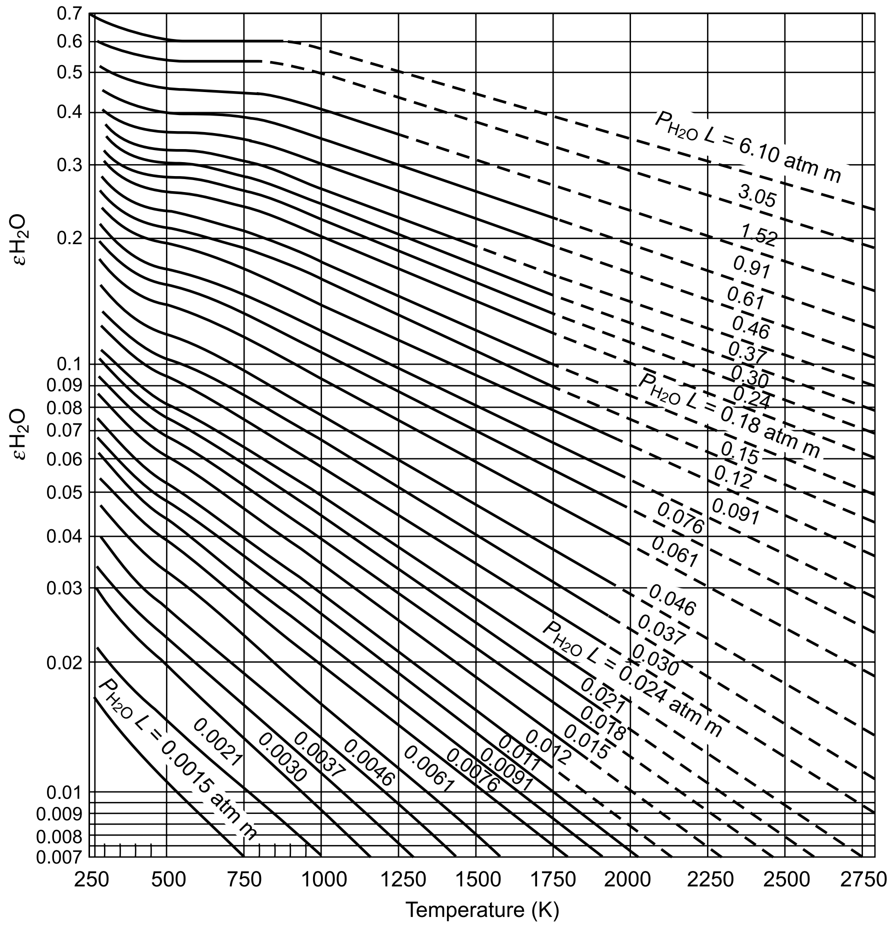


**Fig. 7.53** *Emissivity of $H_2O$ vapour at a total pressure of 1 atm*

**Table 7.5** *Mean beam length for various gas volumes*

| *No.* | *Gas volume shape dimension, D* | *Characteristic* | *Mean beam length, L* |
|---|---|---|---|
| 1. | Sphere | Diameter | 0.67 D |
| 2. | Infinite cylinder radiating to walls | Diameter | D |
| 3. | Cube | Side | 0.67 D |
| 4. | Space between infinite parallel planes | Distance between planes | 1.8 D |
| 5. | Infinite cylinder radiating to elemental surface on, centre of base | Diameter | D |
| 6. | Cylinder of height equal to, diameter radiating to whole surface | Diameter | 0.67 D |
| 7. | Cylinder of height, equal to diameter radiating, to elemental surface at centre of base | Diameter | 0.77 D |
| 8. | Space outside infinite bank, of tubes with centres on, equilateral triangles tube, diameter being equal to, clearance | Clearance | 3.4 D |

(*Contd*)

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**Table 7.5** (*Contd*)

| *No.* | *Gas volume shape dimension, D* | *Characteristic* | *Mean beam length, L* |
|---|---|---|---|
| 9. | Same as (8) except that tube diameter equals 1/2 clearance | Clearance | 4.5 D |
| 10. | Same as (8) except that tube centres are on squares | Clearance | 4.1 D |
| 11. | 1 × 2 × 6 rectangular parallelepiped radiating to | Shortest edge | |
| | 2 × 6 face | | 1.18 D |
| | 1 × 6 face | | 1.24 D |
| | 1 × 2 face | | 1.18 D |
| | all face | | 1.02 D |
| 12. | Any other shape | Hydraulic mean radius | 4 D |

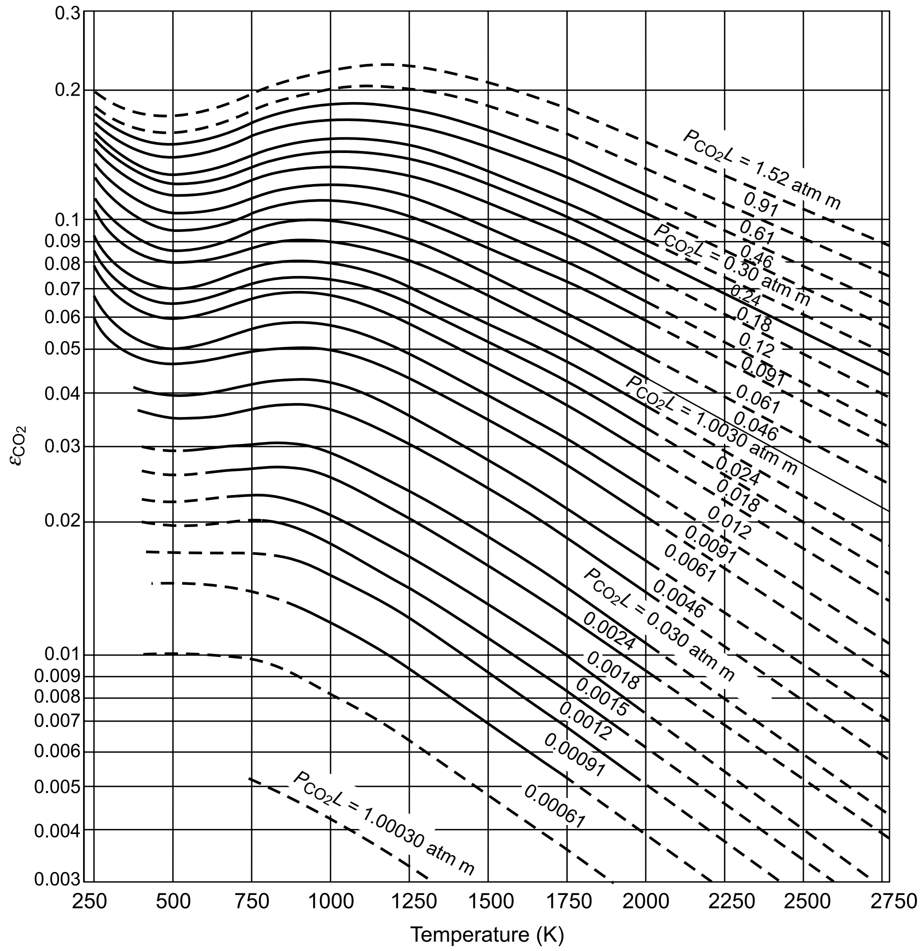


**Fig. 7.54** *Emissivity of $CO_2$ at a total pressure of 1 atm*

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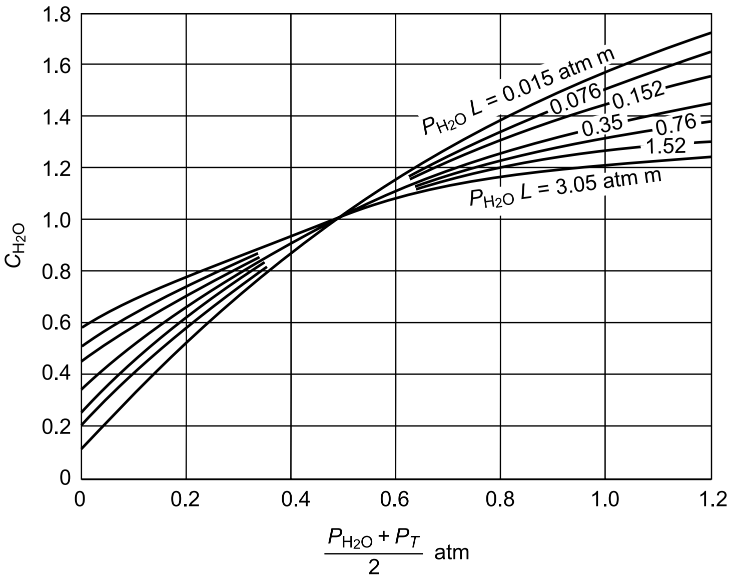


**Fig. 7.55** *Correction factor for the emissivity of water vapour at pressures other than 1 atm*

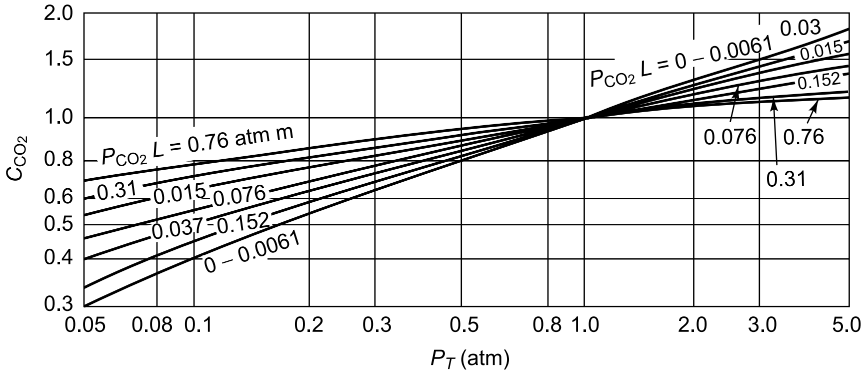


**Fig. 7.56** *Correction factor for the emissivity of carbon dioxide at pressures other than 1 atm*

The curves in Figs 7.53 and 7.54 give the effective emissivity of $CO_2$ and $H_2O$ respectively at different temperatures for various values of $pL$, where $p$ is the partial pressure of $CO_2$ gas in atm and $L$ is the mean beam length in metres. The curves are for a total pressure of 1 atm. If the pressure is different from 1 atm, then correction factors $C$ obtained from Figs 7.55 and 7.56 must be multiplied with the values of $\varepsilon_{CO_2}$ and $\varepsilon_{H_2O}$ obtained from Figs 7.53 and 7.54 respectively.

When both $CO_2$ and $H_2O$ are present in a gas volume, a rough estimate of the effective emissivity of the mixture can be made by adding the emissivities of the constituents $CO_2$ and $H_2O$.

$$\boldsymbol{\varepsilon = \varepsilon_{CO_2} + \varepsilon_{H_2O}}$$

This value is on the higher side, because some of the emission bands of $H_2O$ vapour and $CO_2$ overlap. To determine the emissivity of the mixture correctly, a correction factor $\Delta\varepsilon$, given in Fig. 7.57 should be subtracted from the sum. If suffix $m$ denotes the mixture.

$$\varepsilon_m = \varepsilon_{CO_2} + \varepsilon_{H_2O} - \Delta\varepsilon \tag{7.97}$$

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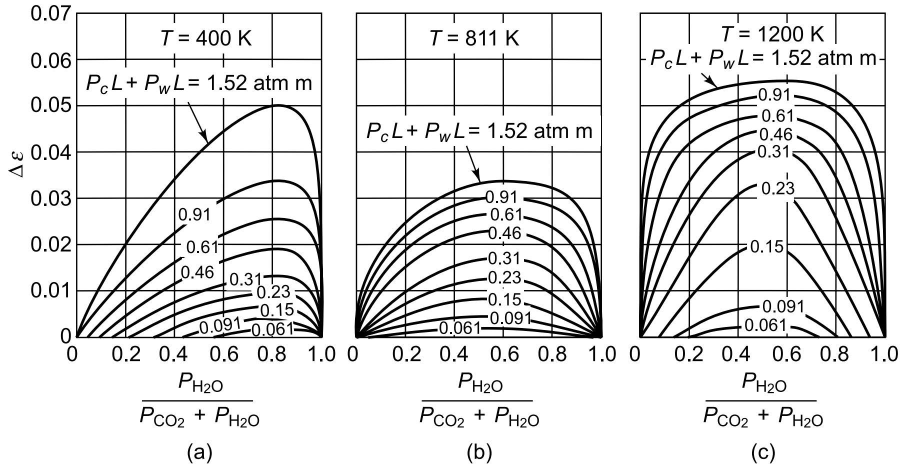


**Fig. 7.57** *Factor $\Delta\varepsilon$ to correct the emissivity of a mixture of water vapour and $CO_2$*

### 7.23.3 Radiant Heat Exchange between a Gas Volume and a Black Enclosure

Let us consider a gas volume at temperature $T_G$ enclosed by a black surface at temperature $T_W$.

Rate of radiation emitted by the gas and falling on the black enclosure walls

$$Q_{G-W} = A\,\varepsilon_G\,\sigma\,T_G^4 \tag{7.98}$$

where $\varepsilon_G$ is the emissivity of the gas at $T_G$. All this energy is absorbed by the black enclosure walls. Rate of energy emitted by the black wall

$$Q_W = \sigma A T_W^4 \tag{7.99}$$

But all this energy is not absorbed by the gas.

Rate of absorption of energy by the gas

$$Q_{W-G} = \alpha'_G\,\sigma A T_W^4 \tag{7.100}$$

where $\alpha'_G$ is the absorptivity of the gas at $T_G$ for incident radiation at $T_W$.

Hence, the net rate of radiant heat exchange between the gas and the black enclosure is

$$(Q_{G-W})_{\text{net}} = Q_{G-W} - Q_{W-G}$$

$$= \boldsymbol{\sigma A\,(\varepsilon_G\,T_G^4 - \alpha'_G\,T_W^4)} \tag{7.101}$$

Since $\alpha'_G$ is not known, an approximation can be made by using $\alpha_G$ i.e., the absorptivity of the gas at $T_W$ for incident radiation from a black body at $T_W$, instead of $\alpha_G$.

For a gray enclosure having an emissivity $\varepsilon_W$ for its wall, the net heat exchange may approximately be obtained by multiplying Eq. (7.101) by a factor of $(\varepsilon_W + 1)/2$, for $\varepsilon_W > 0.8$.

## 7.24 RADIATION COMBINED WITH CONVECTION

Heat is transferred from a hot wall both by convection and radiation.

Total heat transfer = Heat Transferred by convection + Heat transferred by radiative

$$Q_{\text{total}} = Q_{\text{convective}} + Q_{\text{radiative}} = Q_C + Q_r$$

$$= h_c A_1\,(T_1 - T_2) + \sigma A_1\,\mathcal{F}_{12}\,(T_1^4 - T_2^4)$$

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$$= h_c A_1 (T_1 - T_2) + h_r A_1 (T_1 - T_2)$$

$$= \mathbf{(h_c + h_r)\, A_1\, (T_1 - T_2)} \tag{7.102}$$

where $h_r$ is the *radiation heat transfer coefficient* (W/m$^2$ K) given by

$$h_r = \frac{\sigma \mathcal{F}_{12}(T_1^4 - T_2^4)}{T_1 - T_2}$$

$$= \boldsymbol{\sigma \mathcal{F}_{12}\, (T_1 + T_2)\, (T_1^2 + T_2^2)} \tag{7.103}$$

At low temperatures, the effects of radiation being insignificant, owing to a small value of $\sigma$, we are often justified in neglecting radiation effects. At high temperatures, however, radiation predominates because of the fourth power of the absolute temperature, and convective effects are often neglected. In vacuum, radiation is the only mode of heat transfer.

## 7.25 GREENHOUSE EFFECT

Glass transmits over 90% of radiation in the visible range and is almost opaque to infrared wavelengths ($\lambda > 3$ μm). Thus, glass allows the solar radiation to enter, but does not allow infrared radiation from the interior surfaces to exit. This causes a rise in the interior temperature, with heat thus being trapped. This heating effect due to the nongray characteristic of glass or clear plastics is known as the *greenhouse effect* (Fig. 7.58).

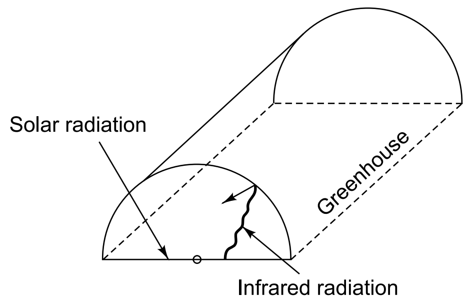


**Fig. 7.58** *Greenhouse which traps energy by allowing the solar radiation to come in, but not allowing the infrared radiation to go out*

The greenhouse effect is also experienced on a larger scale on earth. The surface of the earth, which warms up during the day as a result of the absorption of solar energy, cools down at night by radiating its energy into deep space as infrared radiation. The gases $CO_2$ and $H_2O$ vapour in the atmosphere transmit the bulk of the solar radiation during the day, but absorb the infrared radiation emitted by the surfaces of the earth at night. Thus, the energy trapped on earth by the atmosphere causes *global warming*, and drastic changes in weather conditions.

In coastal areas, where humidity is high, there is not much difference between the day and night temperatures, because water vapour acts as a barrier on the path of infrared radiation emanating from the earth and thus slows down the cooling process at night. On the other hand, in deserts where the air is dry and the skies are clear, there is a large difference between day and night temperatures because of the absence of water vapour barrier for infrared radiation.

## 7.26 SOLAR RADIATION

Our primary source of energy is the sun. Solar energy emanating from the sun reaches us in the form of electromagnetic waves after experiencing considerable interactions with the atmosphere. The radiation energy emitted or reflected by the constituents of the atmosphere form the atmospheric radiation.

The sun has a diameter of about $1.39 \times 10^9$ m and a mass of $2 \times 10^{30}$ kg, and is located at a mean distance of $1.50 \times 10^{11}$ m from the earth. It emits radiation continuously at a rate of $3.8 \times 10^{26}$ W. Less than a billionth of this energy (about $1.7 \times 10^{17}$ W) strikes the earth, which is sufficient to keep the earth warm and to sustain life through the photosynthesis process. The energy of the sun is due to continuous fusion reaction during which two hydrogen atoms fuse to form one atom of helium. Therefore, the sun is

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essentially a nuclear reactor, with temperatures as high as 40,000,000 K in its core region. The temperature drops to about 6000 K in the outer region of the sun as a result of dissipation of this energy by radiation.

The solar energy reaching the earth's atmosphere, called the *solar constant*, is given by

$$\mathbf{G_S = 1353\ W/m^2}$$

It is the rate at which solar energy is incident on a surface normal to the sun's rays at the outer edge of the atmosphere when the earth is at its mean distance from the sun (Fig. 7.59).

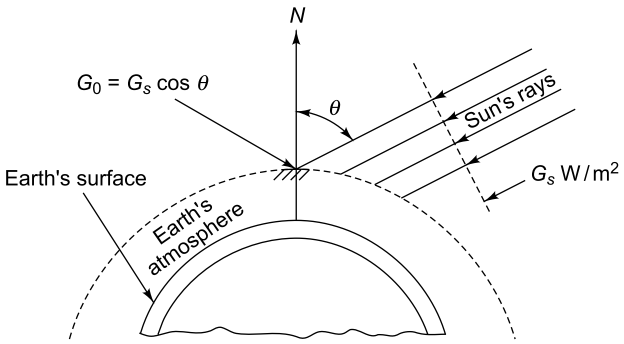


**Fig. 7.59** *Solar radiation reaching the earth's atmosphere and the solar constant*

The measured value of the solar constant can be used to estimate the effective surface temperature of the sun from the energy balance

$$4\pi L^2 G_S = 4\pi r^2\, \sigma T_{sun}^4 \tag{7.104}$$

where $L$ is the mean distance between the sun and the earth and $r$ is the radius of the sun (Fig. 7.60). The effective surface temperature is

$$T_{sun} = \left(\frac{L^2 G_S}{r^2 \sigma}\right)^{1/4} = 5762\ \text{K}$$

The sun can be treated as a black body at a temperature of 5762 K.

The spectral distribution of solar radiation on the ground shows that the solar radiation undergoes considerable attenuation as it passes through the atmosphere as a result of absorption and scattering (Fig. 7.61). Several dips on the spectral distribution on the earth's surface are due to absorption by gases like $O_2$, $O_3$, $H_2O$ and $CO_2$. Absorption by oxygen occurs in the narrow band at $\lambda = 0.76$ µm. The ozone absorbs ultraviolet radiation at wavelengths below 0.3 µm and also some radiation in the range of 0.3 – 0.4 µm. Absorption in the infrared region is dominated by water vapour and carbon dioxide. The dust particles and other pollutants in the atmosphere also absorb radiation at various wavelengths.

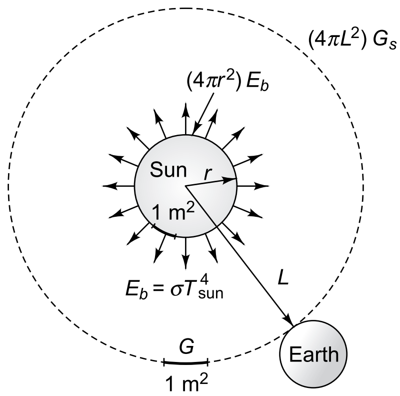


**Fig. 7.60** *Total solar energy passing through concentric spheres remains constant, but the energy falling per unit area decreases with increasing radius*

As a result of these adsorptions the solar energy reaching the earth's surface is weakened considerably to about 950 W/m² on a clear day and much less on cloudy days in the wavelength range of 0.3 – 2.5 µm.

Solar radiation is also attenuated, as it passes through the atmosphere, due to scattering or reflection by air molecules and other particles like dust, smog, water droplets and so on. Scattering is mainly governed

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by the size of particles relative to the wavelength of radiation. The oxygen and nitrogen molecules primarily scatter radiation at very short wavelengths. Therefore, radiation at wavelengths for violet and blue colours is scattered the most in all directions, which gives the sky its bluish colour. The same phenomena occur for red sunrises and sunsets, when the solar radiation passes through greater thicknesses of the atmosphere and is scattered more. Consequently, the light that reaches the earth is of longer wavelengths like red, orange and yellow.

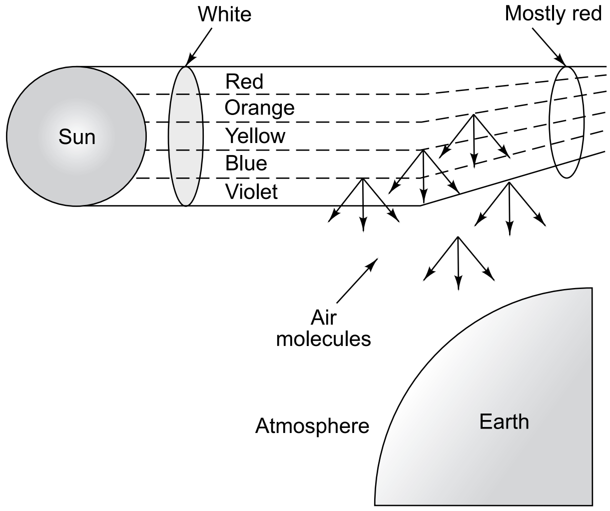


**Fig. 7.61** *Air molecules scatter blue light much more than they do red light. At sunset, the light travels through a thicker layer of atmosphere, which removes much of the blue from the natural light, and letting the red dominate*

Solar energy reaching the earth consists of direct and diffuse parts (Fig. 7.62). Direct radiation does not get scattered or absorbed by the atmosphere. The diffuse radiation reaches from all directions. The total solar energy incident on the unit area of a horizontal surface on the ground is

$$G_{\text{solar}} = G_D \cos\theta + G_d \text{ W/m}^2 \qquad (7.105)$$

where $\theta$ is the angle of incidence of direct radiation. The diffuse radiation varies from about 10% of the total radiation on a clear day to nearly 100% on a very cloudy day.

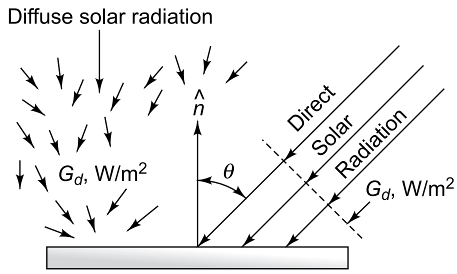


**Fig. 7.62** *Direct and diffuse radiation incident on a horizontal surface at the earth's surface*

The gas molecules and the suspended particles in the atmosphere, primarily $CO_2$ and $H_2O$ molecules, emit as well as absorb radiation in the 5–8 μm and above 13 $\mu$m wavelengths. Assuming the atmosphere as a black body at an effective sky temperature $T_{\text{sky}}$, the radiation emission from the atmosphere to the earth's surface is expressed as

$$G_{\text{sky}} = \sigma T_{\text{sky}}^4 \text{ (W/m}^2\text{)} \qquad (7.106)$$

where value of $T_{\text{sky}}$ ranges from 230 to 285 K, not much different from the room temperature. The energy absorbed by the surface from the sky radiation is

$$E_{\text{sky, absorbed}} = \alpha G_{\text{sky}} = \alpha\sigma T_{\text{sky}}^4 = \varepsilon\sigma T_{\text{sky}}^4 \qquad (7.107)$$

The net rate of radiation heat transfer to a surface exposed to solar and atmospheric radiation (Fig. 7.63) is given by

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$$
\begin{aligned}
Q_{\text{net rad}} &= \sum E_{\text{absorbed}} - \sum E_{\text{emitted}} \\
&= E_{\text{solar, absorbed}} + E_{\text{sky, absorbed}} - E_{\text{emitted}} \\
&= \alpha_s\, G_{\text{solar}} + \varepsilon\, \sigma\, T_{\text{sky}}^4 - \varepsilon\, \sigma\, T_s^4 \\
&= \alpha_s\, G_{\text{solar}} + \varepsilon\, \sigma\, (T_{\text{sky}}^4 - T_s^4) \text{W/m}^2 \qquad (7.108)
\end{aligned}
$$

where $T_s$ is the surface temperature and $\varepsilon$ is its emissivity at room temperature.

The absorption and emission of radiation by the elementary gases like $H_2$, $O_2$ and $N_2$ are almost negligible, and a medium filled with these gases may be treated as a vacuum. Larger molecules of $H_2O$ and $CO_2$ emit and absorb radiation significantly, as stated earlier.

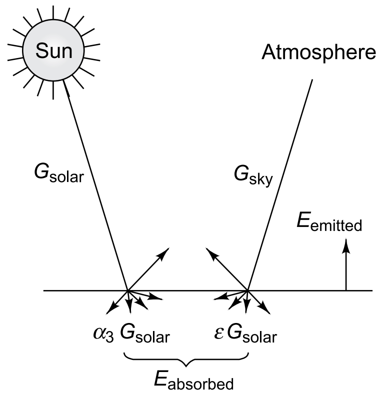


**Fig. 7.63** *Radiation interactions of a surface exposed to solar and atmospheric radiation*

In solar energy applications, the spectral distribution of incident solar radiation is very different from the spectral distribution of emitted radiation by the surfaces, since the former is concentrated in the short wavelength region and the latter in the infrared region. Therefore, the radiation properties of surfaces will be quite different for the incident and emitted radiation, and the surfaces cannot be assumed to be gray. Instead, the surfaces are assumed to have two sets of properties: one for solar radiation and the other for infrared radiation at room temperature. Table 7.6 gives the emissivity $\varepsilon$ and the solar absorptivity $\alpha_s$ of the surfaces of some common materials. Surfaces that are intended to collect solar energy, such as the absorber surfaces of solar collectors, are desired to have high $\alpha$, but low $\varepsilon$ values to maximise the absorption of solar radiation and to minimise the emission of radiation. Surfaces are often given the desired properties by coating them with thin layers of selective materials. A surface can be kept cool, e.g. by simply painting it white.

**Table 7.6** *Comparison of the solar absorptivity $a_s$, of some surfaces with their emissivity $\varepsilon$ at room temperture*

| *Surface* | $\alpha$ | $\varepsilon$ |
|---|---|---|
| Aluminium | | |
| Polished | 0.09 | 0.03 |
| Anodised | 0.14 | 0.84 |
| Foil | 0.15 | 0.05 |
| Copper | | |
| Polished | 0.18 | 0.03 |
| Tarnished | 0.65 | 0.75 |
| Stainless steel | | |
| Polished | 0.37 | 0.60 |
| Dull | 0.50 | 0.21 |
| Plated metals | | |
| Black nickel oxide | 0.92 | 0.08 |
| Black chrome | 0.87 | 0.09 |
| Concrete | 0.60 | 0.88 |
| White marble | 0.46 | 0.95 |
| Red brick | 0.63 | 0.93 |
| Asphalt | 0.90 | 0.90 |
| Black paint | 0.97 | 0.97 |
| White paint | 0.14 | 0.93 |
| Snow | 0.28 | 0.97 |
| Human skin (caucasian) | 0.62 | 0.97 |

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# Solved Examples

**Example 7.1** A 100 W electric bulb has a filament temperature of 3001°C. Assuming the filament to be black, calculate (a) the diameter of the wire if the length is 250 mm and (b) the efficiency of the bulb if visible radiation lies in the range of wavelengths from 0.5 μ to 0.8 μ.

**Solution**

(a) For a black body the rate of emission of radiant energy

$$Q = \sigma A T^4 = \sigma \pi d l T^4$$

where $d$ is the diameter and $l$ is the length of the filament

$$Q = 5.67 \times 10^{-8} \times \pi \times d \times 0.25 \times (32.74)^4 = 100$$

$$d = \frac{100 \times 1000}{5.67 \times (32.74)^4 \times \pi \times 0.25} \text{ mm} = 0.02 \text{ mm} \quad \textit{Ans.}$$

(b) The efficiency of the bulb is defined as (Fig. Ex. 7.1)

$$\eta_{\text{bulb}} = \frac{\text{Energy in the visible range of wavelengths}}{\text{Total energy in the spectrum}}$$

$$= \frac{\int_{0.5\mu}^{0.8\mu} E_{b\lambda}\, d\lambda}{\int_0^{\infty} E_{b\lambda}\, d\lambda} = \frac{\int_{0.5\mu}^{0.8\mu} E_{b\lambda}\, d\lambda}{\sigma T^4}$$

$$= \frac{1}{\sigma T^4}\left( \int_0^{0.8\mu} E_{b\lambda}\, d\lambda - \int_0^{0.5\mu} E_{b\lambda}\, d\lambda \right)$$

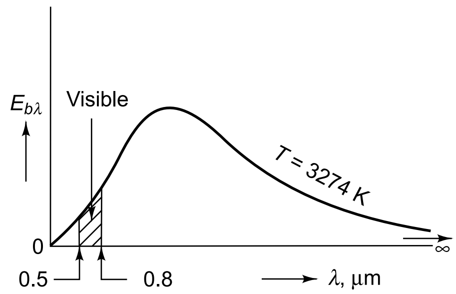


**Fig.** *Ex. 7.1*

Now, $\lambda_1 T = 0.8 \times 3274\ \mu\text{K} = 2619.2\ \mu\text{K}$

and $\lambda_2 T = 0.5 \times 3274 = 1637\ \mu\text{K}$

From Table 7.3

When $\lambda_1 T = 2.6192 \times 10^{-3}$ mK, $\dfrac{E_b(0 \to \lambda_1 T)}{\sigma T^4} = 0.184$

and when $\lambda_2 T = 1.637 \times 10^{-3}$ mK, $\dfrac{E_b(0 \to \lambda_2 T)}{\sigma T^4} = 0.022 = 0.022$

Efficiency of the bulb = 0.184 – 0.022 = 0.162, or 16.2% *Ans.*

About 84% of the energy leaves the bulb as infrared energy.

**Example 7.2** Assuming the sun to be a black body having a surface temperature of 5800 K, calculate (a) the total emissive power, (b) the wavelength at which the maximum spectral intensity occurs, (c) the maximum value of $E_{b_\lambda}$, (d) the percentage of total emitted energy that lies in the visible range of 0.35 μ to 0.76 μ and (e) the total amount of radiant energy emitted by the sun per unit time if its diameter can be assumed to be $1.391 \times 10^9$ m.

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**Solution**

(a) $E_b = \sigma T^4 = 5.67 \times 10^{-8} \times (5800)^4$
$= 6.42 \times 10^7 \text{ W/m}^2 = 64{,}200 \text{ kW/m}^2$ *Ans.*

(b) $\lambda_{max} T = 2.898 \times 10^{-3}$ mK

$$\lambda_{max} = \frac{2.898 \times 10^{-3}}{5800} = 5 \times 10^{-7} \text{ m} = 0.5\ \mu \quad \textit{Ans.}$$

(c) $(E_{b\lambda})_{max} = 1.287 \times 10^{-5}\ T^5 \text{ W/m}^3$
$= 1.287 \times 10^{-5} \times (5800)^5 = 8.45 \times 10^{13} \text{ W/m}^3$
$= 8.45 \times 10^{10} \text{ kW/m}^3$ *Ans.*

(d) $\lambda_1 T = 0.38 \times 10^{-6} \times 5800 = 2.205 \times 10^{-3}$ mK
$\lambda_2 T = 0.76 \times 10^{-6} \times 5800 = 4.407 \times 10^{-3}$ mK
From Table 7.3,

$$\frac{E_b\,(0 \to \lambda_2 T)}{\sigma T^4} = 0.550$$

and
$$\frac{E_b\,(0 \to \lambda_1 T)}{\sigma T^4} = 0.102$$

Percentage of total emitted energy in the visible range
$= 0.550 - 0.102 = 0.448$, or 44.8% *Ans.*

The peak of the sun's energy, $(E_{b\lambda})_{max}$, at $\lambda_{max} = 0.5\mu$ is in the visible portion of the spectrum and about 45% of its total emitted energy can be intercepted by human eye.

(e) $Q = \sigma A T^4 = 6.42 \times 10^7 \times \pi \times (1.391 \times 10^9)^2$ W
$= 4.38 \times 10^{26}$ W *Ans.*

**Example 7.3** The spectral emissivity function of an opaque surface at 800 K is approximated by a step function and is given below:

$$\varepsilon_\lambda = \begin{cases} \varepsilon_1 = 0.3 & \text{for} \quad 0 \le \lambda \le 3\ \mu\text{m} \\ \varepsilon_2 = 0.8 & \text{for} \quad 3\ \mu\text{m} \le \lambda \le 7\ \mu\text{m} \\ \varepsilon_3 = 0.1 & \text{for} \quad 7\ \mu\text{m} \le \lambda \le \infty \end{cases}$$

Determine the average emissivity of the surface and the emissive power.

**Solution** The average emissivity can be determined by breaking the integral

$$\varepsilon(T) = \frac{\int_0^\infty \varepsilon_\lambda(T)\, E_{b\lambda}(T)\, \mathrm{d}\lambda}{\sigma T^4}$$

into three parts

$$\varepsilon(T) = \frac{\varepsilon_1 \int_0^{\lambda_1} E_{b\lambda}(T)\, \mathrm{d}\lambda}{\sigma T^4} + \frac{\varepsilon_2 \int_{\lambda_1}^{\lambda_2} E_{b\lambda}(T)\, \mathrm{d}\lambda}{\sigma T^4} + \frac{\varepsilon_3 \int_{\lambda_2}^{\infty} E_{b\lambda}(T)\, \mathrm{d}\lambda}{\sigma T^4}$$

$$= \varepsilon_1 f_0 - \lambda_1(T) + \varepsilon_2 f_{\lambda_1 - \lambda_2}(T) + \varepsilon_3 f_{\lambda_2 - \infty}(T)$$

$$= \varepsilon_1 f_{\lambda_1} + \varepsilon_2 (f_{\lambda_2} - f_{\lambda_1}) + \varepsilon_3 (1 - f_{\lambda_2})$$

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where $f_{\lambda_1}$ and $f_{\lambda_2}$ are black body radiation functions corresponding to $\lambda_1 T$ and $\lambda_2 T$. From Table 7.3,

when $\lambda_1 T = 3\,\mu\text{m} \times 800\text{ K} = 2400\ \mu\text{m K}, f_{\lambda_1} = 0.140256$

when $\lambda_2 T = 7\mu\text{m} \times 800\text{ K} = 5600\ \mu\text{m K},\ f_{\lambda_2} = 0.701046.$

$$\varepsilon = 0.3 \times 0.140256 + 0.8(0.701046 - 0.140256) + 0.1\,(1 - 0.701046)$$

$$= 0.521 \quad \textit{Ans.}$$

The emissive power of the surface is

$$E = \varepsilon\sigma T^4 = 0.521 \times 5.67 \times 10^{-8} \times (800)^4$$

$$= 12{,}100\text{ W/m}^2 \quad \textit{Ans.}$$

**Example 7.4** A domestic hot water tank (0.5 m diameter and 1 m high) is installed in a large space. The ambient temperature is 25°C. If the tank surface is oxidised copper with an emissivity of 0.8, find the heat loss from the tank surface at temperature 80°C by radiation. What would be the reduction in heat loss if a coating of aluminium paint having an emissivity of 0.3 is given to the tank? What would be the increase in heat loss if a white paint having an emissivity of 0.97 is given to the tank?

**Solution** Since the tank is small compared to the surrounding space, $\mathcal{F}_{12} = \varepsilon_1 = 0.8$.

$$A_1 = \pi\, d\, l + 2\,\frac{\pi}{4}\,d^2 = \pi \times 0.5 \times 1 + 2 \times \frac{\pi}{4} \times (0.5)^2$$

$$= \frac{5\pi}{8}\text{ m}^2$$

Rate of heat loss from the tank by radiation

$$Q_{12} = \sigma A_1\, \mathcal{F}_{12}\,(T_1^4 - T_2^4)$$

$$= 5.67 \times 10^{-8} \times \frac{5\pi}{8} \times 0.8\,[(353)^4 - (298)^4]$$

$$= 690\text{ W}$$

If the tank is coated with aluminium paint ($\varepsilon = 0.3$), the reduction in heat loss

$$= \frac{0.8 - 0.3}{0.8} \times 690 = 430\text{ W}$$

$$\therefore \quad Q_{12} = 260\text{ W}$$

If the tank is coated with white paint ($\varepsilon = 0.97$), the rate of radiant heat loss

$$Q_{12} = \frac{690}{0.8} \times 0.97 = 835.62\text{ W} \quad \textit{Ans.}$$

**Example 7.5** The distance of the sun from the earth is 150 × $10^6$ km. If the radius of the sun is 0.7 × $10^6$ km and its temperature is 6200 K, estimate approximately the mean temperature of the earth. Assume that the rate of radiative transfer from the sun to the earth is equal to the rate of radiant transfer from the earth to the outer space which is at 0 K. Consider the earth and sun as black.

**Solution** The fraction of solar radiation intercepted by the earth is

$$F_{12} = \frac{\pi r_2^2}{4\pi x^2}$$

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where $x$ is the distance of the sun from the earth and $r_2$ is the radius of the earth. Both the sun and the earth are considered to be black bodies (Fig. Ex. 7.5).

The net rate of radiative energy transfer from sun to the earth

$$Q_{12} = \sigma A_1 F_{12} (T_1^4 - T_2^4)$$

$$= \sigma 4\pi r_1^2 \frac{\pi r_2^2}{4\pi x^2}(T_1^4 - T_2^4)$$

$$= \frac{\pi r_1^2 r_2^2}{x^2}\sigma(T_1^4 - T_2^4)$$

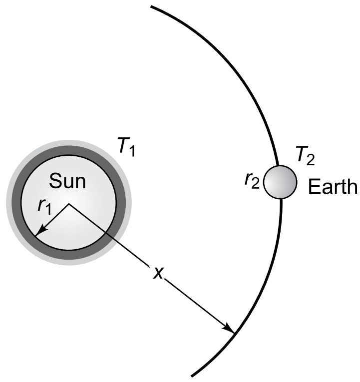


**Fig.** *Ex. 7.5*

The rate of radiant transfer from the earth to outer surface which is at 0 K is

$$Q_{20} = \sigma 4\pi r_2^2 \times 1 \times (T_2^4 - 0)$$

$$= 4\pi r_2^2 \sigma T_2^4$$

At equilibrium, $Q_{12} = Q_{20}$

$$4\pi r_2^2 \sigma T_2^4 = \frac{\pi r_1^2 r_2^2}{x^2}\sigma(T_1^4 - T_2^4)$$

$$4x^2/r_1^2 = (T_1/T_2)^4 - 1$$

Since $T_1 >> T_2$, 1 can be neglected.

$$T_2 = T_1\left(\frac{r_1}{2x}\right)^{1/2} = 6200\left(\frac{0.7\times10^6}{2\times150\times10^6}\right)^{1/2} \cong 300\text{ K} \quad \textit{Ans.}$$

**Example 7.6** Assuming the sun to radiate as a black body, calculate its temperature from the data given below:

Solar constant, i.e. average radiant heat flux incident on the earth's surface = 1380 W/m$^2$.

Radius of the sun = $7 \times 10^8$ m

Distance between the sun and the earth = $15 \times 10^{10}$ m

**Solution** Heat flux from small area d$A_1$ on the surface of the sun to the small area d$A_2$ on the surface of the earth (Fig. Ex. 7.5) is given by Eq. (7.29),

$$dQ_{1-2} = I_1 dA_1 \cos\phi_1 \frac{dA_2 \cos\phi_2}{r^2}$$

or,
$$\frac{dQ_{1-2}}{dA_2\cos\phi_2} = \frac{E_b}{\pi r^2} dA_1 \cos\phi_1$$

where $r$ is the distance of the earth from the sun ($= x$).

On integration,

$$\int \frac{dQ_{1-2}}{dA_2\cos\phi_2} = \frac{E_b}{\pi r^2}\int dA_1\cos\phi_1$$

Now, the left hand side is the solar constant,

$$\therefore \qquad 1380 = \frac{\sigma T^4}{\pi r^2}\cdot \pi r_1^2$$

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where $r_1$ = radius of the sun and $T$ = sun's temperature.

$$\sigma T^4 = 1380 \times \left(\frac{r}{r_1}\right)^2$$

$$5.67 \times 10^{-8}\, T^4 = 1380 \times \left(\frac{15 \times 10^{10}}{7 \times 10^8}\right)^2$$

$$\left(\frac{T}{100}\right)^4 = 243.39 \times (214.28)^2$$

$\therefore$ $$\frac{T}{100} = 3.95 \times 14.64$$

$\therefore$ $$T = 5782 \text{ K} \quad \textit{Ans.}$$

**Example 7.7** A black body emits radiation at 2000 K. Calculate (i) the monochromatic emissive power at 1 µm wavelength, (ii) wavelength at which the emission is maximum, and (iii) the maximum emissive power.

**Solution**

(i) From Planck's law,

$$E_{b\lambda} = \frac{c_1 \lambda^{-5}}{e^{c_2/\lambda T} - 1}$$

$$= \frac{3.74 \times 10^{-16} (1 \times 10^{-6})^{-5}}{e^{1.438 \times 10^{-2}/(10^{-6} \times 2000)} - 1}$$

$$= 2.79 \times 10^{11} \text{ W/m}^3 \quad \textit{Ans.}$$

(ii) From Wien's displacement law,

$$\lambda_{max}\, T = 2.898 \times 10^{-3} \text{ mK}$$

$\therefore$ $$\lambda_{max} = \frac{2.898 \times 10^{-3}}{2000} = 1.449 \times 10^{-6} \text{ m} \quad \textit{Ans.}$$

(iii) $$(E_{b\lambda})_{max} = \frac{c_1 \lambda_{max}^{-5}}{e^{c_2/\lambda_{max} T} - 1}$$

$$= \frac{3.74 \times 10^{-16} (1.449 \times 10^{-6})^{-5}}{e^{0.01433/(1.449 \times 10^{-6} \times 2000)} - 1}$$

$$= 4.1 \times 10^{11} \text{ W/m}^3 \quad \textit{Ans.}$$

**Example 7.8** Assuming the sun as a black body, it emits maximum radiation at 0.5 µm wavelength. Calculate (i) the surface temperature of the sun, (ii) its emissive power, (iii) the energy received by the surface of the earth and (iv) the energy received by a 2 m × 2 m solar collector whose normal is inclined at 60° to the sun. Take the diameter of the sun as $1.4 \times 10^9$ m, diameter of the earth as $13 \times 10^6$ m and the distance of the earth from the sun as $15 \times 10^{10}$ m.

**Solution**

$$\lambda_{max} = 0.5 \text{ µm}$$

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From Wien's displacement law,

$$\lambda_{max} T = 2898 \times 10^{-6} \text{ mK}$$

$\therefore$ Surface temperature of the sun

$$T = \frac{2898 \times 10^{6}}{0.5 \times 10^{-6}} = 5796 \text{ K} \quad \textit{Ans.} \text{ (i)}$$

Emissive power of the sun, a black body, is obtained from Stefan-Boltzmann law:

$$E_b = \sigma T^4 = 5.67 \times 10^{-8} (5796)^4 = 63865 \text{ kW/m}^2 \quad \textit{Ans.} \text{ (ii)}$$

Radiation reaching the earth would be

$$= \text{Emissive power of the sun} \times \left(\frac{\text{Radus of the sun}}{\text{Distance from the earth}}\right)^2$$

$$= 63865 \times \left(\frac{0.7 \times 10^{9}}{15 \times 10^{10}}\right)^2 = 1.39 \text{ kW/m}^2 \quad \textit{Ans.} \text{ (iii)}$$

Surface area of the solar collector in the direction normal to solar radiation

$$= A \cos\theta = 4 \cos 60° = 2 \text{ m}^2$$

$\therefore$ Energy received by the solar collector

$$= 1.39 \times 2 = 2.78 \text{ kW} \quad \textit{Ans.} \text{ (iv)}$$

**Example 7.9** The filament of a 75 W light bulb may be considered a black body radiating into a black enclosure at 70°C. The filament diameter is 0.10 mm and length is 50 mm. Considering the radiation, determine the filament temperature.

**Solution** *Given*: $Q = 75$ W, $T_2 = 70 + 273 = 343$ K, $d = 0.1$ mm, $l = 50$ mm, $T_1$ = filament temperature, $\varepsilon_1 = 1$

$$Q = \sigma \varepsilon A (T_1^4 - T_2^4)$$

$$75 = 5.67 \times 10^{-8} \times 1 \times \pi \times 0.1 \times 10^{-3} \times 50 \times 10^{-3} \times (T_1^4 - 343^4)$$

$$= 8.906 \times 10^{-13} (T_1^4 - 343^4)$$

$$\therefore \quad T_1 = 3029 \text{ K} = 2756°\text{C} \quad \textit{Ans.}$$

**Example 7.10** A long steel rod 20 mm in diameter is to be heated from 427°C to 538°C. It is placed concentrically in a long cylindrical furnace which has an inside diameter of 160 mm. The inner surface of the furnace is at a temperature of 1093°C and has an emissivity of 0.85. If the surface of the rod has an emissivity of 0.6, estimate the time required for the heating operation. Take the density of steel as 7800 kg/m$^3$ and its specific heat 0.67 kJ/kg K.

**Solution** With reference to Fig. Ex. 7.10

$$T_1 = 427 + 273 = 700 \text{ K}$$

$$T_1' = 538 + 273 = 811 \text{ K}$$

$$T_2 = 1093 + 273 = 1366 \text{ K}$$

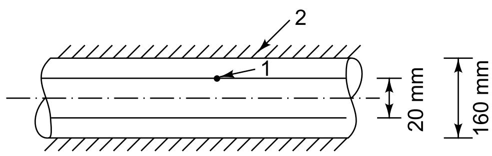


**Fig.** *Ex. 7.10*

$$\mathcal{F}_{12} = \frac{1}{(1/\varepsilon_1 - 1) + (1/F_{12}) + A_1/A_2\,(1/\varepsilon_2 - 1)}$$

where $\varepsilon_1 = 0.6$, $\varepsilon_2 = 0.85$, $F_{12} = 1.0$, $A_1 = \pi \times 0.02 \times L$

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$$A_2 = \pi \times 0.16 \times L, \quad L = \text{length of furnace}$$

$$\mathcal{F}_{12} = \frac{1}{(1/0.6 + 1/8)\,[1/0.85 - 1]} = 0.592$$

$(Q_{1-2})_{\text{beginning}}$ = Rate of heat absorption at the beginning per unit length of rod

$$= \sigma A_1\, \mathcal{F}_{12}\,(T_1^4 - T_2^4)$$

$$= 5.67 \times 10^{-8} \times \pi \times 0.02 \times 1 \times (700^4 - 1366^4)$$

$$= -\,11{,}543.47 \text{ W/m}$$

$(Q_{1-2})_{\text{end}}$ = Rate of heat absorption at the end

$$= \sigma A_1\, \mathcal{F}_{12}\,(T_1'^4 - T_2^4)$$

$$= 5.67 \times 10^{-8} \times \pi \times 0.16 \times 1\,(811^4 - 1366^4)$$

$$= -\,86{,}864.3 \text{ W/m}$$

$$Q_{\text{average}} = \frac{(Q_{1-2})_{\text{beginning}} + Q_{\text{end}}}{2}$$

$$= -\,49{,}203.9 \text{ W/m}$$

= Average rate of heat absorption

Let $t$ represent the time required for heating per unit length of rod. Then

$$Q_{\text{average}} \times t = (\rho V)\, c\,(T_f - T_i)$$

$$= 7800 \times \frac{\pi}{4}\,(0.02)^2 \times 1 \times 0.67\,(538 - 427) \times 10^3 \text{ J}$$

$$49{,}203.9 \times t = 182147 \text{ J}$$

$$t = \frac{182147}{49{,}203.9} = 3.7 \text{ s} \quad \textit{Ans.}$$

**Example 7.11** Liquid air boiling at –153°C is stored in a spherical container of diameter 320 mm. The container is surrounded by a concentric spherical shell of diameter 360 mm in a room at 27°C. The space between the two spheres is evacuated. The surfaces of the spheres are flushed with aluminium ($\varepsilon$ = 0.03). Taking the latent heat of vaporisation of liquid air as 210 kJ/kg, find the rate of evaporation of liquid air.

**Solution** Let suffix 1 denote the inner sphere and suffix 2 the outer sphere.

$$(Q_{12})_{\text{net}} = \sigma A_1\, \mathcal{F}_{12}\,(T_1^4 - T_2^4)$$

where

$$\mathcal{F}_{12} = \frac{1}{(1/\varepsilon_1) + A_1/A_2\,(1/\varepsilon_2 - 1)}$$

$$= \frac{1}{\dfrac{1}{0.03} + \dfrac{4\pi(0.16)^2}{4\pi(0.18)^2}\left(\dfrac{1}{0.03} - 1\right)} = 0.017$$

$$A_1 = 4\pi\,(0.16)^2 \text{ m}^2 = 0.3215 \text{ m}^2$$

$$T_1 = -\,153 + 273 + 120 \text{ K}$$

$$T_2 = 27 + 273 = 300 \text{ K}$$

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$$(Q_{12})_{net} = 5.67 \times 10^{-8} \times 0.3215 \times 0.017\,(120^4 - 300^4)$$
$$= -\,2.446 \text{ W}$$

$$\text{Rate of evaporation} = \frac{2.446 \times 10^{-3} \text{ kW}}{210 \text{ kJ/kg}} \times 3600$$
$$= 0.042 \text{ kg/h} \quad \textit{Ans.}$$

**Example 7.12** An enclosure measures 1.5 m × 1.7 m with a height of 2 m. The walls and ceiling are maintained at 250°C and the floor at 130°C. The walls and ceiling have an emissivity of 0.82 and the floor 0.7. Determine the net radiation to the floor.

**Solution**

Let $A_1$ = total area of walls and ceiling

$= (1.5 + 1.75) \times 2 + 1.5 \times 1.75 = 15.625 \text{ m}^2$

$A_2$ = floor area $= 1.5 \times 1.75 = 2.625 \text{ m}^2$

The floor is completely enclosed by the area $A_1$.

$$F_{21} + F_{22} = 1$$
$$F_{21} = 1$$
$$A_1 F_{12} = A_2 F_{21}$$
$$F_{12} = A_2/A_1 = 2.625/15.625 = 0.168$$

Rate of heat transfer by radiation from the walls and ceiling to the floor

$$Q_{1-2} = \sigma A_1 \mathcal{F}_{12} (T_1^4 - T_2^4)$$

where

$$\mathcal{F}_{12} = \frac{1}{\left(\dfrac{1}{\varepsilon_1} - 1\right) + \dfrac{1}{F_{12}} + \dfrac{A_1}{A_2}\left(\dfrac{1}{\varepsilon_2} - 1\right)}$$

$$= \frac{1}{\left(\dfrac{1}{0.82} - 1\right) + \dfrac{1}{0.168} + \dfrac{15.625}{2.625}\left(\dfrac{1}{0.7} - 1\right)}$$

$$= \frac{1}{8.79} = 0.114$$

$$Q_{1-2} = 5.67 \times 10^{-8} \times 15.625 \times 0.114\,(523^4 - 403^4)$$
$$= 4920 \text{ W} = 4.92 \text{ kW} \quad \textit{Ans.}$$

**Example 7.13** Two parallel rectangular surfaces 1 m × 2 m are opposite to each other at a distance of 4 m. The surfaces are black and at 100°C and 200°C respectively. Calculate the heat exchange by radiation between the two surfaces.

**Solution**

$$Q_{12} = A_1 F_{12}\, \sigma (T_1^4 - T_2^4)$$
$$\varepsilon_1 = \varepsilon_2 = 1$$
$$\frac{X}{L} = \frac{2}{4} = 0.5 \text{ and } \frac{Y}{L} = \frac{1}{4} = 0.25$$

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From Fig. 7.28,

$$F_{12} = 0.042$$

$$Q_{12} = (2 \times 1) \times 0.042 \times 5.67 \left[\left(\frac{473}{100}\right)^4 - \left(\frac{373}{100}\right)^4\right]$$

$$= 149.5 \text{ W} \quad \textit{Ans.}$$

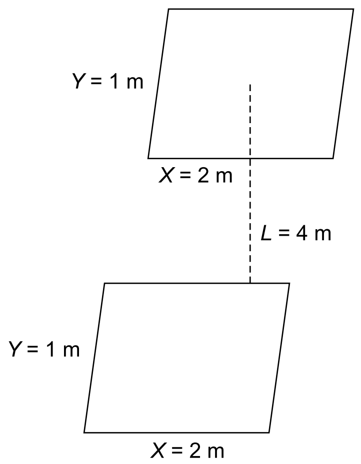


**Fig.** *Ex. 7.13*

**Example 7.14** Two black discs 1 m in diameter are placed directly opposite to each other at a distance of 0.5 m. The discs are maintained at 1000 K and 500 K respectively. Calculate the heat flow between the discs (a) when no other surfaces are present and (b) when the discs are connected by a cylindrical refractory no-flux surface.

**Solution**

(a) The value of $F_{12}$ for $D/x = 1/0.5 = 2$ from Fig. 7.29 is 0.36.

$$Q_{12} = \sigma A_1 \; F_{12} \, (T_1^4 - T_2^4)$$

$$= 5.67 \times 10^{-8} \times \pi \times (0.5)^2 \times 0.36 \, (1000^4 - 500^4)$$

$$= 15021.96 \text{ W} = 15.022 \text{ kW}$$

(b) If the discs are connected by nonconducting and reradiating walls,

$$\bar{F}_{12} = \frac{A_2 - A_1 F_{12}^2}{A_1 + A_2 - 2A_1 F_{12}} = \frac{1 + F_{12}}{2} = \frac{1.36}{2} = 0.68$$

$$\therefore \quad Q_{12} = 5.67 \times 10^{-8} \times \pi \times (0.5)^2 \times 0.68 \, (1000^4 - 500^4)$$

$$= 28374.8 \text{ W} = 28.3748 \text{ kW}$$

**Example 7.15** A brick wall having an emissivity of 0.85 is 6 m wide and 4 m high. It is at a distance of 4 m from a 500 mm × 400 mm opening in a furnace wall. The centre line of the opening lies 1 m lower and 1 m left of the centre of the wall. The furnace temperature is 1500°C and that of the wall is 37°C. Calculate the rate of radiation heat transfer between the opening and the wall.

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**Solution**

$$T_1 = 1773 \text{ K} \qquad T_2 = 310 \text{ K}$$

The brick wall is divided into four areas I, II, III and IV as shown in Fig. Ex. 7.15.

Heat loss from the opening to the wall

$$(Q_{1-2})_{net} = \sigma A_1 \, \varepsilon_e F_{12} (T_1^4 - T_2^4)$$

where $\varepsilon_e$ = effective emissivity = $\varepsilon_1 \, \varepsilon_2$.

The furnace opening is considered to act as a black body.

$$\varepsilon_e = 1 \times 0.85 = 0.85$$

$$A_1 = 0.5 \times 0.4 = 0.2 \text{ m}^2$$

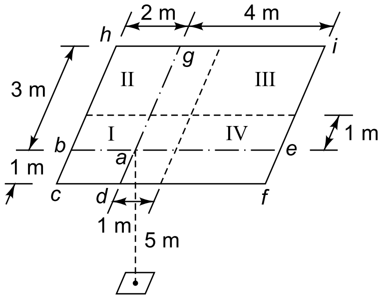


**Fig.** *Ex. 7.15*

Rectangle I, $\quad \dfrac{D}{L_1} = \dfrac{5}{1} = 5.0$

$$\frac{D}{L_2} = \frac{5}{2} = 2.5$$

Using Fig. 7.26, $F = 0.023$

Similarly, for

Rectangle II, $\quad \dfrac{D}{L_1} = \dfrac{5}{3} = 1.67$

$$\frac{D}{L_2} = \frac{5}{2} = 2.5$$

$$F = 0.053$$

Rectangle III, $\quad \dfrac{D}{L_1} = \dfrac{5}{3} = 1.67$

$$\frac{D}{L_2} = \frac{5}{4} = 1.25$$

$$F = 0.09$$

Rectangle IV, $\quad \dfrac{D}{L_1} = \dfrac{5}{1} = 5.0$

$$\frac{D}{L_2} = \frac{5}{4} = 1.25$$

$$F = 0.036$$

$$F_{12} = \sum F = 0.023 + 0.053 + 0.09 + 0.036$$

$$= 0.202$$

$$(Q_{12})_{net} = 5.67 \times 10^{-8} \times 0.2 \times 0.85 \times 0.202\,(1773^4 - 310^4)$$

$$= 19{,}222 \text{ W} = 19.222 \text{ kW} \qquad \textit{Ans.}$$

**Example 7.16** Determine the shape factor $F_{12}$ between a small area $A_1$ and a parallel circular disc $A_2$. $A_1$ is located on the axis of the disc and the semi-vertex angle of the cone formed with the disc as base and $A_1$ as the vertex is $\alpha$.

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**Solution** From Eq. (7.31),

$$A_1 F_{12} = \frac{1}{\pi} \int_{A_1} \int_{A_2} \frac{\cos\phi_1 \cos\phi_2}{r^2} \, dA_1 \, dA_2$$

Considering an elementary ring $dA_2$ of width $dr$ at a radius $r$ (Fig. Ex. 7.16),

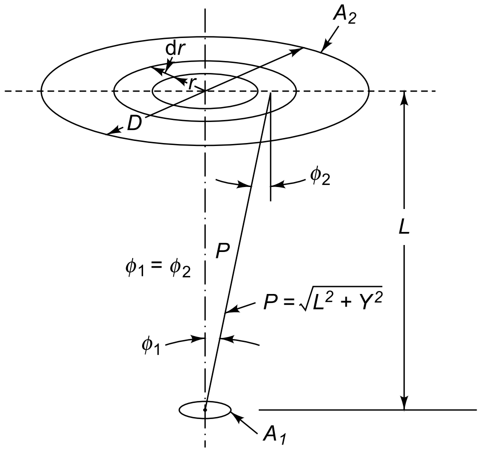


**Fig.** Ex. 7.16

$$dA_2 = 2\pi r \, dr$$

Now,

$$r = L \tan\phi_1$$

$$dr = L \sec^2\phi_1 \, d\phi_1$$

$$dA_2 = 2\pi L^2 \tan\phi_1 \sec^2\phi_1 \, d\phi_1$$

$$\frac{L}{P} = \cos\phi_1 \quad \text{or} \quad P = \frac{L}{\cos\phi_1}$$

$$F_{12} = \frac{1}{\pi} \int_{A_2} \frac{\cos\phi_1 \cos\phi_2}{P^2} \, dA_2$$

$$= \frac{1}{\pi} \int_{\phi_1=0}^{\alpha} \frac{\cos^2\phi_1 \, 2\pi L^2 \tan\phi_1 \sec^2\phi_1 d\phi_1}{L^2} \cos^2\phi_1$$

$$= \int_0^{\alpha} \sin 2\phi_1 d\phi_1 = 1 - \cos 2\alpha$$

$$= \sin^2\alpha = \frac{D^2/4}{D^2/4 + L^2} = \frac{D^2}{D^2 + 4L^2}$$

where $D$ is the diameter of the disc.

**Example 7.17** Two very large parallel planes with emissivities 0.3 and 0.8 exchange radiative energy. Determine the percentage reduction in radiative energy transfer when a polished aluminium radiation shield ($\varepsilon$ = 0.04) is placed between them.

**Solution** The radiant heat transfer rate without shield is given by

$$\frac{Q}{A} = \frac{\sigma(T_1^4 - T_2^4)}{1/\varepsilon_1 + 1/\varepsilon_2 - 1} = \frac{\sigma(T_1^4 - T_2^4)}{1/0.3 + 1/0.8 - 1}$$

$$= 0.279\, \sigma(T_1^4 - T_2^4)$$

The radiation network for two infinite parallel planes separated by one radiation shield is shown in Fig. Ex. 7.17.

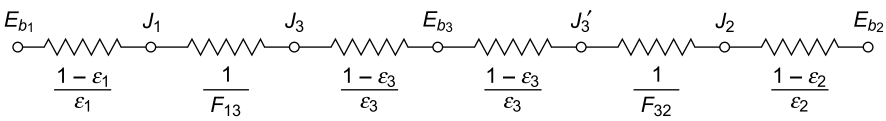


**Fig.** Ex. 7.17

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With shield, the total resistance is

$$R = \frac{1-\varepsilon_1}{\varepsilon_1} + \frac{1}{F_{13}} + 2\frac{1-\varepsilon_3}{\varepsilon_3} + \frac{1}{F_{32}} + \frac{1-\varepsilon_2}{\varepsilon_2}$$

$$= \frac{1-0.3}{0.3} + 1 + 2\frac{1-0.04}{0.04} + 1 + \frac{1-0.08}{0.08}$$

$$= 2.333 + 1 + 48 + 1 + 0.25$$

$$= 52.283$$

Rate of radiant heat transfer with one shield

$$Q_{1-2} = \frac{E_{B_1} - E_{B_2}}{R} = \frac{\sigma(T_1^4 - T_2^4)}{52.283}$$

$$= 0.017\,\sigma(T_1^4 - T_2^4)$$

Percentage reduction in heat transfer $= \dfrac{0.279 - 0.017}{0.279} \times 100 = 93.6\%$ *Ans.*

**Example 7.18** In a cylindrical furnace 0.6 m in diameter and 1 m high, the upper surface is maintained at 727°C and the lower surface is maintained at 427°C. Assuming the cylindrical wall to be a refractory surface, and if the emissivities of the upper and lower surfaces are 0.8 and 0.7 respectively, estimate the net rate of radiative energy transfer from the upper to the lower surface.

**Solution** The upper and lower surfaces (discs) are connected by a nonconducting and reradiating cylindrical wall. From Fig. 7.29, for the ratio $D/L$ as 0.6, $\bar{F}_{12} = 0.4$.

From Eq. (7.74),

$$\mathcal{F}_{12} = \frac{1}{(1/\varepsilon_1 - 1) + (1/\bar{F}_{12}) + A_1/A_2(1/\varepsilon_2 - 1)}$$

$$= \frac{1}{(1/0.8 - 1) + (1/0.4) + 1(1/0.7 - 1)} = 0.314$$

$$Q_{12} = \sigma A_1\, \mathcal{F}_{12}\,(T_1^4 - T_2^4)$$

$$= 5.67 \times 10^{-8} \times \pi(0.3)^2 \times 0.314\,[(1000)^4 - (700)^4]$$

$$= 3823.32 \text{ W} = 3.823 \text{ kW} \quad \textit{Ans.}$$

**Example 7.19** Calculate the shape factor $F_{12}$ for the configuration shown in Fig. Ex. 7.19 and the net heat transfer $Q_{12}$ if $T_1$ = 427°C and $T_2$ = 227°C, and if both surfaces are black (see Fig. Ex. 7.19).

**Solution**

$$F_{12} + F_{13} = F_{16}, \quad A_1F_{16} = A_6F_{61}, \quad A_1F_{13} = A_3F_{31}$$

$$F_{12} = F_{16} - F_{13} = \frac{A_6}{A_1}F_{61} - \frac{A_3}{A_1}F_{31}$$

$$= \frac{A_6}{A_1}(F_{65} - F_{64}) - \frac{A_3}{A_1}(F_{35} - F_{34})$$

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From Fig. 7.27, for $y/x = 4/4 = 1$ and $z/x = 5/4 = 1.25$,

$F_{65} = 0.22$ for $y/x = 4/4 = 1$ and $z/x = 2/4 = 0.5$,

$F_{64} = 0.15$ for $y/x = 2/4 = 0.5$ and $z/x = 5/4 = 1.25$

$F_{35} = 0.30$ and for $y/x = 2/4 = 0.5$ and $z/x = 2/4 = 0.5$

$F_{34} = 0.24$.

By substituting,

$$F_{12} = \frac{4\times 4}{3\times 4}(0.22 - 0.15) - \frac{2\times 4}{3\times 4}(0.30 - 0.24)$$

$$= 0.093 - 0.04 = 0.053$$

Rate of radiant exchange

$$Q_{12} = 5.67\times 10^{-8}\times 3\times 4\times 0.053\,[(700)^4 - (500)^4]$$

$$= 6404.5 \text{ W} = 6.4045 \text{ kW} \quad \textit{Ans.}$$

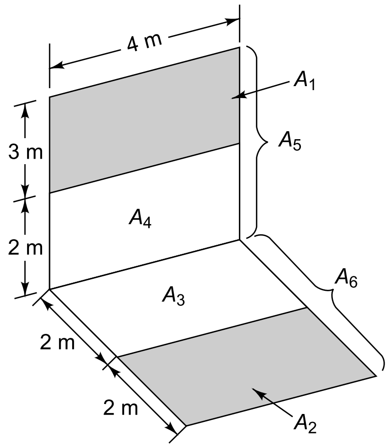


**Fig.** *Ex. 7.19*

**Example 7.20** Two parallel plates 2 m × 1 m are spaced 1 m apart. The plates are at temperatures of 727°C and 227°C and their emissivities are 0.3 and 0.5 respectively. The plates are located in a large room, the walls of which are at 27°C. Determine the rate of radiant heat loss from each plate and the heat gain by the walls.

**Solution** There are three surfaces involved in the problem: two parallel plates having $T_1 = 727 + 273 = 1000$ K, $\varepsilon_1 = 0.3$, $T_2 = 227 + 273 = 500$ K, $\varepsilon_2 = 0.5$, and the surroundings at $T_3 = 27 + 273 = 300$ K. The surroundings are assumed to be black, since no radiation is reflected to the plates and $\varepsilon_3$ is therefore unity. The surface resistance of the walls

$$\frac{1-\varepsilon_3}{\varepsilon_3 A_3} \approx 0$$

or, $E_{b_3} = J_3$. The radiation network is shown in Fig. Ex. 7.20.

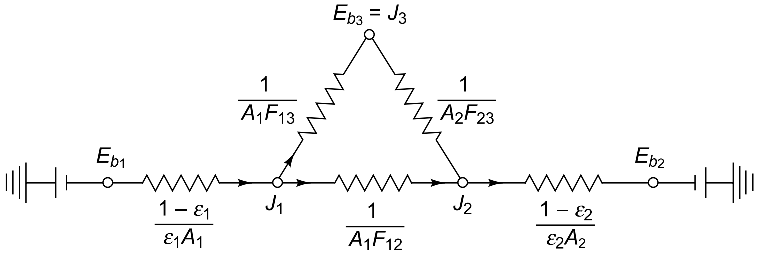


**Fig.** *Ex. 7.20*

For $L_1/D = 2/1 = 2$, $L_2/D = 1 \text{ m}/1 \text{ m} = 1$

From Fig. 7.28, $F_{12} = 0.282 = F_{21}$

$$F_{12} + F_{13} = 1$$

$$F_{13} = 1 - 0.282 = 0.718$$

$$F_{21} + F_{23} = 1$$

$$F_{23} = 1 - 0.282 = 0.718$$

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Surface resistances:

$$\frac{1-\varepsilon_1}{\varepsilon_1 A_1} = \frac{1-0.3}{0.3\times 2} = \frac{7}{6} = 1.67 \text{ m}^{-2}$$

$$\frac{1-\varepsilon_2}{\varepsilon_2 A_2} = \frac{1-0.5}{0.5\times 2} = 0.5 \text{ m}^{-2}$$

Space resistances:

$$\frac{1}{A_1F_{12}} = \frac{1}{2\times 0.282} = 1.773 \text{ m}^{-2}$$

$$\frac{1}{A_1F_{13}} = \frac{1}{2\times 0.718} = 0.7 \text{ m}^{-2}$$

$$\frac{1}{A_2F_{23}} = 0.7 \text{ m}^{-2}$$

$$E_{b_1} = \sigma T_1^4 = 5.67\times 10^{-8}\times (1000)^4$$

$$= 5.67\times 10^4 \text{ W/m}^2 = 56.7 \text{ kW/m}^2$$

$$E_{b_2} = \sigma T_2^4 = 5.67\times\left(\frac{500}{100}\right)^4 = 5.67\times 625$$

$$= 3544 \text{ W/m}^2 = 3.544 \text{ kW/m}^2$$

$$E_{b_3} = \sigma T_3^4 = 5.67\times\left(\frac{300}{100}\right)^4 = 459.27 \text{ W/m}^2$$

$$= 0.459 \text{ kW/m}^2$$

For each node, the algebraic sum of the currents is zero.

$$\text{Node } J_1: \frac{E_{b_1}-J_1}{1.67} + \frac{J_2-J_1}{1.773} + \frac{E_{b_3}-J_1}{0.7} = 0$$

$$\text{Node } J_2: \frac{E_{b_2}-J_2}{0.5} + \frac{J_1-J_2}{1.773} + \frac{E_{b_3}-J_2}{0.7} = 0$$

Substituting,

$$\frac{56.7-J_1}{1.67} + \frac{J_2-J_1}{1.773} + \frac{0.459-J_1}{0.7} = 0$$

and
$$\frac{3.544-J_2}{0.5} + \frac{J_1-J_2}{1.773} + \frac{0.459-J_2}{0.7} = 0$$

or,
$$2.593\, J_1 - 0.564\, J_2 = 34.606$$

and
$$3.993\, J_2 - 0.564\, J_1 = 7.744$$

Solving the two equations,

$$J_1 = 14.246 \text{ kW/m}^2$$

$$J_2 = 4.127 \text{ kW/m}^2$$

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Heat lost by plate 1

$$Q_1 = \frac{E_{b_1} - J_1}{1.67} = \frac{56.7 - 14.246}{1.67} = 25.42 \text{ kW}$$

Heat lost by plate 2

$$Q_2 = \frac{E_{b_2} - J_2}{0.5} = 2(3.544 - 4.127) = -1.166 \text{ kW}$$

Heat gained by the walls

$$Q_3 = \frac{J_1 - J_3}{1/(A_1 F_{12})} + \frac{J_2 - J_3}{1/(A_2 F_{23})}$$

$$= \frac{14.246 - 0.459}{0.7} + \frac{4.127 - 0.459}{0.7} = 24.93 \text{ kW}$$

Alternatively,

$$Q_3 = Q_1 + Q_2 = 24.254 \text{ kW}$$

**Example 7.21** A hemispherical cavity of radius 0.75 m is covered with a plate having a hole of 0.25 m diameter drilled at its centre. The inner surface of the plate is maintained at 550 K by a heater embedded in the surface. Assuming the surfaces to be black and the hemisphere to be well insulated, calculate (i) the temperature of the surface of the hemisphere and (ii) the power input to the heater.

**Solution** Let the inner surface of the plate be 1, the surface of the hemisphere be 2, and the projected surface of the hole be 3. Since the surface 1 is completely surrounded, we have

$$F_{1-1} + F_{1-2} + F_{1-3} = 1$$

Since the surface 1 can neither see itself nor surface 3, therefore,

$$F_{1-1} = 0 = \text{F}_{1-3}, \quad \therefore F_{1-2} = 1.0$$

By reciprocity theorem,

$$A_1 F_{12} = A_2 F_{21}$$

$$\therefore \quad F_{21} = \frac{A_1}{A_2} = \frac{\left[\pi(0.75)^2 - (0.125)^2\right]}{2\pi \times (0.75)^2}$$

$$\therefore \quad F_{21} = 0.4861$$

Again, for surface 3,

$$F_{3-3} + F_{3-2} + F_{3-1} = 1$$

$$\therefore \quad F_{3-2} = 1 \quad (\because F_{3-3} = F_{31} = 0)$$

$$F_{23} = \frac{A_3}{A_2} = \frac{\pi(0.125)^2}{2\pi(0.75)^2} = 0.0139$$

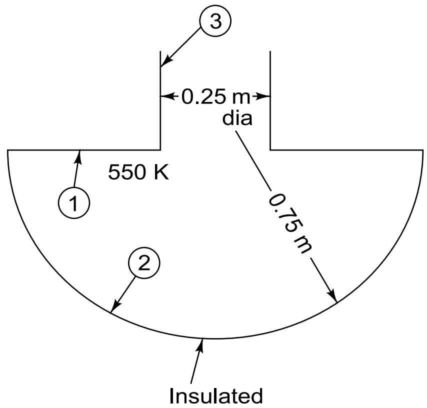


**Fig.** *Ex. 7.21*

Rate of energy incident on surface of the hemisphere (the rate of energy entering through the hole from outside being negligible since the surroundings are very large and at normal temperature)

$$= A_1 F_{12} \sigma T_1^4 = A_1 \sigma T_1^4$$

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Rate of energy emitted by surface 2 would be

$$A_2 F_{21} \sigma T_2^4 + A_2 F_{2-3} \sigma T_2^4 = A_2 \sigma T_2^4 (0.4861 + 0.0139) = 0.5 A_2 \sigma T_2^4$$

Under steady-state conditions,

$$A_1 \sigma T_1^4 = 0.5 A_2 \sigma T_2^4$$

$$\left(\frac{T_2}{T_1}\right)^4 = \frac{A_1}{A_2} \times 2 = 0.4861 \times 2 = 0.9722$$

$\therefore$ $$T_2 = (0.9722)^{1/4} \times 550 = 546.1 \text{ K} \quad \textit{Ans.} \text{ (i)}$$

Heat input to the heater,

$$Q_1 = A_1 F_{12} \sigma (T_1^4 - T_2^4)$$

$$= \pi[(0.75)^2 - (0.125)^2] \times 1 \times 5.67 (5.5^4 - 5.4^4) = 256.3 \text{ W} \quad \textit{Ans.} \text{ (ii)}$$

**Example 7.22** A spherical ball 6 cm in diameter and 310 K is placed inside a large spherical furnace at 600 K. Estimate the diameter of the spherical furnace such that 20% of the energy emitted by the furnace reaches the spherical ball. Assume the surfaces as black. What is the net exchange of energy between the two surfaces?

**Solution**

$$T_1 = 310 \text{ K}, T_2 = 600 \text{ K}, r_1 = 3 \text{ cm}, r_2 = ?$$

$$F_{11} + F_{12} = 1.0$$

Since $$F_{11} = 0, F_{12} = 1.0$$

$$A_1 F_{12} = A_2 F_{21}, F_{21} = A_1/A_2$$

Energy emitted by the inside surface of the sphere = $A_2 E_2$ and the fraction reaching the spherical ball is 0.2 $A_2 E_2$.

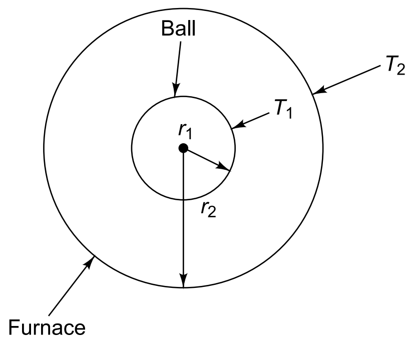


**Fig.** *Ex. 7.22*

$\therefore$ $$F_{21} = 0.2 = \frac{A_1}{A_2} = \frac{4\pi r_1^2}{4\pi r_2^2}$$

$$r_2 = r_1/\sqrt{0.2} = \frac{3 \text{ cm}}{0.447} = 6.71 \text{ cm}$$

$\therefore$ Diameter of the furnace = 13.42 cm *Ans.*

Net exchange of energy between the two surfaces

$$Q_{1-2} = \sigma A_1 F_{12} (T_1^4 - T_2^4)$$

$$= 5.67 \times 4\pi (0.3)^2 \times 1 \times (3.1^4 - 6^4)$$

$$= -77.195 \text{ W} \quad \textit{Ans.}$$

**Example 7.23** A small sphere (outside radius = 60 mm) with a surface temperature of 300°C is located at the geometric centre of a large sphere (inside diameter = 360 mm) with an inner surface temperature of 15°C. Calculate how much of heat emitted from the large sphere inner surface is incident upon the outer surface of the small sphere, assuming that both surfaces approach black body behaviour. What is the net exchange of heat between the two spheres?

**Solution** Radius of small sphere, $r_1$ = 30 mm = 0.03 m, Radius of large sphere $r_2$ = 180 mm = 0.18 m.

Here $$F_{12} = 1$$

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By reciprocity theorem,

$$A_1F_{12} = A_2F_{21}$$

$$\therefore \qquad F_{21} = \frac{A_1}{A_2} = \frac{r_1^2}{r_2^2} = \left(\frac{0.03}{0.18}\right)^2 = 0.0278 \quad \textit{Ans.}$$

Thus, 2.78% of the emission from the inner surface of the large sphere is incident upon the small sphere and absorbed by it.

Also, $F_{21} + F_{22} = 1$

$$\therefore \qquad F_{22} = 1 - F_{21} = 1 - 0.0278 = 0.9722$$

Thus, 97.22% of emission from the large sphere is absorbed by the inner surface of the sphere itself.

Radiant heat exchange between the two spheres

$$Q_{12} = \sigma A_1 F_{12}\,(T_1^4 - T_2^4)$$

$$= 5.67 \times 4\pi(0.03)^2 \times 1 \times \left[\left(\frac{573}{100}\right)^4 - \left(\frac{288}{100}\right)^4\right]$$

$$= 0.0113 \times 5.67 \times 1009.2 = 64.66 \text{ W} \quad \textit{Ans.}$$

**Example 7.24** Two parallel discs of 1 m diameter are situated 2 m apart in the surroundings at a temperature of 20°C. One side of one disc has an emissivity of 0.5 and is maintained at 500°C by electrical resistance heating and the other side is insulated. The other disc is open to radiation on both sides. Determine the equilibrium temperature of the second disc and the heat flow rate from the first disc.

Discuss the effect on the solution if both sides of the second disc are perfect mirrors.

**Solution** The surroundings are assumed to be black and so $E_{b_3} = J_3$. Since there is no net heat transfer from the second disc at steady state, the heat flow from 1 to 2 is equal to the heat flow from 2 to 3. The radiation network is shown in Fig. Ex. 7.24(b).

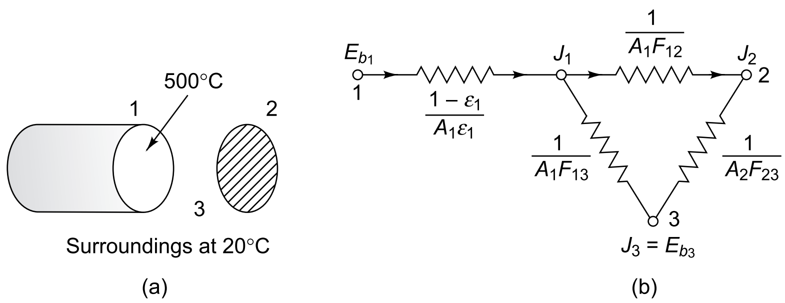


**Fig.** *Ex. 7.24*

$$E_{b_1} = \sigma T_1^4 = 5.67 \times \left(\frac{773}{100}\right)^4 = 20240 \text{ W/m}^2$$

$$= 20.24 \text{ kW/m}^2$$

$$E_{b_3} = \sigma T_3^4 = 5.67 \times \left(\frac{293}{100}\right)^4 = 418 \text{ W/m}^2$$

$$= 0.418 \text{ kW/m}^2$$

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$$\frac{1-\varepsilon_1}{\varepsilon_1 A_1} = \frac{1-0.5}{0.5\times\pi/4\times 1} = 1.27 \text{ m}^{-2}$$

From Fig. 7.29, $F_{12} = 0.06$,

$$\frac{1}{A_1F_{12}} = \frac{1}{\pi/4\times 1\times 0.06} = 21.22 \text{ m}^{-2}$$

$$F_{13} = 1 - F_{12} = 0.94$$

$$2F_{23} = F_{2'3} + F_{2''3} = 0.94 + 1 = 1.94$$

where suffix 2 indicates both sides, 2′ for the left side and 2″ for the right side, of the disc 2.

$$F_{23} = \frac{1.94}{2} = 0.97$$

$$\frac{1}{A_1F_{13}} = \frac{4}{\pi\times 0.94} = 1.35 \text{ m}^{-2}$$

$$\frac{1}{2A_2F_{23}} = \frac{4}{\pi\times(2\times 0.97)} = 0.65 \text{ m}^{-2}$$

Algebraic sum of currents at each node is zero.

At node $J_1$:
$$\frac{20.24-J_1}{1.27} + \frac{E_{b_2}-J_1}{21.22} + \frac{0.418-J_1}{1.35} = 0$$

At node 2:
$$\frac{J_1-E_{b_2}}{21.22} + \frac{0.418-E_{b_2}}{0.65} = 0$$

Solving the two equations,

$$E_{b_2} = 0.715 \text{ kW/m}^2$$

and $J_1 = 10.34 \text{ kW/m}^2$

Now, $\sigma T_2^4 = 0.715 \text{ kW/m}^2$

$$T_2 = 335.1 \text{ K} = 62°\text{C} \quad \textit{Ans.}$$

Heat flow rate from the disc 1

$$Q_{1-J_1} = \frac{E_{b_1}-J_1}{(1-\varepsilon_1)/\varepsilon_1 A_1} = \frac{20.24-10.34}{1.27}$$

$$= 7.79 \text{ kW} \quad \textit{Ans.}$$

Alternatively,

Since $Q_{1-J_1} = Q_{J_1-3} + Q_{2-3}$

$$= \frac{10.34-0.42}{1.35} + \frac{0.715-0.42}{0.65} = 7.79 \text{ kW}$$

The second disc reradiates to the surroundings all the energy, incident on it from the first disc, irrespective of its emissivity (for $\varepsilon$ varying from 0 to 1) and does not affect the heat flow from the first disc, when it has reached the state of thermal equilibrium.

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If both sides of the second disc behave as perfect mirrors, $\varepsilon = 0$, and all the energy incident on it is re-radiated at thermal equilibrium, and would have no effect on solution of the problem.

**Example 7.25**

(a) Derive an expression for the time required to cool a body of mass $m$, surface area $A$, emissivity $\varepsilon$ and specific heat $c_p$ from an initial temperature $T_1$ to the final temperature $T_2$ in a large enclosure, the walls of which are at temperature $T_w$. Neglect convective losses and temperature gradients inside the body.

(b) A solid copper sphere 0.1 m in diameter is heated to 1000°C and suspended in a large room, the walls of which are at 30°C. Calculate the time taken by the sphere to cool to 500°C. Consider only radiative energy transfer and neglect the internal thermal resistance of the sphere. For copper take $\rho$ = 8680 kg/m$^3$, $c_p$ = 0.41 kJ/kg K and $\varepsilon$ = 0.1.

**Solution**

(a) Let $T$ denote the temperature of the body at instant $t$. In time d$t$, let the temperature of the body drop by d$T$. By energy balance,

Energy loss of the body = Energy transfer to the surroundings by radiation

$$-mc_p\,\mathrm{d}T = \sigma A\varepsilon (T^4 - T_w^4)\,\mathrm{d}t$$

or,
$$\frac{\sigma A\varepsilon}{mc_p}\int_0^t \mathrm{d}t = -\int_{T_1}^{T_2} \frac{\mathrm{d}T}{T^4 - T_w^4}$$

$$\frac{\sigma A\varepsilon}{mc_p}\,t = \int_{T_1}^{T_2} \frac{\mathrm{d}T}{(T_w^2 + T^2)(T_w^2 - T^2)}$$

$$= -\frac{1}{2T_w^2}\int_{T_2}^{T_1}\left(\frac{1}{T_w^2 + T^2} + \frac{1}{T_w^2 - T^2}\right)\mathrm{d}T$$

$$= -\frac{1}{2T_w^2}\left[\frac{1}{T_w}\left(\tan^{-1}\frac{T}{T_w}\right)_{T_2}^{T_1} + \frac{1}{2T_w}\int_{T_2}^{T_1}\left(\frac{1}{T_w + T} + \frac{1}{T_w - T}\right)\mathrm{d}T\right]$$

$$= -\frac{1}{2T_w^3}\left[\tan^{-1}\frac{T_1}{T_w} - \tan^{-1}\frac{T_2}{T_w} + \frac{1}{2}\left(\ln\frac{T_w + T_1}{T_w - T_1} - \ln\frac{T_w + T_2}{T_w - T_2}\right)\right]$$

$$= -\frac{1}{2T_w^3}\left[\tan^{-1}\frac{(T_1/T_w) - (T_2/T_w)}{1 + (T_1/T_w)(T_2/T_w)} + \frac{1}{2}\ln\frac{(T_w + T_1)(T_w - T_2)}{(T_w - T_1)(T_w + T_2)}\right]$$

$$t = \frac{mc_p}{\sigma A\varepsilon}\frac{1}{2T_w^3}\left[\frac{1}{2}\ln\frac{(T_2 + T_w)(T_1 - T_w)}{(T_1 + T_w)(T_2 - T_w)} - \tan^{-1}\frac{T_w(T_1 - T_2)}{T_w^2 + T_1T_2}\right]$$

(b)
$$T_1 = 1273\text{ K},\ T_2 = 773\text{ K},\ T_w = 303\text{ K}$$

$$A = 4\pi r^2 = 4\pi \times (0.05)^2 = 0.0314\text{ m}^2$$

$$m = \rho v = 8680 \times (3/4) \times \pi \times (0.05)^3 = 4.542\text{ kg}$$

$$\ln\frac{(T_2 + T_w)(T_1 - T_w)}{(T_1 + T_w)(T_2 - T_w)} = \ln\frac{(773 + 303)(1273 - 303)}{(1273 + 303)(773 - 303)}$$

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$$= \ln \frac{1076 \times 970}{1576 \times 470}$$

$$= \ln 1.409 = 0.34288$$

$$\tan^{-1} \frac{T_w (T_1 - T_2)}{T_w^2 + T_1 T_2} = \tan^{-1} \frac{303(1273 - 773)}{303^2 + 1273 \times 773}$$

$$= \tan^{-1} 0.1408 = 8.0145$$

$$= 0.1396 \text{ rad}$$

$$t = \frac{4.542 \times 0.41}{5.67 \times 10^{-11} \times 0.0314 \times 0.1} \frac{1}{12 \times 303^3} \left( \frac{1}{2} \times 0.34288 - 0.1396 \right)$$

$$= 188.12 \times 10^3 \times 0.0318 = 5982.2 \text{ s}$$

$$= 1 \text{ h } 40 \text{ min} \quad \textit{Ans.}$$

**Example 7.26** If the inside surface temperature of a hemispherical cavity of 0.5 m diameter is 400°C and its emissivity is 0.6, calculate the rate of radiant heat transfer from the cavity.

**Solution** Net rate of radiant emission from the cavity is given by Eq. (7.88),

$$Q = A_1 \varepsilon_1 \sigma T_1^4 \frac{1 - F_{11}}{1 - F_{11}(1 - \varepsilon_1)}$$

From Fig. 7.48(b),

$$F_{11} + F_{12} = 1$$

$$A_1 F_{12} = A_2 F_{21}$$

where suffix 1 denotes the cavity surface and suffix 2 denotes a plate above the cavity.

$$F_{21} + \overset{= 0}{F_{22}} = 1$$

$$\therefore \quad F_{12} = A_2 / A_1$$

$$F_{11} = 1 - F_{12} = 1 - \frac{A_2}{A_1} = 1 - \frac{\pi r^2}{2\pi r^2} = 0.5$$

On substitution,

$$Q = 2\pi \left( \frac{0.5}{2} \right)^2 \times 0.6 \times 5.67 \times 10^{-8} \times 673^4 \frac{1 - 0.5}{1 - 0.4 \times 0.5}$$

$$= 1397.5 \text{ W} = 1.3975 \text{ kW} \quad \textit{Ans.}$$

**Example 7.27** A cubical oven has inside sides equal to 0.4 m. One of the faces of the oven forms the door. If the five other inside faces are black and maintained at 600°C, find the rate of heat loss if the oven door is kept open.

**Solution**

$$F_{21} + F_{22} = 1 \text{ (Fig. Ex. 7.27)}$$

$$F_{22} = 0$$

$$\therefore \quad F_{12} = \frac{A_2}{A_1} F_{21} = \frac{A_2}{A_1}$$

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$$F_{11} = 1 - F_{12} = 1 - \frac{A_2}{A_1}$$

where $A_1$ = area of the five inside faces

$= 5 \times 0.4 \times 0.4 = 0.8 \text{ m}^2$

$A_2$ = area of the door through which energy streams out

$= 0.4 \times 0.4 = 0.16$

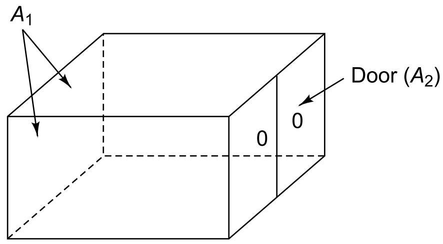


**Fig.** *Ex. 7.27*

$$F_{11} = 1 - \frac{0.16}{0.8} = 0.8$$

Rate of heat loss when the door is open

$$Q = \sigma A_1 \varepsilon_1 T_1^4 \frac{1 - F_{11}}{1 - (1 - \varepsilon_1)F_{11}}$$

$$= 5.67 \times 10^{-8} \times 0.8 \times 1 \times (873)^4 \frac{1 - 0.8}{1}$$

$= 5274 \text{ W} = 5.274 \text{ kW}$ *Ans.*

**Example 7.28** A thermocouple placed in a gas stream flowing in a duct measures a temperature which is somewhere between the actual gas temperature and the duct wall temperature. At steady state the heat flux from gas stream to the thermocouple by convection is balanced by the heat flux from the thermocouple to the duct wall by radiation.

(a) A bare chromel-alumel thermocouple ($\varepsilon$ = 0.8) wire of diameter 0.25 mm is in a gas flowing in a long 250 mm diameter duct (wall emissivity = 0.7). The gas flow rate produces a convection heat transfer coefficient of 120 W/m$^2$ K. This thermocouple in the gas stream reads 800°C and a thermocouple attached to the duct wall reads 500°C. Calculate the actual temperature of the gas.

(b) The thermocouple error can be substantially reduced by a radiation shield. Suppose a thin cylindrical shield ($\varepsilon$ = 0.2) having inside diameter of 2.5 mm is placed around the thermocouple. If the thermocouple reads 800°C and $h$ remains the same, what is the true gas temperature?

**Solution**

Let $T_g$ = true temperature of the gas,

$T_c$ = temperature recorded by the thermocouple,

$T_w$ = wall temperature of the duct

$A_c$ = area of the couple

By energy balance,

$$Q = hA_c(T_g - T_c) = \sigma A_c \mathcal{F}_{cw}(T_c^4 - T_w^4)$$

where $\mathcal{F}_{cw} = \varepsilon_c \varepsilon_w$

Substituting the values,

$$120\,(T_g - 1073) = 5.67 \times 10^{-8} \times 0.8 \times 0.7 \times (1073^4 - 773^4)$$

$\therefore$ $T_g = 1329$ K, or 1056°C

The error is 256°C.

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(b) When the thermocouple is shielded, at steady state,

1. heat transfer by convection from gas to thermocouple = heat transfer by radiation from couple to shield.
2. heat transfer by convection from gas to shield + heat transfer by radiation from couple to shield = heat transfer by radiation from shield to the wall.

Thus,

$$hA_c(T_g - T_c) = \sigma A_c \mathcal{F}_{cs}(T_c^4 - T_s^4) \quad (1)$$

$$h2A_s(T_g - T_s) + \sigma A_c \mathcal{F}_{cs}(T_c^4 - T_s^4) = \sigma A_s \mathcal{F}_{sw}(T_s^4 - T_w^4) \quad (2)$$

$$\frac{A_s}{A_c} = \frac{\pi \, d_s \times 1}{\pi \, d_c \times 1} = \frac{2.5 \text{ mm}}{0.5 \text{ mm}} = 5$$

where the diameter of the couple, $d_c = 2 \times 0.25 = 0.5$ mm

$$\mathcal{F}_{cs} = \frac{1}{(1/\varepsilon_c) + (A_c/A_s)\,(1/\varepsilon_s - 1)} = \frac{1}{(1/0.2) + (1/5)\,(1/0.2 - 1)} = 0.49$$

$$\mathcal{F}_{sw} = \frac{1}{(1/\varepsilon_s) + (A_s/A_w)\,(1/\varepsilon_w - 1)} = \frac{1}{(1/0.2) + (2.5/250)\,(1/0.7 - 1)} = 0.2$$

From Eq. (1)

$$T_g - 1073 = \frac{5.67 \times 0.49}{120}\left[\left(\frac{1073}{100}\right)^4 - \left(\frac{T_s}{100}\right)^4\right]$$

or
$$T_g - 1073 = 0.023\left[13255 - \left(\frac{T_s}{100}\right)^4\right] \quad (3)$$

From Eq. (2)

$$T_g - T_s = \frac{\sigma A_c}{h2A_s}\left[\frac{A_s}{A_c}\mathcal{F}_{sw}(T_s^4 - T_w^4) - \mathcal{F}_{cs}\,(T_c^4 - T_s^4)\right]$$

$$= \frac{5.67}{1200}\left\{5 \times 0.2\left[\left(\frac{T_s}{100}\right)^4 - \left(\frac{773}{100}\right)^4\right] - 0.49\left[\left(\frac{1073}{100}\right)^4 - \left(\frac{T_s}{100}\right)^4\right]\right\} \quad (4)$$

Equations (3) and (4) have to be solved for $T_g$ and $T_s$. This may be done by trial and error. Let us assume $T_s = 1000$ K.

From Eq. (3), $T_g = 1147.9$ K

Substituting in Eq. (4) L.H.S. = 247.9, R.H.S. = 22.845

When $T_s = 1050$ K, $T_g = 1098$ K

By substituting in Eq. (4)

LHS = 48, RHS = 38.2

When $T_s = 1060$ K, $T_g = 1087$ K

In Eq. (4), LHS = 27 K and RHS = 41 K

Therefore, the required gas temperature is about 1090 K or 817°C. The error is reduced to 17°C.

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**Example 7.29** A gas turbine combustion chamber is 0.35 m in diameter and the walls are maintained at 500°C. The products of combustion are at 1000°C and a pressure of 1 atm and contain 12% $CO_2$ and 10% $H_2O$ vapour by volume. Determine the net radiant heat transfer per unit surface area.

**Solution** From Table 7.5, the mean beam length, $L = D = 0.35$ m

$$P_{CO_2} = 0.12 \text{ atm}, \quad P_{CO_2} L = 0.12 \times 0.35 = 0.042 \text{ m atm}$$

$$P_{H_2O} = 0.10 \text{ atm}, \quad P_{H_2O} L = 0.10 \times 0.35 = 0.035 \text{ m atm}$$

From Figs 7.53 and 7.54

$$\varepsilon_{CO_2} = 0.078, \quad \varepsilon_{H_2O} = 0.05$$

From Figs 7.55 and 7.56, the correction factors are

$$C_{CO_2} = 1 \text{ and } C_{H_2O} = \left(\text{at } \frac{P_{H_2O} + P}{2} = \frac{1.1}{2} = 0.55 \text{ atm}\right) = 1.03$$

$$\varepsilon_{CO_2} = 0.078 \text{ and } \varepsilon_{H_2O} = 0.05 \times 1.03 = 0.0515$$

From Fig. 7.57(c), when $\dfrac{P_{H_2O}}{P_{H_2O} + P_{CO_2}} = \dfrac{0.10}{0.10 + 0.12}$

$$= \frac{10}{22} = 0.45$$

and $\quad (P_{CO_2} + P_{H_2O})L = 0.077$ m atm

$$\Delta\varepsilon = 0.003$$

The emissivity of the mixture at 1000°C

$$\varepsilon_g = \varepsilon_{CO_2} + \varepsilon_{H_2O} - \Delta\varepsilon = 0.078 + 0.0515 - 0.003 = 0.1265$$

The absorptivity has to be found at wall temperature of 500°C.

From Figs 7.53 and 7.54 at 500°C for $P_{CO_2} L = 0.042$ m atm and $P_{H_2O} L = 0.035$ m atm,

$$\varepsilon_{CO_2} = \alpha_{CO_2} = 0.081, \; \varepsilon_{H_2O} = \alpha_{H_2O} = 0.085$$

From Figs 7.55 and 7.56,

$$C_{CO_2} = 1 \text{ and } C_{H_2O} = 1.03$$

Corrected values:

$$\varepsilon_{CO_2} = 0.081 \text{ and } \varepsilon_{H_2O} = 1.03 \times 0.085 = 0.0876$$

From Fig. 7.57(c),

$$\Delta\varepsilon = 0.003$$

For the mixture at $T_w = 500$°C,

$$\alpha_g = \varepsilon_g = \varepsilon_{CO_2} + \varepsilon_{H_2O} - \Delta\varepsilon$$

$$= 0.081 + 0.0876 - 0.003 = 0.1656$$

Radiant heat exchange

$$Q = \sigma A \, (\varepsilon_g T_g^4 - \alpha_g T_w^4)$$

$$= 5.67 \times 10^{-8} \times 1\,[(0.1265 \times (1273)^4 - 0.1656 \times (773)^4] \text{ W}$$

$$= 15483.5 \text{ W} = 15.4835 \text{ kW} \quad \textit{Ans.}$$

**Example 7.30** Two infinitely long parallel plates of widths $x$ = 12 cm and $y$ = 6 cm are located at a distance $z$ = 7 cm apart as shown in Fig. Ex. 7.30. Determine the view factor $F_{12}$.

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**Solution** We label the end points of both the surfaces and draw straight dashed lines between the end points, as shown in Fig. Ex. 7.30. Using the crossed-string method

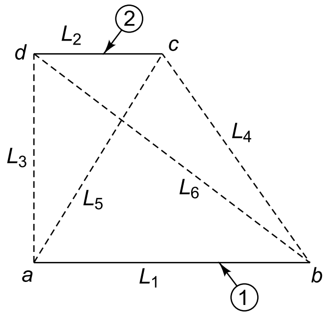


**Fig.** *Ex. 7.30*

$$F_{12} = \frac{\sum \text{(crossed strings)} - \sum \text{(uncrossed strings)}}{2 \times \text{(string on surface 1)}}$$

$$= \frac{(L_5 + L_6) - (L_3 + L_4)}{2L_1}$$

where
$L_1 = x = 12$ cm
$L_2 = y = 6$ cm
$L_3 = z = 7$ cm
$L_4 = (7^2 + 6^2)^{1/2} = 9.22$ cm,
$L_5 = (6^2 + 7^2)^{1/2} = 9.22$ cm,
$L_6 = (12^2 + 7^2)^{1/2} = 13.89$ cm

$$F_{12} = \frac{(9.22 + 13.89) - (7 + 9.22)}{2 \times 12}$$

$$= 0.287 \quad \textit{Ans.}$$

**Example 7.31** In a cylindrical furnace of diameter 2 m and height 1 m, the base and the top having emissivities 0.4 and 0.8, respectively are maintained at 700 K and 500 K, while the lateral surface approximating a black body is maintained at 400 K. Determine the net rate of radiation heat transfer at each surface during steady-state operation.

**Solution** The furnace and the radiation network are shown in Fig. Ex. 7.31. Writing the energy balance for the nodes 1 and 2,

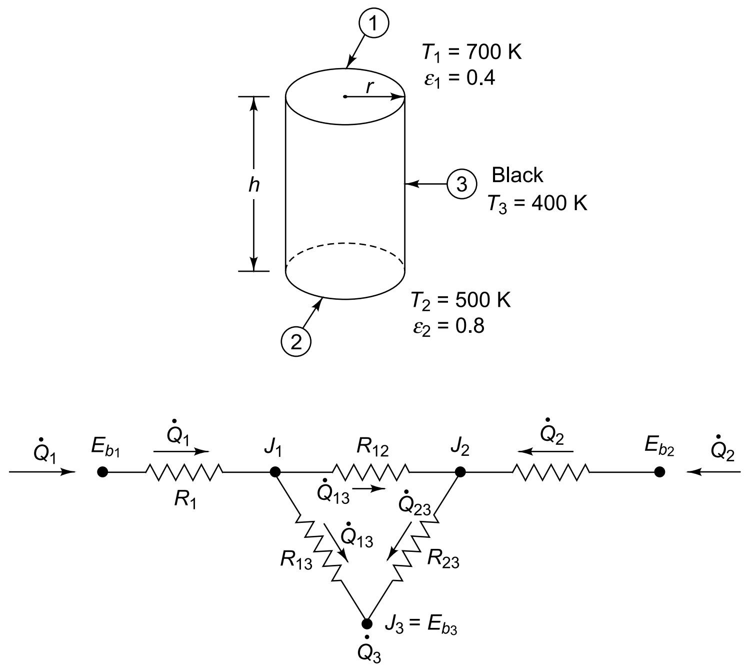


**Fig.** *Ex. 7.31*

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$$\frac{E_{b_1} - J_1}{R_1} = \frac{J_1 - J_2}{R_{12}} + \frac{J_1 - J_3}{R_{13}}$$

$$\frac{E_{b_2} - J_2}{R_2} = \frac{J_2 - J_1}{R_{12}} + \frac{J_2 - J_3}{R_{23}}$$

Here, $E_{b_1} = \sigma T_1^4 = 5.67 \times 10^{-8}\,(700)^4 = 13{,}614\ \text{W/m}^2$

$E_{b_2} = \sigma T_2^4 = 5.67 \times 10^{-8}\,(500)^4 = 3{,}544\ \text{W/m}^2$

$E_{b_3} = J_3 = \sigma T_3^4 = 5.67 \times 10^{-8}\,(400)^4 = 1452\ \text{W/m}^2$

$A_1 = A_2 = \pi r^2 = \pi(1)^2 = 3.14\ \text{m}^2$

From Fig. 7.29, the view factor from the base to the top is found to be $F_{12} = 0.38$.

Now, $F_{11} + F_{12} + F_{13} = 1.0$

$F_{13} = 1 - F_{12} = 0.62$

$$R_1 = \frac{1-\varepsilon_1}{A_1\,\varepsilon_1} = \frac{1-0.8}{3.14 \times 0.8} = 0.0796\ \text{m}^{-2}$$

$$R_2 = \frac{1-\varepsilon_2}{A_2\,\varepsilon_2} = \frac{1-0.4}{3.14 \times 0.4} = 0.4777\ \text{m}^{-2}$$

$$R_{12} = \frac{1}{A_2\,F_{12}} = \frac{1}{3.14 \times 0.38} = 0.8381\ \text{m}^{-2}$$

$$R_{23} = \frac{1}{A_2\,F_{23}} = \frac{1}{3.14 \times 0.62} = 0.5137\ \text{m}^{-2} = R_{13} \quad \text{(by symmetry)}$$

On substitution,

$$\frac{13614 - J_1}{0.0796} = \frac{J_1 - J_2}{0.8381} + \frac{J_1 - 1452}{0.5137}$$

$$\frac{3544 - J_2}{0.0777} = \frac{J_2 - J_1}{0.8381} + \frac{J_2 - 1452}{0.5137}$$

Solving these two equations,

$J_1 = 11418\ \text{W/m}^2$ and $J_2 = 4562\ \text{W/m}^2$

$$Q_1 = \frac{E_{B_1} - J_1}{R_1} = \frac{13614 - 11418}{0.0796} = 27{,}588\ \text{W} \quad \textit{Ans.}$$

$$Q_2 = \frac{E_{B_2} - J_2}{R_2} = \frac{3544 - 4562}{0.4777} = -\,2132\ \text{W} \quad \textit{Ans.}$$

$$Q_3 + \frac{J_1 - J_3}{R_{13}} + \frac{J_2 - J_3}{R_{23}} = 0$$

$$Q_3 = \frac{1452 - 11418}{0.5137} + \frac{1452 - 4562}{0.5137} = -\,25455\ \text{W} \quad \textit{Ans.}$$

$Q_1 + Q_2 + Q_3 = 27588 - 2132 - 25455 = 0.$

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**Example 7.32** Calculate the shape factor $F_{12}$ for the configuration of Fig. Ex. 7.32 and the net radiant exchange $Q_{12}$, if $T_1$ = 427°C, $T_2$ = 227°C, $\varepsilon_1$ = 0.8 and $\varepsilon_2$ = 0.9.

**Solution**

$$F_{21} = F_{24} - F_{23}$$

$$F_{12} = \frac{A_2}{A_1} F_{21} = \frac{A_2}{A_1} (F_{24} - F_{23})$$

$$\frac{y}{x} = \frac{1.5}{5} = 0.3, \quad \frac{z}{x} = \frac{6}{5} = 1.2, \quad F_{24} = 0.35$$

$$\frac{y}{x} = \frac{1.5}{5} = 0.3, \quad \frac{z}{x} = \frac{2}{5} = 0.4, \quad F_{23} = 0.29$$

$$F_{12} = \frac{5 \times 1.5}{5 \times 4} (0.35 - 0.29) = 0.0225$$

$$Q = \frac{5.67 \times 10^{-8} \times 20 \left[ (700)^4 - (500)^4 \right]}{\left( \frac{1}{0.8} - 1 \right) + \frac{1}{0.0225} + \frac{20}{7.5} \left( \frac{1}{0.9} - 1 \right)}$$

$$= \frac{201398.4}{44.986}$$

$$= 4476.9 \text{ W} = 4.4769 \text{ kW}$$

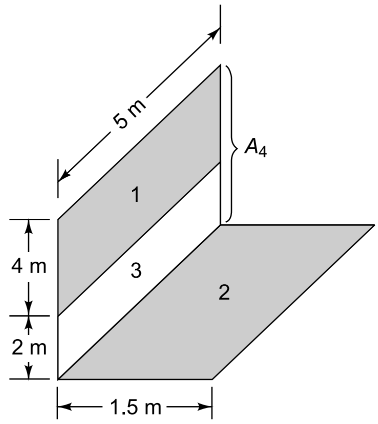


**Fig.** *Ex. 7.32*

**Example 7.33** A bed of burning coal in a furnace radiates as a plane rectangular black surface, 3 m by 2 m, at 1500°C, to an opaque bank of black tubes of the same projected area. These are at a surface temperature of 300°C and at such a distance from the fire bed that the shape factor is 0.5. Determine the net radiant heat flow to the tube bank and show that enclosing the furnace with adiabatic vertical black walls increases the heat flow by 50 per cent.

**Solution**

$$Q_{12} = 5.67 \times 10^{-8} \times 6 \times 0.5 \times (1773^4 - 573^4)$$

$$= 5.67 \times 10^{-3} \times 3 \times (17.73^4 - 5.73^4)$$

$$= 1662.5 \text{ kW}$$

$$Q'_{12} = \frac{\sigma A_1 (T_1^4 - T_2^4)}{(1/\varepsilon_1 - 1) + 1/\bar{F}_{12} + A_1/A_2 (1/\varepsilon_2 - 1)}$$

$$\bar{F}_{12} = \frac{A_2 - A_1 F_{12}^2}{A_1 + A_2 - 2 A_1 F_{12}}, \quad A_1 = A_2$$

$$\bar{F}_{12} = \frac{A(1 - F_{12}^2)}{2A(1 - F_{12})} = \frac{1 + F_{12}}{2} = \frac{1.5}{2} = 0.75$$

$$\frac{Q'_{12} - Q_{12}}{Q_{12}} \times 100 = \frac{0.75 - 0.5}{0.5} \times 100 = 50\% \quad Ans.$$

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**Example 7.34** Find the shape factor of a cylindrical cavity of diameter $d$ and depth $H$ with respect to itself. If $d$ = 200 mm, $H$ = 500 mm, $T_1$ = 600 K and $\varepsilon_1$= 0.8, find the rate at which energy streams out from the cavity.

**Solution**

$$F_{11} = \frac{4H}{4H+d} = \frac{4 \times 0.5}{4 \times 0.5 + 0.2} = \frac{2}{2.2} = 0.91$$

$$Q = \pi \times 0.2 \times 0.5 \times 0.8 \times 5.67 \times 10^{-8} \times (600^4) \left[\frac{1-0.91}{1-(1-0.8)\times 0.91}\right]$$

$$= \pi \times 0.008 \times 5.67 \times 6^4 \times \frac{0.09}{0.918} \text{ W}$$

$$= 18.1 \text{ W}$$

**Example 7.35** A 100 mm diameter disc ($\varepsilon$ = 0.8) at 800 K is at a distance of 2 m from a disc ($\varepsilon$ = 0.7) of 2 m diameter maintained at 300 K. Find the net rate of radiant exchange.

**Solution**

$$F_{12} = \frac{D^2}{4L^2 + D^2} = \frac{2^2}{4(2)^2 + 2^2} = 0.2$$

$$Q_{12} = \frac{\sigma A_1 (T_1^4 - T_2^4)}{(1/\varepsilon_1 - 1) + 1/F_{12} + A_1/A_2\,(1/\varepsilon_2 - 1)}$$

$$= \frac{5.67 \times \pi \times (0.05)^2 \left[(800/100)^4 - (300/100)^4\right]}{(1/0.8-1)+1/0.2+(0.05/1)^2(1/0.7-1)}$$

$$= \frac{5.67 \times 25 \times 10^{-4} (8^4 - 3^4)}{5.251} \text{ W}$$

$$= 34.03 \text{ W}$$

**Example 7.36** A cylindrical rod ($\varepsilon$ = 0.7) of 50 mm diameter is maintained at 1000°C by electrical resistance heating and kept in a room, the walls ($\varepsilon$ = 0.6) of which are at 15°C. Determine the energy which must be supplied per metre length of the rod. If an insulated half-circular reflector of 0.45 m is placed around the rod, estimate the energy supply to the rod per metre length.

**Solution**

Without reflector:

$$Q = \frac{\sigma (T_R^4 - T_w^4)}{(1-\varepsilon_R)/(\varepsilon_R A_R) + 1/A_R F_{Rw} + (1-\varepsilon_w)/(\varepsilon_w A_w)}$$

$$= \frac{5.67\,[(1273/100)^4 - (288/100)^4]}{[0.3/0.7\,\pi\,(0.05)] + 1/\pi\,(0.05)\,(1) + 0} \text{ watts}$$

$$= 16319.5 \text{ W} = 16.32 \text{ kW}$$

With reflector:

$$R_1 = \frac{0.3}{0.7\,\pi\,(0.05)} = 2.72 \text{ m}^{-2},\; R_2 = 0$$

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$$R_3 = \frac{1}{A_R\, F_{R\,\text{ref}}} = \frac{1}{\pi(0.05)(0.5)}\ \text{m}^{-2} = 12.73\ \text{m}^{-2}$$

$$R_4 = \frac{1}{A_w\, F_{w\,\text{ref}}} = \frac{1}{(450-50)\times 10^{-3}\ \text{m}\times 1\ \text{m}}$$

$$= \frac{1}{0.4} = 2.5\ \text{m}^{-2}$$

$$R_5 = \frac{1}{A_R\, F_{Rw}} = \frac{1}{\pi\,(0.05)\,(0.5)}\ \text{m}^{-2}$$

$$\frac{1}{R_5} = 0.0785\ \text{m}^2$$

Total resistance,
$$R = R_1 + R_2 + \frac{1}{(1/R_5) + 1/(R_3 + R_4)}$$

$$= 2.73 + 0 + \frac{1}{0.0785 + 0.0657} = 9.66\ \text{m}^{-2}$$

$$Q = \frac{5.67\left[(1273/100)^4 - (288/100)^4\right]}{9.66}\ \text{W/m}$$

$$= 15373.5\ \text{W} = 15.374\ \text{kW} \quad \textit{Ans.}$$

**Example 7.37** A pipe carrying steam, having an outside diameter of 20 cm runs in a large room and is exposed to air at a temperature of 30°C. The pipe surface temperature is 200°C. Find the heat loss per metre length of the pipe by convection and radiation taking the emissivity of the pipe surface as 0.8.

**Solution** Heat lost by radiation per unit length of pipe

$$Q_r = \sigma A_1\, \varepsilon\, (T_1^4 - T_2^4)$$

$$= 5.67 \times \pi \times 0.02 \times 1 \times 0.8 \times [(4.73)^4 - (3.03)^4]$$

$$= 1185\ \text{W/m}$$

To determine the heat transfer by natural convection,

$$T_f = \frac{T_w + T_\infty}{2} = \frac{200 + 30}{2} = 115°\text{C}$$

At 115°C, the properties of air are:

$$\beta = \frac{1}{T_f} = \frac{1}{388}\ \text{K}^{-1},\ k = 33.06 \times 10^{-3}\ \text{W/mK},$$

$$\nu = 24.93 \times 10^{-6}\ \text{m}^2/\text{s},\ \text{Pr} = 0.687$$

$$\text{Gr}_d \cdot \text{Pr} = \frac{g\beta\theta d^3}{\nu^2},\ \text{Pr} = 0.687 \times \frac{9.81 \times \dfrac{1}{3.88} \times (200 - 30) \times (0.2)^3}{(24.93 \times 10^{-6})^2}$$

$$= 0.38 \times 10^8$$

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$$\text{Nu}_d = 0.53\,(\text{Gr}_d\,\text{Pr})^{1/4}$$

$$= 0.53\,(0.38 \times 10^8)^{1/4} = 41.61 = \frac{hd}{k}$$

$$\therefore \qquad h_c = \frac{0.03306 \times 41.61}{0.2} = 6.878 \text{ W/mK}$$

Heat lost by natural convection

$$Q_c = h_c A\theta = 6.878 \times \pi \times 0.2 \times 170$$

$$= 734.3 \text{ W/m}$$

$\therefore$ Total heat lost per metre length

$$= Q_r + Q_c = 1185 + 734.3 = 1919.3 \text{ W/m} \quad \textit{Ans.}$$

**Example 7.38** The overall heat transfer coefficient due to convection and radiation for a steam main at 200°C running in a large room at 30°C is 17.95 W/m$^2$K. Calculate the heat transfer coefficients due to convection and radiation taking the emissivity of the pipe surface as 0.8.

**Solution**

$$h = h_c + h_r = 17.95 \text{ W/m}^2\text{K}$$

By radiation,

$$\frac{Q}{A} = \sigma\varepsilon\,(T_1^4 - T_2^4)$$

$$= 5.67 \times 10^{-8} \times 0.8\,(473^4 - 303^4)$$

$$= 1886.94 \text{ W/m}^2$$

$$h_r = \frac{Q}{A(T_1 - T_2)} = \frac{1886.94}{170} = 11.1 \text{ W/m}^2\text{K} \quad \textit{Ans.}$$

$$h_c = h - h_r$$

$$= 17.95 - 11.1 = 6.85 \text{ W/m}^2\text{K} \quad \textit{Ans.}$$

## Summary

The physics of radiation which is quite different from that of conduction and convection is first explained. The laws of thermal radiation such as Planck's law, Wien's displacement law, Stefan-Boltzmann law, and Kirchhoff's law are introduced along with the basic definitions and concepts such as intensity of radiation, diffuse and specular surfaces, absorptivity, reflectivity, emissivity and transmissivity, black body and gray body. Subsequently, the concepts of view factor, the reciprocity theorem and the view factor algebra are explained along with electrical analogy. Radiation heat transfer between gray bodies, infinite parallel planes, radiosity and irradiation along with radiation network are explained. Hottel's cross-string method for estimating shape factor and the application of radiation shields are discussed. Radiation from gases and vapours, Hottel's curves for radiation and radiation heat transfer coefficient are introduced. Finally, the chapter concludes with some discussion on greenhouse effect and solar radiation.

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## Important Formulae and Equations

| *Equation Number* | *Equation* | *Remarks* |
|---|---|---|
| (7.2) | $E_b = \sigma T^4$ | Stefan–Boltzmann law-radiant energy emitted by a black body per unit area and per unit surface area is proportional to the fourth power of its absolute temperature |
| (7.3) | $E_{b\lambda} = \dfrac{dE_b}{d\lambda}$ (W/m$^3$) | Spectral or radiation intensity of a black body |
| (7.6) | $\varepsilon = \alpha$ | Kirchhoff's law–emissivity of a body is equal to its absorptivity at thermal equilibrium |
| (7.9) | $E_b = \int_0^\infty E_{b\lambda}\, d\lambda$ | Total rate of energy emission by a black body per unit area |
| (7.10) | $E_{b\lambda} = \dfrac{C_1\lambda^{-5}}{e^{C_2}/\lambda T - 1}$ | Planck's law stating $E_{b\lambda} = f(\lambda, \tau)$, where $C_1 = 3.74 \times 10^{-16}$ Wm$^2$, $C_2 = 1.438 \times 10^{-2}$ mK |
| (7.11) | $E_{b\lambda} = \dfrac{C_1 T}{C_2 x^4}$ | Rayleigh–Jeans law—valid for large wavelengths |
| (7.13) | $E_{b\lambda} = \dfrac{C_1}{\pi^5} e^{-C_2/\lambda T}$ | Wien's law—valid for short wavelengths |
| (7.14) | $\lambda_{\max} T = 2.898 \times 10^{-3}\,\text{mK}$ | Wien's displacement law |
| (7.15) | $(E_{b\lambda})_{\max} = C_3 T^5$ | Combining Planck's law and Wien's displacement law, where $C_3 = 1.287 \times 10^{-5}$ W/m$^3$K$^5$ |
| (7.19) | $f = \dfrac{\int_{\lambda_1}^{\lambda_2} E_{b\lambda}\, d\lambda}{\sigma T^4} = \dfrac{1}{\sigma T^4}\left[\int_0^{\lambda_2} E_{b\lambda}\, d\lambda - \int_0^{\lambda_1} E_{b\lambda}\, d\lambda\right]$ | Fraction of radiation at temperature $T$ for the wavelength range between $\lambda_1$ and $\lambda_2$ |
| (7.22) | $\sigma = \int_0^\infty \dfrac{E_{b\lambda}}{T^5}\, d(\lambda T)$ | Stefan–Boltzmann constant is equal to the total area under the curve $\dfrac{E_{b\lambda}}{T^5}$ vs $\lambda T$ at $T = 1$ K |
| (7.27) | $E_b = \pi I$ | Emissive power of a black body is $\pi$-times the radiation intensity |
| (7.28) | $I = \dfrac{\sigma T^4}{\pi}$ | Radiation intensity depends only on temperature |

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| Equation Number | Equation | Remarks |
|---|---|---|
| (7.31) | $A_1F_{12} = \dfrac{1}{\pi}\int_{A_1}\int_{A_2} \dfrac{\cos\phi_1 . \cos\phi_2}{r^2} \mathrm{d}A_1\mathrm{d}A_2$ | Radiation leaving $A_1$ and absorbed by $A_2$, both being black |
| (7.36) | $A_1F_{12} = A_2F_{21}$ | Reciprocity theorem |
| (7.45) | $(Q_1)_{\text{net}} = \sigma A_1\bar{F}_{12}(T_1^4 - T_2^4)$ where $A_1\bar{F}_{12} = A_1F_{12} + \dfrac{1}{\dfrac{1}{A_1F_{1R}} + \dfrac{1}{A_2F_{2R}}}$ | Radiant heat transfer between two black surfaces connected by non-conducting and reradiating walls |
| (7.47) | $\bar{F}_{12} = \dfrac{A_2 - A_1F_{12}^2}{A_1 + A_2 - 2A_1F_{12}}$ | View factor between two black surfaces connected by non-conducting and re-radiating walls |
| (7.52) | $\bar{\varepsilon} = \varepsilon_1\varepsilon_2 = f_{12}$ | Equivalent emissivity $\bar{\varepsilon}$ of two small gray bodies and the view factor |
| (7.55) | $f_{12} = \dfrac{1}{\dfrac{1}{\varepsilon_1} + \dfrac{1}{\varepsilon_2} - 1}$ | View factor for radiant exchange between two infinite parallel gray planes |
| (7.56) | $J = \varepsilon E_B + (1 - \varepsilon)G$ | Radiosity, the total radiant energy leaving a surface, is the sum of energy emitted and the energy reflected |
| (7.58) | $Q_{\text{net}} = \dfrac{E_b - J}{(1-\varepsilon)/A\varepsilon}$ | Net radiant energy leaving a surface per unit time per unit surface area. The denominator is called 'surface resistance' |
| (7.59) | $Q_{1-2} = \dfrac{J_1 - J_2}{1/A_1F_{12}}$ | Net radiant exchange between two bodies. The denominator is called 'space resistance' |
| (7.63) | $\mathcal{F}_{1-2} = \dfrac{1}{\left(\dfrac{1}{\varepsilon_1} - 1\right) + \dfrac{1}{F_{12}} + \dfrac{A_1}{A_2}\left(\dfrac{1}{\varepsilon_2} - 1\right)}$ | Radiant exchange between two gray surfaces: the view factor |
| (7.68) | $(Q_{12})_{\text{net}} = \dfrac{\sigma A_1(T_1^4 - T_2^4)}{\left(\dfrac{1}{\varepsilon_1} - 1\right) + \dfrac{1}{F_{12}} + \dfrac{A_1}{A_2}\left[\dfrac{1}{\varepsilon_2} - 1\right]}$ where $\bar{F}_{12} = \dfrac{A_2 - A_1F_{12}^2}{A_1 + A_2 - 2A_1F_{12}}$ | Radiant exchange between two gray surfaces connected by non-conducting and reradiating walls |

(*Contd*)

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| *Equation Number* | *Equation* | *Remarks* |
|---|---|---|
| (7.69) | $(Q_{12})_{\text{net}} = \dfrac{\sigma A_1(T_1^4 - T_2^4)}{\dfrac{1}{\varepsilon_1} + \dfrac{A_1}{A_2}\left[\dfrac{1}{\varepsilon_2} - 1\right]}$ | Radiant heat exchange between two concentric cylindrical gray surfaces |
| (7.72) | $\mathcal{F}_{12} = \dfrac{1}{\dfrac{1}{\varepsilon_1} + \dfrac{r_1^2}{r_2^2}\left[\dfrac{1}{\varepsilon_2} - 1\right]}$ | View factor for radiant heat exchange between two concentric gray spheres |
| (7.84) | $\left(\dfrac{Q}{A}\right)_{\text{with } N \text{ shields}} = \dfrac{1}{N+1}\left(\dfrac{Q}{A}\right)_{\text{without shields}}$ | Resistance to radiation heat transfer with the shields in place is $(N + 1)$ times as large as when the shields are absent |
| (7.88) | $Q = \sigma A_1 \varepsilon_1 T_1^4 \dfrac{1 - F_{11}}{1 - F_{11}(1 - \varepsilon_1)}$ | Radiant energy streaming out from a conical cavity |
| (7.93) | $I_{\lambda L} = I_{\lambda_0} e^{-k_\lambda L}$ | Beer's law stating radiation intensity $I_{\lambda L}$ decreases exponentially with the thickness of gas layer $L$ |
| (7.101) | $(Q_{G-W})_{\text{net}} = \sigma A(\varepsilon_G T_G^4 - \alpha'_G T_w^4)$ | Net radiant heat exchange between the gas and the black walls |
| (7.102) | $Q_{\text{total}} = (h_r + h_c) A_1 (T_1 - T_2)$ | Total heat transfer by convection and radiation |
| (7.103) | $h_r = \sigma F_{12}(T_1 + T_2)(T_1^2 + T_2^2)$ | Radiation heat transfer coefficient |
| (7.108) | $Q_{\text{net}} = \alpha_s G_{\text{solar}} + \varepsilon\sigma(T_{\text{sky}}^4 - T_s^4)$ | Net radiation heat transfer to a surface exposed to solar and atmospheric radiation |

## Review Questions

7.1 Explain how thermal radiation exhibits wave particle duality.

7.2 How are the wavelength and frequency of radiation propagating in a medium related?

7.3 What portion of electromagnetic spectrum is covered by thermal radiation?

7.4 What is the range of wavelength for visible radiation, i.e. light?

7.5 What do you mean by infrared and ultra-violet radiation?

7.6 Why are microwave ovens suitable for cooking?

7.7 What is the speed of energy propagation between two bodies when the intervening space is vacuum?

7.8 What do you understand by participating and non-participating mediums?

7.9 Explain Prevost's theory of heat exchange.

7.10 Explain what you mean by absorptivity, reflectivity and transmissivity.

7.11 What is an opaque body? How can its absorptivity be increased or decreased?

7.12 What is a black body? Give examples of some surfaces which do not appear black, but have high values of absorptivities.

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7.13 Define total emissive power and monochromatic emissive power of a body.

7.14 What is spectral or radiation intensity of a black body? How is it related to the total emissive power at a certain temperature?

7.15 Define emissivity of a surface. Explain spectral, directional, hemispherical and total emissivity.

7.16 What is a gray body? How does $\varepsilon_\lambda$ vary for a gray body and for a real surface?

7.17 Explain Kirchhoff's law. What do you mean by the statement: A perfect absorber of radiant energy is also a perfect emitter?

7.18 How does an enclosure with a small hole in it behave as a black body?

7.19 Why is Planck's law the basic law of thermal radiation? Explain graphically how $E_{b\lambda}$ and $T$ are related.

7.20 Show that Rayleigh–Jeans law and Wien's law can be derived from Planck's equation.

7.21 Derive Wien's displacement law

$$\lambda_{max} T = 2.898 \times 10^{-3} \text{ m K}$$

from Planck's equation.

7.22 Explain how the fraction of radiation at temperature $T$ for a range of wavelengths between $\lambda_1$ and $\lambda_2$ can be expressed as an area ratio.

7.23 How is Stefan–Boltzmann law derived from Planck's law of thermal radiation? What is the value of Stefan–Boltzmann constant?

7.24 Plot $E_{b\lambda}/T^5$ vs. $\lambda T$. What does the curve signify?

7.25 Explain how the hemispherical emissivity of a real surface varies with wavelength, temperature, degree of roughness and degree of oxidation.

7.26 Define intensity of radiation. What is a solid angle? What is its unit? What is a steragon?

7.27 Show that the emissive power of a black body is $\pi$-times the intensity of emitted radiation.

7.28 On what factors does the radiant heat exchange between two bodies depend?

7.29 What is shape factor? Show that

$$A_1 F_{12} = \frac{1}{\pi} \int_{A_1} \int_{A_2} \frac{\cos \phi_1 \cos \phi_2}{r^2} \, dA_1 \, dA_2$$

7.30 State and explain the reciprocity theorem.

7.31 Explain how the shape factor is determined by decomposing one or both the surfaces into subdivisions.

7.32 What is the shape factor with respect to itself if the surface is concave, convex or flat?

7.33 Show that the hemispherical black cavity with a flat cover over it emits 50% of radiation to the surface itself and is absorbed.

7.34 Explain the electrical analogy for radiative heat transfer in a black enclosure. Draw the equivalent electrical network for radiative flux between four walls of a black enclosure.

7.35 What do you mean by a nonconducting and reradiating wall? Why is it a no-net-flux surface? Give some examples.

7.36 Show that the shape factor for two surfaces 1 and 2 connected by a refractory surface is given by

$$\bar{F}_{12} = \frac{A_2 - A_1 F_{12}^2}{A_1 + A_2 - 2A_1 F_{12}}$$

What will be its value (a) if $A_1 = A_2$ and (b) if the surface 2 does not see the surface 1?

7.37 Explain how the shape factors for different surfaces evaluated?

7.38 Explain the radiant energy exchange between two small gray surfaces. Show that

$$\mathcal{F}_{12} = \varepsilon_1 \, \varepsilon_2$$

Why is the real value greater than this value?

7.39 For two infinite parallel gray planes exchanging radiant energy,

$$\mathcal{F}_{12} = \frac{1}{1/\varepsilon_1 + 1/\varepsilon_2 - 1}$$

7.40 Explain what you mean by radiosity and irradiation.

7.41 Explain the "surface resistance" and "space resistance". How can you construct a radiation network for two gray surfaces exchanging radiant energy?

7.42 What is a "floating node"? Where does this exist?

7.43 Show that radiative flux $(Q_{12})_{net}$ between two gray surfaces 1 and 2 connected by a

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nonconducting and reradiating surface is given by

$$(Q_{12})_{net} = \sigma A_1 \mathcal{F}_{12} (T_1^4 - T_2^4)$$

where $\mathcal{F}_{12} = \dfrac{1}{(1/\varepsilon_1 - 1) + 1/\bar{F}_{12} + A_1/A_2 (1/\varepsilon_2 - 1)}$

and $\bar{F}_{12} = \dfrac{A_2 - A_1 F_{12}^2}{A_1 + A_2 - 2A_1 F_{12}}$

7.44 What will be the value of $\mathcal{F}_{12}$ (a) when the surfaces are black, (b) when the surfaces are infinite parallel planes, (c) for two long concentric cylinders, (d) for two concentric spheres and (e) when a small body is enclosed by a large body?

7.45 Explain Hottel's crossed string method for estimating shape factor for infinitely long surfaces. Derive the expression for $F_{12}$ in terms of areas and lengths of surfaces.

7.46 What do you mean by a radiation shield? Where is it used?

7.47 By using one radiation shield between two surfaces and if all the three surfaces have the same emissivity, show that the net radiant heat transfer is reduced by 50%.

7.48 Show that

$$\left(\frac{Q}{A}\right)_{\text{with } N \text{ shields}} = \frac{1}{N+1}\left(\frac{Q}{A}\right)_{\text{without shields}}$$

7.49 Explain the radiation error in high temperature measurement.

7.50 What is thermocouple error? How is this error reduced when the thermocouple is shielded?

7.51 Show that the net radiative energy transfer from a conical cavity of diameter $D$, height $H$, lateral length $L$, semi-vertex angle $\alpha$ and surface area $A_1$ is given by

$$Q = A_1 \varepsilon_1 \sigma T_1^4 \frac{1 - F_{11}}{1 - F_{11}(1 - \varepsilon_1)}$$

If a flat surface of area $A_2$ covers the cavity, show that

$$F_{11} = 1 - \sin \alpha$$

7.52 Explain the differences between radiation from gases and that from solids. Why are gases called selective radiators?

7.53 What is Beer's law? How does the radiation intensity $I_{\lambda_L}$ decrease with the thickness of gas layer $L$?

7.54 What do you mean by monochromatic absorptivity and effective absorptivity?

7.55 What is beam length? How is it estimated?

7.56 Explain how Hottel'curves can help in estimating radiant heat exchange between a gas volume and an enclosure.

7.57 How will heat transfer be estimated when both convection and radiation are significant? How do you contribute to total energy transfer when the temperature is low and when the temperature is high?

7.58 Explain what do you understand by greenhouse effect and global warming.

7.59 What is solar insolation? Give the value of solar constant. What is the effective surface temperature of the sun regarded as a black body?

7.60 How is solar radiation attenuated as it passes through the atmosphere? Why is the light reaching the earth of longer wavelengths like red, orange and yellow?

7.61 What is sky temperature? What is its approximate value?

7.62 Give the net rate of radiation heat transfer to a surface exposed to solar and atmospheric radiation.

## Objective Type Questions

7.1 A surface that reflects all the incident radiation appears
(a) yellow (b) white
(c) black (d) red

7.2 Radiation emitted from heated bodies comprises rays with wavelengths
(a) 0.4 – 800 μm (b) 0.4 – 0.8 μm
(c) 0.76 – 100 μm (d) 0.01 – 0.4 μm

7.3 The large part of thermal radiation between 0.8 μm to 800 μm is called the
(a) ultraviolet radiation
(b) visible light

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(c) infrared radiation
(d) microwave radiation

7.4 Most solids are
(a) highly absorptive
(b) highly transmittive
(c) opaque
(d) highly reflective

7.5 The absorptivity of a solid can be increased by
(a) polishing the surfaces
(b) roughening the surfaces
(c) whitening the surfaces
(d) none of the above

7.6 Gases have poor
(a) tranmissitivity (b) absorptivity
(c) reflectivity (d) emissivity

7.7 Ice is very close to a
(a) gray body (b) black body
(c) white body (d) specular body

7.8 A hollow sphere with uniform temperature and a small hole in it behaves very nearly as a
(a) black body (b) opeque body
(c) white body (d) gray body

7.9 A gray body is the one whose absorptivity
(a) varies with temperature
(b) varies with the wavelength of incident ray
(c) varies with temperature and wavelength of incident ray
(d) does not vary with temperature and wavelength of incident ray

7.10 A body which partly absorbs and partly reflects, but does not allow any radiation to pass through it is called
(a) specular (b) gray
(c) opaque (d) white

7.11 The emissivity of a real surface varies with
(a) the temperature
(b) the wavelength
(c) the direction of emitted radiation
(d) all of the above

7.12 When a body is in thermal equilibrium with its surroundings, the emissivity of its surface is equal to its absorptivity. This is called
(a) Kirchhoff's law (b) Wien's law
(c) Planck's law (d) Lambert's law

7.13 The radiant energy emitted by a body per unit time and per unit surface area at a particular wavelength and temperature is called
(a) intensity of radiation
(b) monochromatic emissive power
(c) total emissive power
(d) none of the above

7.14 The value of the maximum monochromatic emissive power of a black body, $(E_{b\lambda})_{max}$ shifts with increasing temperature towards
(a) the shorter wavelengths
(b) the longer wavelengths
(c) the same wavelength
(d) none of the above

7.15 The maximum monochromatic emissive power of a black body is proportional to
(a) $T^3$ (b) $T^4$
(c) $T^5$ (d) $T^6$

7.16 The area under the curve, $E_{b\lambda}/T^5$ vs $\lambda T$, at $T = 1$ K is equal to
(a) Planck's constant
(b) Stefan–Boltzmann constant
(c) Boltzmann constant
(d) Radiation constant

7.17 The intensity of radiation is obtained by multiplying the emissive power by a factor
(a) $\pi$ (b) $1/\pi$
(c) $\frac{1}{\sqrt{2}}\pi$ (d) $\frac{\sqrt{2}}{\pi}$

7.18 The reciprocity theorem states that
(a) $A_1F_{12} = A_2F_{21}$ (b) $A_2F_{12} = A_1F_{21}$
(c) $F_{12} = F_{21}$ (d) $\alpha_1F_{12} = \alpha_2F_{21}$

7.19 A surface has a shape factor with respect to itself if it is
(a) flat (b) convex
(c) concave (d) cylindrical

7.20 In two concentric black cylinders, if 1 represents the outer surface of the inner cylinder of diameter $d_1$ and 2 represents the inner surface of the outer cylinder of diameter $d_2$, then $F_{22}$ is given by
(a) $\frac{d_1}{d_2}$ (b) $\frac{d_1^2}{d_2^2}$
(c) $1-\frac{d_2}{d_1}$ (d) $1-\frac{d_1}{d_2}$

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7.21 For two black parallel planes of equal areas connected by reradiating walls at a constant temperature the view factor. $\bar{F}_{12}$ is given by

(a) $1 + F_{12}$ (b) $\dfrac{1+F_{12}}{2}$

(c) $\dfrac{F_{12}}{2}$ (d) $\dfrac{1-F_{12}}{2}$

7.22 For two infinite parallel planes with emissivities $\varepsilon_1$ and $\varepsilon_2$. The interchange view factor from surface 1 to surface 2 is given by

(a) $\dfrac{1}{\varepsilon_1+\varepsilon_2}$ (b) $\varepsilon_1+\varepsilon_2$

(c) $\varepsilon_1-\varepsilon_2$ (d) $\dfrac{\varepsilon_1\varepsilon_2}{\varepsilon_1+\varepsilon_2-\varepsilon_1\varepsilon_2}$

7.23 Match List I and List II and select the correct answers using the codes given below:

| **List I** | **List II** |
|---|---|
| A. Infinite parallel planes | 1. $\varepsilon_1$ |
| B. A small body completely enclosed in a large enclosure | 2. $\varepsilon_1\varepsilon_2$ |
| C. Two rectangles with a common edge | 3. $\dfrac{1}{\dfrac{1}{\varepsilon_1}+\dfrac{1}{\varepsilon_2}-1}$ |
| D. Concentric cylinders | 4. $\dfrac{1}{\dfrac{1}{\varepsilon_1}+\dfrac{A_1}{A_2}\left(\dfrac{1}{\varepsilon_2}-1\right)}$ |

**Codes:**

| | A | B | C | D |
|---|---|---|---|---|
| (a) | 1 | 2 | 4 | 3 |
| (b) | 3 | 1 | 4 | 2 |
| (c) | 2 | 1 | 3 | 4 |
| (d) | 3 | 1 | 2 | 4 |

7.24 Match List I with List II using the codes given below selecting the correct answers:

| **List I** | **List II** |
|---|---|
| A. Stefan–Boltzmann law | 1. $E_{b\lambda} = \dfrac{C_1\lambda^{-5}}{e^{C_2/\lambda T}-1}$ |
| B. Kirchhoff's law | 2. $E_{b\lambda} = C_1T/C_2\lambda^4$ |
| C. Planck's law | 3. $\varepsilon = \alpha$ |
| D. Rayleigh–Jeans law | 4. $Q = \mathcal{F}A\sigma(T_1^4 - T_2^4)$ |

**Codes:**

| | A | B | C | D |
|---|---|---|---|---|
| (a) | 4 | 1 | 3 | 2 |
| (b) | 4 | 3 | 1 | 2 |
| (c) | 2 | 1 | 3 | 4 |
| (d) | 2 | 3 | 1 | 4 |

7.25 The sum of the energy emitted and the energy reflected from a solid is called

(a) total emissive power
(b) irradiation
(c) radiosity
(d) total radiant energy

7.26 A radiation shield of emissivity $\varepsilon$ on both sides is placed between two infinite parallel planes of emissivity $\varepsilon$ and temperatures $T_1$ and $T_2$. The ratio of radiant energy flux with shield and without shield would be

(a) 0.25 (b) 0.5
(c) 0.75 (d) 1.0

7.27 It is desired to reduce the radiant energy exchange between two infinite parallel planes by inserting radiation shields of the same emissivity. The number of shields required for 75% reduction would be

(a) two (b) three
(c) four (d) five

7.28 The mean beam length of gas radiation is defined as

(a) Volume $V$/Surface area $A$
(b) Volume $V$/3.6 (Surface area $A$)
(c) 3.4 $V/A$
(d) 4 $V/A$

7.29 **Assertion (A):** Gases like $CO_2$ and $H_2O$ are called selective radiators.

**Reasoning (R):** Because these gases emit and absorb radiation only between narrow ranges of wavelengths called bands.

**Codes:**

(a) Both A and R are false
(b) Both A and R are true and R is the correct reason for A
(c) A is true, R is false

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(d) A is false, R is true

7.30 When monochromatic radiation of intensity $I_{\lambda L}$ passes through a gas layer of thickness $L$, the radiant energy absorption $I_{\lambda L}$ decreases exponentially with the thickness of gas layer $L$. This is known as
(a) Beer's law
(b) Kirchhoff's law
(c) Wien's law
(d) Hottel's law

7.31 How can radiation heat transfer between two surfaces be reduced?
(a) By bringing the surfaces closer
(b) By introducing radiation shield between them
(c) By polishing the surfaces
(d) By roughening the surfaces

7.32 The absorptivity of a gas volume depends upon
(a) pressure only
(b) temperature only
(c) shape and size
(d) all of the above

7.33 The ratio of radiation heat transfer coefficient at room temperature to the surface emissivity is about
(a) 3 (b) 4
(c) 6 (d) 8

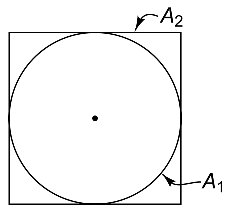
7.34 The intensity of solar radiation in kW/m$^2$ is about
(a) 1 (b) 3
(c) 6 (d) 8

7.35 For effective greenhouse effect, glass should be
(a) transparent to long wavelength radiation
(b) opaque to short wavelength radiation
(c) transparent to short wavelength radiation
(d) transparent to all wavelength

7.36 A radiation shield is used around thermocouples in order to measure more accurately the temperature of
(a) solids
(b) gases
(c) boiling liquids
(d) condensing vapour

7.37 A body at 500K cools down to 400K by radiation to atmosphere at 300K. What will be the ratio of heat loss rate at 500K to the heat loss rate at 400K?
(a) 3.11 (b) 4.02
(c) 5.1 (d) 2.78

7.38 There is a sphere within a cube as shown in figure below. If $F_{11} = 0.1$, $F_{12} = 0.4$, $F_{13} = 0.2$, $A_1 = 1\text{m}^2$, $A_4 = 3\ \text{m}^2$. What will be the value of $F_{41}$?




(a) 0.2 (b) 0.3
(c) 0.4 (d) 0.14

## Answers

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 7.1 | (b) | 7.2 | (a) | 7.3 | (c) | 7.4 | (c) | 7.5 | (b) |
| 7.6 | (c) | 7.7 | (b) | 7.8 | (a) | 7.9 | (d) | 7.10 | (c) |
| 7.11 | (d) | 7.12 | (a) | 7.13 | (b) | 7.14 | (a) | 7.15 | (c) |
| 7.16 | (b) | 7.17 | (b) | 7.18 | (a) | 7.19 | (c) | 7.20 | (d) |
| 7.21 | (b) | 7.22 | (d) | 7.23 | (d) | 7.24 | (b) | 7.25 | (c) |
| 7.26 | (b) | 7.27 | (b) | 7.28 | (c) | 7.29 | (b) | 7.30 | (a) |
| 7.31 | (b) | 7.32 | (d) | 7.33 | (c) | 7.34 | (a) | 7.35 | (c) |
| 7.36 | (b) | 7.37 | (a) | 7.38 | (a) | | | | |

$$\left[\frac{Q_{500}}{Q_{44}} = \frac{500^4 - 300^4}{500^4 - 400^4} = 3.11\right]$$

7.38 (c)

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## Open Book Problems

7.1 An optical pyrometer is an instrument to measure high temperatures. It works on the principle of comparing the radiation emitted by the body whose temperature is to be measured with that of an incandescent filament. The instrument is calibrated in such a way that it measures the temperatures which a black body would have if it were emitting radiation equal to that of the body under investigation.

A pyrometer records the temperature of a body as 1600 K with a red light filter ($\lambda$ = 0.65 μm). Find the time temperature of the body if its emissivity $e_\lambda$ at 0.65 μm is 0.7.

*Hints:* If $T$ is the temperature of the body at wavelength $\lambda$ and emissivity $\varepsilon_\lambda$, we have $E_{\theta\lambda} = E_x$. Neglecting unity in the denominator of Planck's equation (7.10).

$$c_1\lambda^{-5}(e^{-c_2/\lambda T_b} - 1) = \varepsilon_\lambda c_1 \lambda^{-5}(e^{-c_2/\lambda T} - 1)$$

Taking logarithm of both sides

$$\exp(-c_2/\lambda T_b) = \varepsilon_\lambda \exp(-c_2/\lambda T)$$

$$\frac{1}{T} = \frac{1}{T_b} + \frac{\lambda}{c_2}\ln \varepsilon_\lambda = \frac{1}{1600} + 0.65\times10^{-6}\times\frac{1}{1.438\times10^{-2}}\ln 0.7$$

7.2 Estimate the surface temperature of the sun and its emissive power, assuming it to be a black body and emits maximum radiation at $\lambda$ = 0.5 μm. Also, calculate (i) the energy received by the surface of the earth, and (ii) the energy received by a 2 m × 2 m solar collector whose normal is inclined at 60º to the sun. Take the diameter of the sun as 1.4 × $10^9$ m, the diameter of the earth as 1.3 × $10^6$ m and the distance of the earth from the sun as 15 × $10^{10}$ m.

*Hints:* From Wien's displacement law, Eq. (7.14), $\lambda_{max}\ T = 2898 \times 10^{-3}$ mK, find $T$ of the sun. From Stefan-Boltzmann law, Eq. (7.2) find $E_b$. Radiation reaching the earth's atmosphere,

$$Q = E_b \times \frac{\text{Radius of the sun}}{\text{Distance of sun from the earth}}$$

Energy received by the solar collector = $qA \cos\theta$.

7.3 Two parallel black plates 0.5 × 1.0 m are spaced 0.5 m apart. One plate is maintained at 1000ºC and the other at 500ºC. What is the net radiant heat exchange between the two plates?

*Hints:* From Fig. 7.28, for $L_1/D$ and $L_2/D$, find $F_{12}$. The estimate $Q = \rho A_1 F_{12}\,(T_1^4 - T_2^4)$.

7.4 Two black square plates of size 1 m by 1 m are placed parallel to each other at a distance of 0.4 m. One plate is maintained at 1000ºC and the other at 500ºC. Find the net radiant heat exchange between the two plates.

*Hints:* Use Fig. 7.29 to find $F_{12}$ and then $Q_{12}$.

7.5 A pipe carrying steam has an OD of 20 cm and run in a large room. It is exposed to air at a temperature of 30ºC. Calculate the loss of heat to surroundings per metre length of pipe due to thermal radiation. The emissivity of the pipe surface is 0.8. Find the reduction in that loss if the pipe is enclosed in a 40 cm brick conduct of emissivity 0.9.

*Hints:* The pipe is a small gray body in a large gray surroundings. $Q_{12} = \sigma A_1 \varepsilon\,(T_1^4 - T_2^4)$. When the pipe is enclosed in the brick conduit,

from Eq. (7.70), $\mathcal{F}_{12} = \dfrac{1}{\dfrac{1}{\varepsilon_1} + \dfrac{A_1}{A_2}\left(\dfrac{1}{\varepsilon_2} - 1\right)}$.

Then estimate $Q_{12}^1 = \sigma A_1 \mathcal{F}_{12}(T_1^4 - T_2^4)$ and $Q_{12} - Q_{12}^1$.

7.6 A thermos flask consists of two thin walled coaxial cylinders with a narrow gap such that the radius ratio $R_1/R_2$ = 0.95. The surfaces that face each other are silvered and have emissivity of 0.05 each. The annulus is evacuated. The length of flask is $L$ = 0.3 m and the radius $R_1$ = 0.05 m. Calculate the heat loss when the thermos flask is filled with hot water such that $T_1$ = 373 K while the outer cylinder at a temperature of $T_2$ = 300 K. Also calculate the instantaneous rate of change of temperature of hot water.

*Hints:* Eq. (7.69) is used to find

$$\mathcal{F}_{12} = \frac{1}{\dfrac{1}{\varepsilon_1} + \dfrac{A_1}{A_2}\left(\dfrac{1}{\varepsilon_2} - 1\right)} \text{ and then calculate}$$

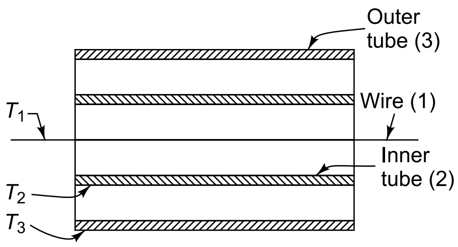
$$Q_{12} = \sigma A_1\, \mathcal{F}_{12}\, (T_1^4 - T_2^4).$$

Initial cooling rate of hot water at 100ºC is obtained by finding the properties of water at 373 K like $\rho$ and $c_p$. Then mass of water in the flask $m = \rho\pi R_1^2 L$ kg and the rate of cooling is

$$\left(\frac{dT}{dt}\right)_{t=0} = \frac{Q \quad W}{m \text{ kg} \times c_p \text{ J/kgK}} \,{}^\circ\text{C/s}$$

7.7 An electric wire 0.25 mm diameter $\varepsilon_1 = 0.4$ is placed within a tube of 2.5 mm diameter, $\varepsilon = 0.6$ having negligible thickness. This tube is placed concentrically within a tube of 5 mm diameter, $\varepsilon = 0.7$. Annular spaces can be assumed to be evacuated completely. If the surface temperature of the outer tube is maintained at 5ºC, what must be the temperature of the wire so as to maintain the inner tube at 120ºC?

*Hints:*




Heat lost by the wire to the inner tube = Heat lost by the inner to the outer tube.

$$\frac{\sigma A_1 (T_1^4 - T_2^4)}{\dfrac{1}{\varepsilon_1} + \dfrac{A_1}{A_2}\left(\dfrac{1}{\varepsilon_2} - 1\right)} = \frac{\sigma A_2 (T_2^4 - T_3^4)}{\dfrac{1}{\varepsilon_2} + \dfrac{A_2}{A_3}\left(\dfrac{1}{\varepsilon_3} - 1\right)}$$

$$\frac{A_1}{A_2} = \frac{d_1}{d_2}, \frac{A_2}{A_3} = \frac{d_2}{d_3}$$

Substituting the given values, find $T_1$.

7.8 For a hemispherical furnace, the flat floor is at 700 K and has an emissivity of 0.5. The hemispherical roof is at 1000 K and has an emissivity of 0.25. Find the net radiative heat transfer from the roof to the floor.

*Hints:* $$Q_{12} = \frac{A_1\sigma(T_1^4 - T_2^4)}{\dfrac{1-\varepsilon_1}{\varepsilon_1} + \dfrac{1}{F_{12}} + \dfrac{A_1}{A_2}\left(\dfrac{1-\varepsilon_2}{\varepsilon_2}\right)}$$

where $A_1 = \pi r^2$, $A_2 = 2\pi r^2$ and $F_{12} = 1$

7.9 Determine the radiant heat exchange in W/m$^2$ between two large parallel steel plates of emissivities 0.8 and 0.5 held at temperatures of 1000 K and 500 K respectively, if a thin copper plate of emissivity 0.1 is placed as a radiation shield between the two plates.

*Hints:* $$Q_{12} = \frac{\sigma A(T_1^4 - T_2^4)}{\left(\dfrac{1}{\varepsilon_1} + \dfrac{1}{\varepsilon_2} - 1\right) + \left(\dfrac{1}{\varepsilon_3} + \dfrac{1}{\varepsilon_2} - 1\right)}$$

where $A = 1$ m$^2$.

7.10 Consider two large parallel plates, one at $T_1$ = 727ºC with emissivity $\varepsilon_1 = 0.8$ and the other at $T_2$ = 227ºC and $\varepsilon_2 = 0.4$. An aluminium radiation shield ($\varepsilon_3 = 0.05$ on both sides) is placed between the plates. Calculate the percentage reduction in heat transfer rate between the plates as a result of the shield.

*Hints:* Without shield, $$\frac{Q_{12}}{A}\,\frac{\sigma(T_1^4 - T_2^4)}{\dfrac{1}{\varepsilon_1} + \dfrac{1}{\varepsilon_2} - 1}$$

With shield $\dfrac{Q_{1-3}}{A} = \dfrac{Q_{3-2}}{A}$,

or $$\frac{\sigma(T_1^4 - T_3^4)}{\dfrac{1}{\varepsilon_1} + \dfrac{1}{\varepsilon_2} - 1} = \frac{\sigma(T_3^4 - T_2^4)}{\dfrac{1}{\varepsilon_3} + \dfrac{1}{\varepsilon_2} - 1}$$

Find $$T_3 \cdot \frac{Q_{13}}{A} = \frac{\sigma(T_1^4 - T_3^4)}{\dfrac{1}{\varepsilon_1} + \dfrac{1}{\varepsilon_3} - 1}$$

$$\% \text{ reduction} = \frac{Q_{12} - Q_{13}}{Q_{12}} \times 100$$

7.11 A spherical ball 6 cm in diameter and at 300 K is placed inside a large spherical furnace at

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600 K. Estimate the diameter of the spherical furnace such that 1/5th of the energy emitted by the furnace reaches the spherical ball and the rate of radiant heat exchange.

*Hints:* $A_1F_{12} = 0.2\ A_2F_{21}$ where $F_{21} = 0.2$, $F_{12} = 1$,

$A_2 = \dfrac{A_1}{F_{21}} = \dfrac{4\pi r_1^2}{0.2}$ and $A_2 = 4\pi r_2^2$. Find $r_2$ and then $Q = \sigma A_1 F_{12} (T_1^4 - T_2^4)$.

7.12 A pipe carrying steam having an OD of 200 mm runs in a large room, and is exposed to air at a temperature of 30ºC. The pipe surface temperature is 200ºC. Find the heat loss per metre length of the pipe by convection and radiation taking the emissivity of the pipe surface as 0.8.

*Hints:* Heat loss by radiation, $Q_r = \sigma A_1 \varepsilon (T_1^4 - T_2^4)$ W/m.

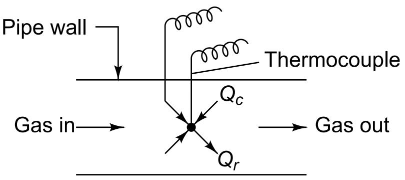
Find properties of air at $\dfrac{200+30}{2}$ or 115ºC from the Appendix. Find $Gr_d.Pr$ and check if it is less than $10^9$. Use $Nu_d = 0.53(Gr_dPr)^{1/4}$ to find $h_c$.

Heat loss by covection, $Q_c = h_cA\ (T_w - T_\infty)$.

Then, $Q_{tot} = Q_r + Q_c$.

7.13 A thermocouple indicates a temperature of 800ºC when placed in a pipeline where a hot gas is flowing at 870ºC. If the convective heat transfer coefficient between the thermocouple and gas is 60 W/m$^2$K, find the duct wall temperature. Take $\varepsilon$ for thermocouple as 0.5.

*Hints:*




There is convective heat flow from gas to thermocouple junction, but a part of this is lost by radiation to the pipe wall.

Convective heat flow from gas to themocouple

$$Q_c = h_cA\ (T_{gas} - T_{couple})$$

Heat radiated by thermocouple to pipe wall.

$$Q_r = F_gA\sigma\ (T_{couple}^4 - T_{wall}^4)$$

where $F_g = \dfrac{1}{\dfrac{1-\varepsilon_1}{\varepsilon_1} + \dfrac{1}{F_{12}} + \left(\dfrac{1-\varepsilon_2}{\varepsilon_2}\right)\dfrac{A_1}{A_2}}$,

Since $F_{12} = 1$ and $A_1 << A$, $F_g = \varepsilon_1$

Under steady state, $Q_c = Q_r$

$60\ A(T_{gas} - T_{couple}) = 0.5\ A\sigma\ (T_{gas}^4 - T_{wall}^4)$

where $T_{gas} = 870 + 273 = 1143$ K,

$T_{couple} = 1073$ K.

Find, $T_{wall}$.

## Problems for Practice

7.1 An annealing furnace is depicted in the Fig. P. 7.1 The fire in the firebox acts as a gray plane, $\varepsilon = 0.7$ at 1500°C, whereas the steel on the hearth acts as a gray plane, $\varepsilon = 0.8$ at 800°C. The firebox and hearth are separated by a bridgewall, and all other surfaces act as nonconducting and reradiating walls. Find the rate of heat transfer between the fire and steel.

(*Hints:* $F_{12} = 0$, since bodies 1 and 2 do not see each other $\mathcal{F}_{12} = 1\Big/\left(\dfrac{1}{\varepsilon_1} + \dfrac{1}{\varepsilon_2}\dfrac{A_1}{A_2}\right)$ $= 0.488$, $Q = 4160$ kW)

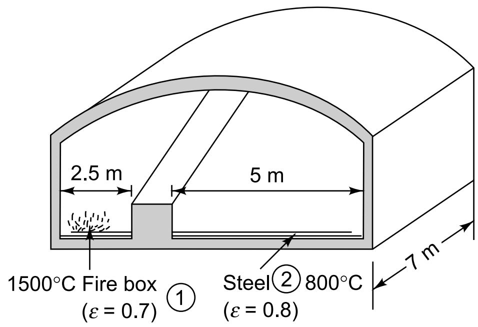



**Fig.** P. 7.1

7.2 Two parallel gray planes have emissivities of 0.8 and 0.7 and are maintained at 800°C

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and 1500°C. What is the net radiant energy exchange? What would be the reduction in heat transfer if a radiation shield of polished aluminium ($\varepsilon$ = 0.04) is placed between them? (*Ans.* 289.0 kW/m$^2$, 96.7%)

7.3 A furnace consists essentially of a long refractory conduit having rectangular cross-section 0.305 m by 0.203 m. The furnace encloses a heat resisting steel pipe of 7.62 cm outside diameter. The furnace wall temperature is maintained at 872°C and the pipe surface is at 371°C. Assuming both surfaces to be gray, calculate the net rate of heat transfer by radiation. Emissivity of furnace wall is 0.8 and that of steel is 0.4.
(*Ans.* $\mathcal{F}_{12}$ = 0.391, $Q_{12}$ = 8.2 kW/m$^2$)

7.4 Determine the heat lost by radiation per metre length of a 75 mm oxidised steel pipe at 327°C if (a) located in a large room with red brick walls at a temperature of 27°C, (b) enclosed in a 150 mm × 150 mm red brick walls at a temperature of 27°C. Emissivities of oxidised steel and red brick are 0.79 and 0.93 respectively.

7.5 The inner sphere of a Dewar flask is of 300 mm diameter and outer sphere is of 360 mm diameter. Both spheres are plated for which $\varepsilon$ = 0.5. The space between them is evacuated. Determine the rate at which liquid oxygen would evaporate at –183°C when the outer sphere temperature is 20°C. The latent heat of vaporisation of liquid oxygen is 14.2 kJ/kg.

7.6 A rectangular enclosure has sides 1 m × 2 m × 4 m. The 1 × 2 m faces are black and at temperatures of 200°C and 100°C respectively. Calculate the net radiative energy exchange between the two surfaces, if the other four faces of the enclosure are reradiating surfaces.
using Eq. (7.66), $Q_{12}$ = 1.8 kW
using Fig. 7.21, $\bar{F}_{12}$ = 0.2, $Q_{12}$ = 418 W

7.7 A hot air duct having an outside diameter of 250 mm and a surface temperature of 95°C is located in a large room whose walls are at 21°C. The air in the room is at 28°C and the heat transfer coefficient for free convection between the duct and the air is 5.82 W/m$^2$ K. Estimate the rate of heat transfer per metre of duct if (a) the duct is bare tin ($\varepsilon$ = 0.1) and (b) the duct is painted with white lacquer ($\varepsilon$ = 0.9). (*Ans.* 354.47 W)

7.8 A small electrically heated element of surface area 3500 mm$^2$ has its surface temperature maintained at 90°C when the voltage across the element is 15 V and the current through the element is 2 A. If the emissivity of the element is 0.8 and the surrounding air temperature is 30°C, determine the heat exchange by radiation and natural convection. What is the convective heat transfer coefficient?
(*Ans.* 136 W/m$^2$ K)

7.9 An exhaust duct 1.7 m in diameter conveys gas at 1000°C and 1 atm pressure with the composition 15% $CO_2$, 10% $H_2O$ and 75% $N_2$ by volume. If the duct walls have an emissivity of 0.8 and operate at a temperature of 250°C, estimate the net radiant heat transfer per unit area.

7.10 Show that $\bar{F}_{12}$ for two black parallel planes of equal areas connected by reradiating walls at a constant temperature is

$$\bar{F}_{12} = \frac{1 + F_{12}}{2}$$

where $F_{12}$ is the shape factor of the black bodies.

7.11 Three thin hollow cylinders of 100, 200, and 300 mm diameters are arranged concentrically. If the inner (100 mm diameter) and the outer (300 mm diameter) cylinders are maintained at 727°C and 27°C respectively, and assuming vacuum in the annular spaces, estimate the steady-state temperature of the middle cylinder (200 mm diameter). Take $\varepsilon_1 = \varepsilon_2 = \varepsilon_3$ = 0.05. Determine the rate of heat loss per metre length of the composite cylinder and the convective heat transfer coefficient on the outer surface, if the surrounding air temperature is 10°C.
(*Ans.* 775 K, 0.39 kW/m, 178 W/m$^2$ K)

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7.12 Two large parallel planes having emissivities of 0.3 and 0.5 are maintained at temperatures of 800°C and 300°C respectively. Determine the net radiant heat exchange per unit area between the planes. If a radiation shield having an emissivity of 0.05 on both sides is placed between the two planes, calculate the temperature of the shield and the heat transfer rate per unit area.

(*Ans.* 960 K, 16 kW/$m^2$, 1.64 kW/$m^2$)

7.13 A long cylindrical heater 25 mm in diameter is maintained at 627°C and has a surface emissivity of 0.8. The heater is located in a large room whose walls are at 27°C. How much will the radiant heat transfer from the heater be reduced if it is surrounded by a 0.2 m diameter radiation shield having an emissivity of 0.2. What is the temperature of the shield?

7.14 Calculate the rate at which radiant energy streams out from a conical cavity having a semi-vertex angle of 45° and height 300 mm, if the surface temperature of the cavity is 600 K and the emissivity is 0.8.

(*Ans.* 1.77 kW)

7.15 Two parallel plates measuring 1 m × 1 m are spaced 2 m apart. The inner surface of each plate radiates as a black body and the outer surfaces are perfectly insulated. A radiation shield with an emissivity of 0.05 on each side and measuring 1 m × 1 m is situated equidistant between the plates. The temperature of one plate is maintained at 727°C, the other plate at 227°C and the surroundings are at 27°C. Sketch the radiation network and calculate the radiative heat transfer from the hot plate. The configuration factor for two square parallel planes at distance apart equal to one of the sides is 0.2. (*Ans.* 54 kW)

7.16 Two parallel plates measuring 2 m × 1 m are spaced 1 m apart facing each other. The outer surfaces of the plates are perfectly insulated. The inner surface of one plate has emissivity of 0.8 and temperature of 500°C and that of the other plate has emissivity of 0.5 and temperature of 300°C. Find the net heat transfer by radiation between the two plates.

A third plate of the same dimensions having an emissivity of 0.2 on each side is placed equidistant between the two plates. Determine the equilibrium temperature of the third plate and radiative heat transfer from the hot plate.

(*Ans.* 6.93 kW, 335°C, 4.16 kW)

7.17 A room measuring 3 m × 4 m × 2 m high has the ceiling covered with heating panels. Under steady-state conditions, the ceiling is at a temperature of 50°C and the walls and floor at a temperature of 20°C. Assuming all the surfaces have an absorptivity of unity, calculate the net radiant heat transfer from the ceiling. (*Ans.* 2.37 kW)

7.18 A room has a radiating wall panel fitted along the entire length of one wall. The panel extends from the ground to a height of 1 m and has a surface temperature of 60°C and an emissivity of 0.9. The room measures $4 \text{ m} \times 4 \text{ m} \times 2\frac{1}{2}$ m high, and the walls are effectively black with a temperature of 15°C. Calculate the heat radiated to the floor and the ceiling.

(*Ans.* 0.41 kW, 0.17 kW)

7.19 A spherical satellite of 1 m diameter encircles the earth at an altitude of 483 km. Estimate the shape factor of the earth from the satellite and hence calculate the equilibrium temperature of the satellite on the "dark" side and on the "bright" side of the earth. Assume that the diameter of the earth is 12,880 km and its black body temperature is 20°C. The temperature of outer space may be taken as 0 K and the satellite is irradiated with a heat flux of 1.3 kW/$m^2$ from the sun when on the bright side.

(*Ans.* 147 K, 297 K)

7.20 An electric arc considered as a black sphere of 5 mm diameter at a temperature of 5000°C is situated 250 mm away from a metal disc of 100 mm diameter. The disc

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is oriented at right angle to the normal to the arc and is open to radiation on both sides. Assuming that the heat exchange with the disc is entirely by radiation, estimate the temperature of the disc when the room temperature is 20°C.

(*Ans.* 147°C)

7.21 A room measuring 5 m × 5 m × 2 m high has the celling covered with electric heating panels. If the ceiling surface temperature is maintained at 45°C and the walls are at 25°C in equilibrium condition, estimate the total heat loss from the ceiling by radiation. The emissivities of the ceiling and the walls are 0.75 and 0.65 respectively. Assume the floor to be non-sensitive to radiation.

($\mathcal{F}_{12}$ = 0.1435, $Q$ = 19.5 kW)

7.22 A rectangular furnace wall has the dimensions of 1 m × 1 m × 2 m high. The 1 m × 1 m surfaces are black and their temperature are 600 K and 400 K respectively. The remaining four surfaces are refractory (nonconducting and reradiating) no-flux surfaces. Calculate the net radiative exchange between the two black surfaces.

(*Ans.* 3.17 kW)

7.23 In a vertical cylindrical furnace 0.6 m in diameter and 1 m high, the upper surface ($\varepsilon$ = 0.8) is maintained at 1000 K and the lower surface ($\varepsilon$ = 0.7) is maintained at 700 K. Assuming the cylindrical wall as a refractory no-flux surface, estimate the net radiative energy transfer between the upper and lower surfaces.

7.24 A tungsten filament of a lamp is at 2500°C. What percent of the total radiant energy is in the visible range?

7.25 A package of electronic equipment is enclosed in a sheet metal cubical box of side 0.3 m. If the equipment used 1500 W of electrical power, what is the average temperature of the container walls if the wall temperature is assumed uniform all around the package? The emissivity of the walls is 0.80 and the room air temperature is 20°C.

7.26 A radiant heating system is installed in the plastered ceiling of a room 5 m × 5 m × 3 m high. The temperature of the concrete floor is maintained at 27°C. Assume that no heat flows through the walls, which are coated with reradiating material. If the required heat supply to the room is 1.2 kW, determine the necessary temperature of the ceiling surface.

7.27 A thermocouple, used to measure the temperature of a gas stream in a duct, records a value of 300°C. If the duct walls are at a temperature of 100°C, calculate the true gas temperature. Assume that the thermocouple has an emissivity of 0.85 and that the heat transfer coefficient of the gas flowing past it is 75 W/m$^2$ K.

7.28 A small disc shaped earth satellite of 1 m diameter is circling the earth (radius 6250 km) at a distance of 300 km from the terrestrial surface. The flat surface of the disc is oriented tangential to the earth's surface. The satellite surface is oxidised aluminium ($\varepsilon$ = 0.3) and is at –18°C. Assuming that (i) the earth is black having an average surface temperature of 27°C, (ii) the satellite is in the shadow of the earth and (iii) the temperature of the outer space is at 0 K, calculate the net rate of energy emission from the satellite.

(*Ans.* 14.53 W)

7.29 An internal combustion engine burns fuel octane ($C_8H_{18}$) with 150% stoichiometric air. The combustion products at a temperature of 1000°C and a pressure of 1.2 atm pass through an exhaust pipe, the walls of which are at 240°C. Approximating the exhaust pipe as an infinite black cylinder 75 mm in diameter, estimate the net radiant exchange between the hot combustion gases and the pipe wall.

7.30 Two parallel discs both 2 m in diameter and 2 m apart are maintained uniformly at 560°C and 280°C respectively. Determine the net rate of heat loss from the hotter disc if (a) the surroundings are black at 0 K and (b) a reradiating surface extends between the discs.

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7.31 Determine the view factor between a 1 $m^2$ skylight, oriented as shown, above a floor which measures 20 m × 28 m.

(*Ans.* 0.608)

7.32 If a 25 mm hole is drilled completely through a 50 mm thick metal plate which is maintained at a uniform temperature of 150°C, what is the rate of energy loss from the drilled surface when the surroundings are at 15°C? Assume both the metallic surface and the surroundings to be black.

7.33 Two large parallel plates, 1 and 2, having emissivities on their inner faces of 0.5 and 0.8 are maintained at 300°C and 100°C respectively. A third plate having unknown emissivities on its faces *A* and *B* is placed between the other two plates. When face *A* is pointing towards plate 1, the third plate reaches an equilibrium temperature of 278°C. When the third plate is turned around so that face *B* is pointing toward plate 1, its equilibrium temperature drops to 140°C. Determine the emissivities of the two faces *A* and *B*.

(*Ans.* $\varepsilon_A = 0.916$, $\varepsilon_B = 0.102$)

7.34 A furance is shaped like a long equilateral triangular duct. The width of each side is 1 m. The base surface has an emissivity of 0.7 and is maintained at a uniform temperature of 600 K. The heated left side surface closely approximates a black body at 1000 K. The right side surface is well-insulated. Determine the rate at which energy must be supplied to the heated side externally per unit length of the duct.

(*Ans.* 28 kW)

7.35 A butt-welded thermocouple (Fig. P. 7.35) having an emissivity of 0.8 is used to measure the temperature of a transparent gas flowing in a large duct whose walls are at 227°C. The temperature indicated by the thermocouple is 504°C. If the convection heat transfer coefficient between the surface of the couple and the gas is 142 $W/m^2$ K, estimate the true gas temperature.

(*Ans.* 602°C)

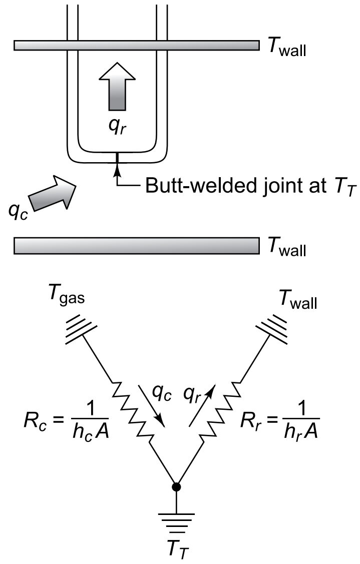


**Fig. P. 7.35** *Butt-welded thermocouple without radiation shield*

7.36 Determine the correct gas temperature in P. 7.35 if the thermocouple is shielded by a thin cylindrical radiation shield having a $d_i$ four times as large as the $d_o$ of the thermocouple (Fig. P. 7.36). Assuming that the convection heat transfer coefficient of the shield is 114 $W/m^2$ K on both sides and that the emissivity of the shield (stainless steel) is 0.3.

(*Ans.* 520°C)

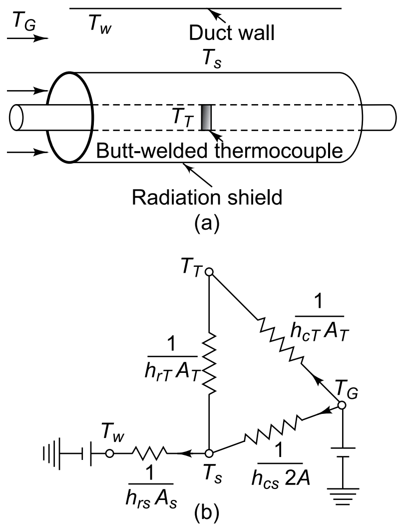


**Fig. P. 7.36** *Butt-welded thermocouple with radiation shield*

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7.37 Evaluate the percentage of daylight entering through a store window $A_1$ that impinges on the floor area $A_4$ located relative to $A_1$ as shown in Fig. P. 7.37. Assume that the light through the window is diffuse.

(*Ans.* $F_{1-4} = 0.097$, i.e. about 10% of the light passing through the window will impinge on the floor area $A_4$)

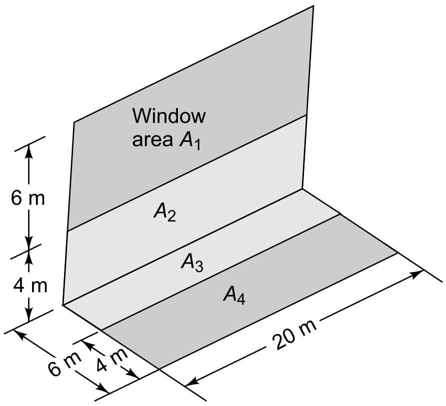


**Fig.** *P. 7.37*

7.38 The temperature of the filament of an incandescent light bulb is 2500 K. Assuming the filament to be a black body, determine the fraction of the radiant energy emitted by the filament that falls in the visible range. What is the wavelength at which the emission of radiation is maximum?

(*Ans.* 0.0527, 1.16 μm)

7.39 A furnace is shaped like a long equilateral duct. The width of each side is 1 m. The base surface has an emissivity of 0.7 and is maintained at a uniform temperature of 600 K. The heated left side surface closely approximates a black body at 1000 K. The right side surface is well insulated. Determine the rate at which energy must be supplied to the heated side externally per unit length of the duct in order to maintain these operating conditions.

(*Ans.* 28 kW)

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# Heat Exchangers

# 8

A heat exchanger is a device in which heat is transferred from one fluid to another. The hot fluid gets cooled, and the cold fluid is heated. The principles of heat transfer discussed so far in the earlier chapters are applied in the thermal design of a heat exchanger. Many types of heat exchangers have been developed for diverse applications in steam power plants, chemical process plants, refrigerators and air conditioners, radiators in cars, space vehicles and so on.

## 8.1 TYPES OF HEAT EXCHANGERS

Heat exchangers can be grouped into three broad classes:

1. Transfer type heat exchangers or recuperators,
2. Storage type heat exchangers or regenerators,
3. Direct contact type heat exchangers or mixers.

In a transfer type heat exchanger or a recuperator, the two fluids are kept separate and they do not mix as they flow through it. Heat is transferred through the separating walls. A concentric double pipe recuperator is shown in Fig. 8.1.

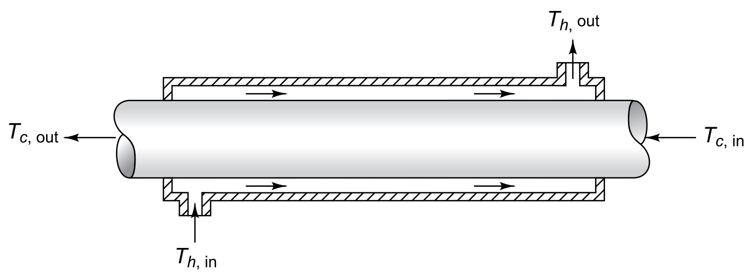


**Fig. 8.1** *Concentric double pipe heat exchanger*

In a storage type heat exchanger or a regenerator, the hot and cold fluids flow alternately through a solid matrix of high heat capacity. When the hot fluid flows through the matrix in an interval of time, heat is transferred from the fluid to the matrix which stores it in the form of an increase in its internal energy. This stored energy is then transferred to the cold fluid as it flows through the matrix in the next interval of time. The matrix is thus subjected to periodic heating and cooling.

Storage type heat exchangers may have matrices which are either (i) stationary or (ii) rotating. Figure 8.2 shows a typical regenerator with a stationary matrix. During the heating period of the cycle when the hot fluid flows through the matrix, valves *A* and *B* are kept open and *C* and *D* are kept closed. During the cooling period,

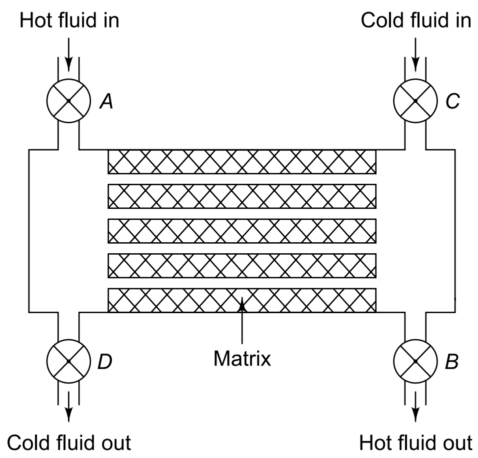


**Fig. 8.2** *Single matrix storage type heat exchanger*

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valves $A$ and $B$ are kept closed and $C$ and $D$ are kept open. A regenerator with a stationary matrix is used in a Stirling refrigerator, such as the Philips refrigerating machine for liquefaction of air, and in a gas turbine power plant.

A rotary regenerator has a matrix rotating at a low rpm, driven by a motor through reduction gears. The heat transfer surfaces provided in the regenerator are alternately exposed to the hot and cold fluids (Fig. 8.3). A typical application of this type of heat exchanger is found in a steam power plant for preheating of air, called Ljungström air preheater.

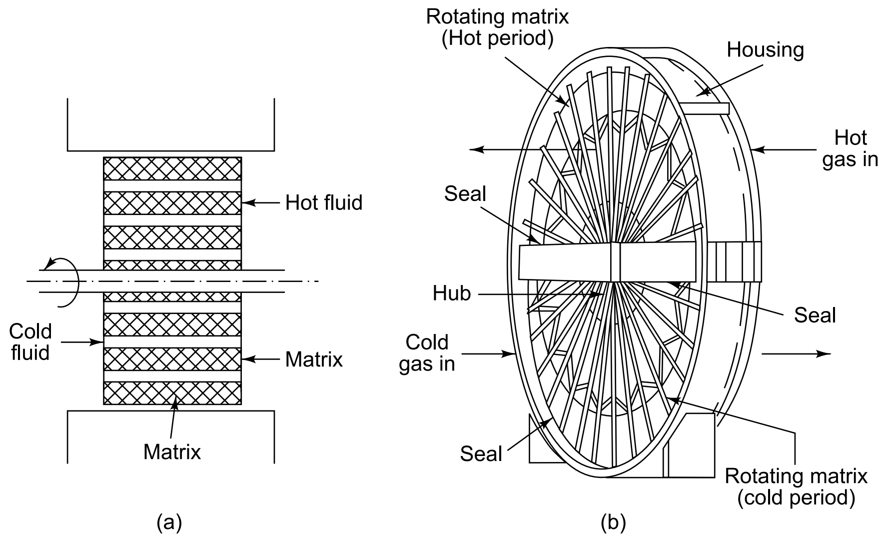


**Fig. 8.3** *Rotary storage type heat exchanger*

A storage type heat exchanger provides a more compact arrangement than the transfer type with more surface area offered per unit volume. The major disadvantage is that some mixing of hot and cold fluids becomes inevitable, and it is quite difficult to seal the hot side from the cold side in the rotary regenerator. There are also more pressure drops in both the fluids.

In a direct contact heat exchanger, the two fluids mix together and transfer heat by direct contact. Open feedwater heaters, desuperheaters, cooling towers and jet condensers are examples of such heat exchangers. The heat transfer is usually accompanied by interphase mass transfer. It cannot be used for transferring heat between two gases or between two miscible liquids. A typical direct contact heat exchanger is shown in Fig. 8.4, which gives a section through a natural draught cooling tower.

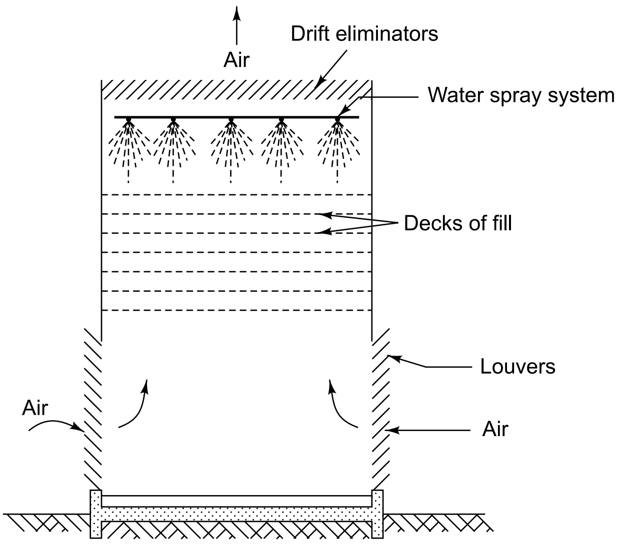


**Fig. 8.4** *Direct contact heat exchanger: A natural draught cooling tower*

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### 8.1.1 Flow Arrangements in Recuperative Heat Exchangers

There are three basic flow arrangements in recuperative heat exchangers: (1) Parallel flow, (2) Counterflow, and (3) Cross flow. If both the fluids move in the same direction, it is a *parallel flow heat exchanger*. If the fluids move in opposite direction, it is a *counterflow heat exchanger*. If they flow normal to each other, it is a *cross flow heat exchanger* (Fig. 8.5). The temperatures of the two fluids usually vary from inlet to exit of the heat exchanger, except in the case of phase change on either side when the temperature remains constant. The tubes may be in the form of coils also, as shown in Fig. 8.6.

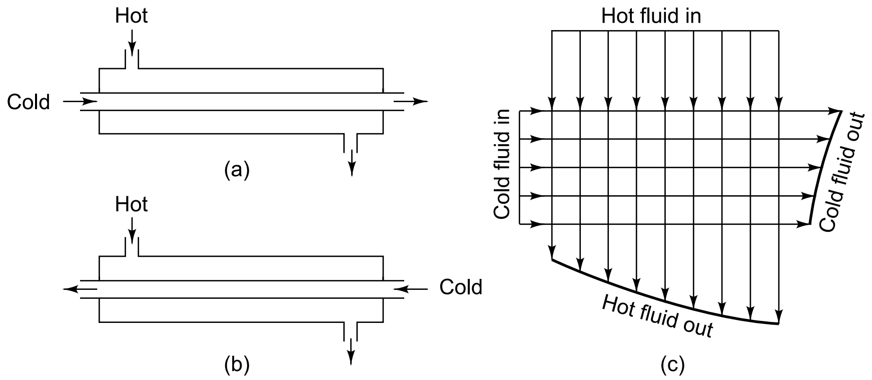


**Fig. 8.5** *Schematic drawing of (a) parallel flow, (b) counterflow and (c) cross flow heat exchanger*

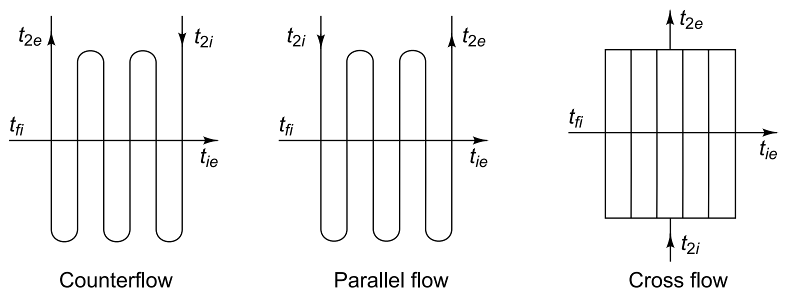


**Fig. 8.6** *Counterflow, parallel flow and cross flow on tube bundles*

## 8.2 COMPACT, SHELL-AND-TUBE AND PLATE HEAT EXCHANGERS

A heat exchanger having a large surface area per unit volume is called a *compact heat exchanger*. The ratio of the heat transfer surface area to the volume is called the area density $\beta$. A heat exchanger with $\beta >$ 700 m$^2$/m$^3$ is said to be compact, e.g. car radiators ($\beta$ = 1000 m$^2$/m$^3$), ceramic regenerators in gas turbines ($\beta$ = 6000 m$^2$/m$^3$), and Stirling engine regenerator ($\beta$ = 15,000 m$^2$/m$^3$). The large surface area is obtained by attaching closely spaced thin plates or corrugated fins to the walls separating the two fluids. Compact heat exchangers are commonly used in gas-to-gas or gas-to-liquid heat transfer, with limitations on their weight and volume, with fins, if any, being used on the gas side where heat transfer coefficient is low.

In compact heat exchangers, the two fluids usually move perpendicular to each other, and such flow configuration is called *cross-flow*, as stated earlier. The cross-flow is further classified as unmixed flow and mixed flow. In Fig. 8.7(a), the cross-flow is said to be *unmixed*, since the plate fins force the fluid to flow through a particular interfin spacing and prevent it from moving in the transverse direction (i.e. flowing

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parallel to the tubes). The cross-flow in (b) is said to be *mixed* since the fluid flow is free to move in the transverse direction.

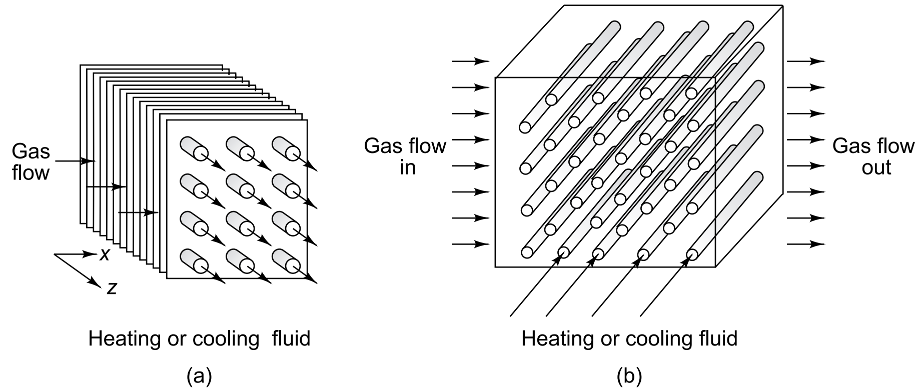


**Fig. 8.7** *Cross-flow heat exchanger: (a) Fluids unmixed; (b) one fluid (gas) mixed, the other unmixed*

Perhaps the most common type of heat exchanger in industrial application is the *shell-and-tube heat exchanger* (Fig. 8.8). Here, a large number of tubes are packed inside a shell with their axes parallel to that of the shell. Heat transfer takes place as one fluid flows inside the tubes while the other fluid flows outside the tubes through the shell. *Baffles* are commonly placed in the shell to force the shell-side fluid to flow across the shell to enhance heat transfer (by increasing residence time) and to maintain uniform spacing between the tubes. Because of their relatively large size and weight, shell-and-tube heat exchangers are not suitable for use in automotive, aircraft and marine applications. At both ends of the shell there are headers where the fluid accumulates before entering the tubes and after leaving them.

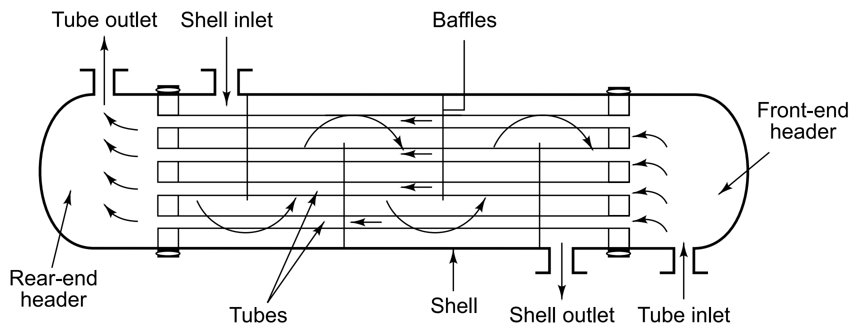


**Fig. 8.8** *Schematic of a shell-and-tube heat exchanger (one-shell pass and one-tube pass)*

Shell-and-tube heat exchangers are further classified according to the number of shell and tube passes involved. Heat exchangers in which all the tubes make one U-turn in the shell, are called *one-shell pass and two-tube pass* heat exchangers. Likewise, a heat exchanger that involves two passes in the shell and four passes in the tubes is called a *two-shell pass and four-tube pass* heat exchanger (Fig. 8.9).

An innovative type of heat exchanger which has found widespread use is the *plate heat exchanger*, which consists of a series of plates with corrugated flow passages (Fig. 8.10). The hot and cold fluids flow in alternate passages, and thus each cold fluid stream is surrounded by two hot fluid streams, resulting in very effective heat transfer. The heat transfer capacity can be enhanced by simply adding more plates in series. They are well suited for *liquid-to-liquid heat transfer* applications, provided that the hot and cold fluid streams are at about the same pressure.

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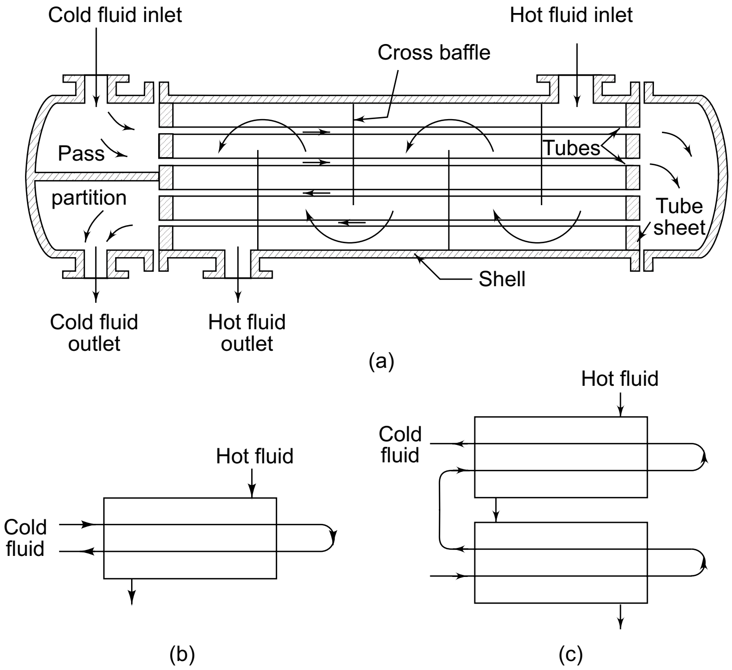


**Fig. 8.9** *Multiple pass heat exchangers: (a), (b) One-shell pass and two-tube pass; (c) two-shell pass, four-tube pass*

Heat exchangers are often given specific names to reflect the specific application for which they are used. For example, a *condenser* is a heat exchanger in which one of the fluids gives up heat and condenses as it flows through the heat exchanger. A *boiler* is another heat exchanger in which one of the fluids absorbs heat and vaporises. A *space radiator* is a heat exchanger that transfers heat from the hot fluid to the surrounding space by radiation.

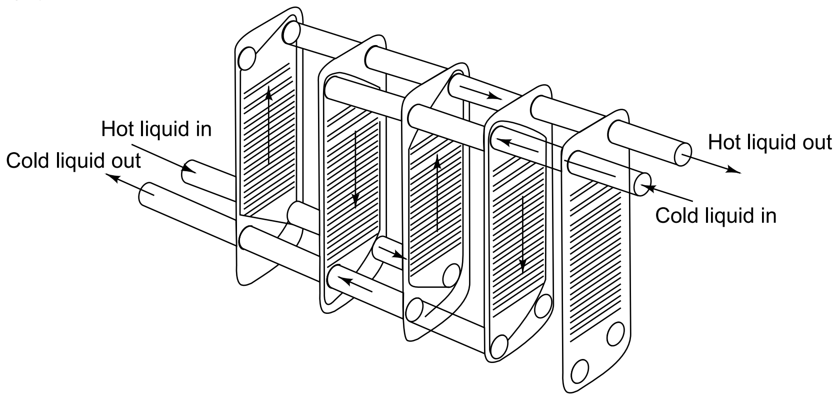


**Fig. 8.10** *Plate heat exchanger*

## 8.3 OVERALL HEAT TRANSFER COEFFICIENT AND FOULING FACTOR

If $T_h$ and $T_c$ represent the bulk mean temperatures of the two fluids on either side of the plane wall, then

$$Q = UA\,\Delta T = UA\,(T_h - T_c) \tag{8.1}$$

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where $$\frac{1}{UA} = \sum R = \frac{1}{h_1 A} + \frac{x_w}{k_w A} + \frac{1}{h_2 A} \tag{8.2}$$

$x_w$ being the thickness of the wall, $k_w$ the thermal conductivity and $h_1$ and $h_2$ the heat transfer coefficients on the two sides.

For heat transfer through a cylindrical wall,

$$Q = U_0 A_0 (T_h - T_c) = U_0 A_0 \Delta T \tag{8.3}$$

where, $$\frac{1}{U_0 A_0} = \frac{1}{h_i A_i} + \frac{x_w}{k_w \; A_{l \cdot m}} + \frac{1}{h_0 A_0} \tag{8.4}$$

$$A_0 = \pi D_0 L, A_i = \pi D_i L, A_{1m} = \frac{A_0 - A_i}{\ln (A_0/A_i)}$$

The value of $U_0$ is dominated by the smaller value of convection coefficient. If $h_i << h_0$ we have $1/h_i >> 1/h_0$, and thus $U_0 \approx h_i$. Therefore, the smaller heat transfer coefficient creates a bottleneck in the path of heat flow, and seriously impedes heat transfer. This situation arises when one of the fluids is a gas and the other a liquid. Since $h_{gas} << h_{liquid}$, fins are provided on the gas side to compensate for low $h$ and enhance $UA$.

The range of values of overall heat transfer coefficient in different heat exchangers is given in Table 8.1. It may be noted that it varies from about 10 W/m$^2$ K for gas-to-gas heat exchangers to about 10,000 W/m$^2$ K for phase-change ones. When the tube is finned on one side to enhance heat transfer, the total heat transfer surface area on the finned side becomes

$$A = A_{\text{total}} = A_{\text{fin}} + A_{\text{unfinned}}$$

where $A_{\text{fin}}$ is the surface area of the fins and $A_{\text{unfinned}}$ is the area of unfinned portion of the tube surface. The effective surface area $A$ can be estimated from

$$A = A_{\text{unfinned}} + \eta_{\text{fin}} A_{\text{fin}}$$

where $\eta_{\text{fin}}$ is the fin efficiency.

**Table 8.1** *Representative values of the overall heat transfer coefficients in heat exchangers*

| *Type of heat exchanger* | $U[W/(m^2K)]$ |
|---|---|
| Water-to-water | 850 – 1700 |
| Water-to-oil | 100 – 350 |
| Water-to-gasoline or kerosene | 300 – 1000 |
| Feedwater heaters | 1000 – 8500 |
| Steam-to-light fuel oil | 200 – 400 |
| Steam-to-heavy fuel oil | 50 – 200 |
| Steam condenser | 1000 – 6000 |
| Freon condenser (water cooled) | 300 – 1000 |
| Ammonia condenser (water cooled) | 800 – 1400 |
| Alcohol condensers (water cooled) | 250 – 700 |
| Gas-to-gas | 10 – 40 |
| Water-to-air in finned tubes (water in tubes) | 30 – 60 |
| | 400 – 850 |
| Steam-to-air in finned tubes (steam in tubes) | 30 – 300 |
| | 400 – 4000 |

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Equation (8.4) holds for clean surfaces on both sides of the cylindrical wall. After a period of operation, scales are formed on the surfaces which offer additional resistances to heat transfer, so that

$$\frac{1}{U_0' A_0} = \frac{1}{h_{s_i} A_i} + \frac{1}{h_i A_i} + \frac{x_w}{k_w A_{1m}} + \frac{1}{h_{s_0} A_0} + \frac{1}{h_0 A_0} \tag{8.5}$$

where $h_{s_i}$ and $h_{s_0}$ are the scale heat transfer coefficients and $U_0'$ is the overall heat transfer coefficient with scaled surfaces. The reciprocal of the scale heat transfer coefficient is called the *fouling factor* $R_f$. Fouling factors which reduce the performance of heat exchangers can be determined experimentally by estimating $U_0$ values for clean and scaled surfaces, so that

$$R_f = \frac{1}{U_0'} - \frac{1}{U_0} \tag{8.6}$$

Table 8.2 gives the fouling factors for certain applications.

**Table 8.2** *Fouling factors*

| *Fluid* | *Fouling factor $R_f$ ($m^2$ K/W)* |
|---|---|
| Sea water | 0.000172 |
| Treated boiler feed water | 0.000172 |
| Well water | 0.00036 |
| Fuel oil | 0.0009 |
| Quenching oil | 0.0007 |
| Diesel exhaust gas | 0.0018 |
| Refrigerant vapours | 0.000344 |
| Brine | 0.000172 |
| Steam, alcohol | 0.00009 |

The most common type of fouling is the precipitation of solid deposits in a fluid on the heat transfer surfaces. When water is hard, scales are formed, and hence water requires to be treated. Corrosion and chemical fouling by chemical reactions and biofouling due to growth of algae, and deposit of ash particles in the flue gases on air preheater surfaces are some other forms of fouling.

## 8.4 PARALLEL FLOW HEAT EXCHANGER

An insulated double pipe parallel flow heat exchanger along with the temperature profiles is shown in Fig. 8.11. The temperatures of the fluids vary from point to point as heat is transferred from the hot to the cold fluid.

Let $\dot{m}_h$ = mass flow rate of the hot fluid, kg/s

$\dot{m}_c$ = mass flow rate of the cold fluid, kg/s

$c_h$ = specific heat of the hot fluid, kJ/kg K

$c_c$ = specific heat of the cold fluid, kJ/kg K

$T_{h_1}$ = inlet temperature of the hot fluid, °C

$T_{h_2}$ = exit temperature of the cold fluid, °C

$T_{c_1}$ = inlet temperature of the cold fluid, °C

$T_{c_2}$ = exit temperature of the cold fluid, °C

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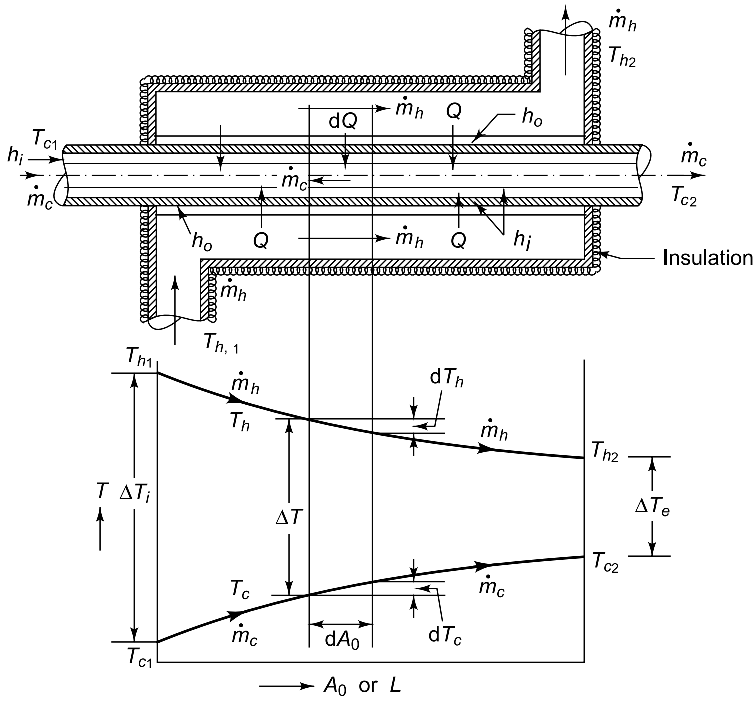


**Fig. 8.11** *Parallel flow heat exchanger*

Let us consider a differential surface area $\mathrm{d}A_0$ ($=\pi D_0\, \mathrm{d}x$) of the heat exchanger where $\mathrm{d}Q$ amount of heat is transferred, the hot fluid temperature decreases by $\mathrm{d}T_h$ and the cold fluid temperature increases by $\mathrm{d}T_c$. By writing an energy balance,

$$\mathrm{d}Q = -\,\dot{m}_h c_h\, \mathrm{d}T_h = \dot{m}_c\, c_c\, \mathrm{d}T_c = U_0\, \mathrm{d}A_0\, \Delta T \tag{8.7}$$

where $\Delta T = T_h - T_c$, $T_h$ and $T_c$ being the mean temperatures of the hot and cold fluids at that section respectively. This temperature difference $\Delta T$ between the two fluids changes from $\Delta T_i$ at inlet to $\Delta T_e$ at exit of the heat exchanger.

Now, $\qquad \Delta T = T_h - T_c$

or, $\qquad \mathrm{d}(\Delta T) = \mathrm{d}T_h - \mathrm{d}T_c \qquad (8.8)$

From Eqs (8.7) and (8.8),

$$\mathrm{d}(\Delta T) = -\frac{\mathrm{d}Q}{\dot{m}_h c_h} - \frac{\mathrm{d}Q}{\dot{m}_c c_c}$$

$$= -\,\mathrm{d}Q\left(\frac{1}{\dot{m}_h c_h} + \frac{1}{\dot{m}_c c_c}\right)$$

$$= -\,\mathrm{d}Q\, \mu_p \tag{8.9}$$

where $\qquad \mu_p = \dfrac{\mathbf{1}}{\dot{\mathbf{m}}_\mathbf{h}\mathbf{c}_\mathbf{h}} + \dfrac{\mathbf{1}}{\dot{\mathbf{m}}_\mathbf{c}\mathbf{c}_\mathbf{c}}$

By integrating Eq. (8.9) from inlet to exit, $\mu_p$ being constant,

$$\Delta T_i - \Delta T_e = \mu_p Q \tag{8.10}$$

Again, from Eqs (8.7) and (8.9),

$$dQ = U_0\, dA_0\, \Delta T$$

$$-\frac{d(\Delta T)}{\mu_p} = U_0\, dA_0\, \Delta T$$

$$-\int_i^e \frac{d(\Delta T)}{\Delta T} = \int_i^e U_0\, dA_0\, \mu_p$$

or,
$$\ln \frac{\Delta T_i}{\Delta T_e} = U_0 A_0 \mu_p \tag{8.11}$$

where $U_0$ has been assumed to be constant. From Eqs (8.10) and (8.11),

$$\ln \frac{\Delta T_i}{\Delta T_e} = U_0 A_0 \frac{\Delta T_i - \Delta T_e}{Q}$$

$$\therefore \quad Q = U_0 A_0 (\Delta T)_{\text{lm}} = \dot{m}_c c_h (T_{h_1} - T_{h_2})$$

$$= \dot{m}_c c_c (T_{c_2} - T_{c_1}) \tag{8.12}$$

where,
$$\Delta T_{1m} = \frac{\Delta T_i - \Delta T_e}{\ln \Delta T_i / \Delta T_e}$$

= logarithmic mean temperature difference, or LMTD

and
$$\frac{1}{U_0 A_0} = \frac{1}{h_i A_i} + \frac{x_w}{k_w A_{1m}} + \frac{1}{h_0 A_0}$$

Here,
$$\Delta T_i = T_{h_1} - T_{c_1} \text{ and } \Delta T_e = T_{h_2} - T_{c_2}$$

## 8.5 COUNTERFLOW HEAT EXCHANGER

Figure 8.12 shows an insulated double pipe counterflow heat exchanger along with the temperature profiles. The energy balance for the differential surface area $dA_0$ ($\pi D_0\, dx$) gives

$$dQ = -\dot{m}_h c_h\, dT_h = -\dot{m}_c c_c\, dT_c = U_0\, dA_0\, \Delta T \tag{8.13}$$

where both the hot and cold fluids undergo temperature decreases $dT_h$ and $dT_c$ (both being negative) in flowing the distance $dx$.

Since,
$$\Delta T = T_h - T_c$$

$$d(\Delta T) = dT_h - dT_c = -\frac{dQ}{\dot{m}_h c_h} + \frac{dQ}{\dot{m}_c c_c}$$

$$= -dQ\left(\frac{1}{\dot{m}_h c_h} - \frac{1}{\dot{m}_c c_c}\right)$$

$$= -dQ\, \mu_c \tag{8.14}$$

where
$$\mu_c = \frac{1}{\dot{m}_h c_h} - \frac{1}{\dot{m}_c c_c} \tag{8.15}$$

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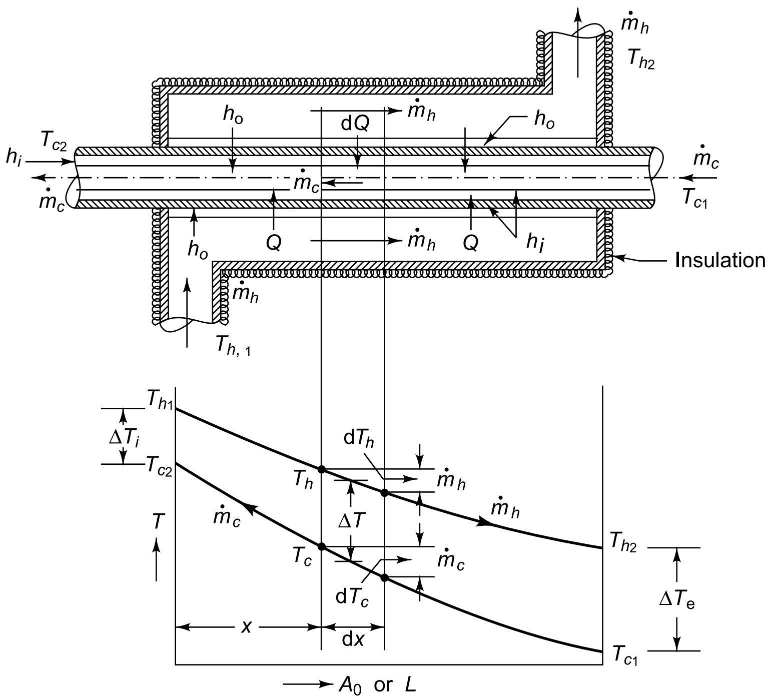


**Fig. 8.12** *Counterflow heat exchanger*

Since $\mu_c$ is constant, by integrating over the whole surface,

$$\Delta T_i - \Delta T_e = \mu_c Q \tag{8.16}$$

Again, $$\mathrm{d}Q = U_0\, \mathrm{d}A_0\, \Delta T$$

Using Eq. (8.14),

$$-\frac{d(\Delta T)}{\mu_c} = U_0\, \mathrm{d}A_0\, \Delta T$$

or, $$\int_i^e -\frac{d(\Delta T)}{\Delta T} = \int_i^e U_0\, \mathrm{d}A_0\, \mu_c$$

With $U_0$ being assumed constant,

$$\mathbf{\ln \frac{\Delta T_i}{\Delta T_e} = U_0 A_0 \mu_c} \tag{8.17}$$

From Eqs (8.16) and (8.17),

$$\begin{aligned} Q &= U_0 A_0 (\Delta T)_{1m} \\ &= \dot{m}_h c_h (T_{h_1} - T_{h_2}) \\ &= \mathbf{\dot{m}_c c_c (T_{c_2} - T_{c_1})} \end{aligned} \tag{8.18}$$

where $$\mathbf{T_{1m} = \frac{\Delta T_i - \Delta T_e}{\ln \Delta T_i / \Delta T_e} = LMTD} \tag{8.18a}$$

$$\Delta T_i = T_{h_1} - T_{c_2} \text{ and } \Delta T_e = T_{h_2} - T_{c_1}$$

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## 8.6 USE OF LMTD

The temperature profiles of the two fluids as shown in Figs 8.11 and 8.12 are curved, having logarithmic variations. If they were straight, arithmetic mean temperature difference, $\Delta T_{am}$, could be used, where

$$\Delta T_{am} = \frac{\Delta T_i + \Delta T_e}{2}$$

Now,

$$\Delta T_{1m} = \frac{\Delta T_i - \Delta T_e}{\ln \Delta T_i / \Delta T_e}$$

$$= \frac{\Delta T_i - \Delta T_e}{\ln \dfrac{1 + (\Delta T_i - \Delta T_e)/(\Delta T_i + \Delta T_e)}{1 - (\Delta T_i - \Delta T_e)/(\Delta T_i + \Delta T_e)}}$$

If $\dfrac{\Delta T_i - \Delta T_e}{\Delta T_i + \Delta T_e} = x$, then $x$ is less than 1. Now,

$$\ln \frac{1+x}{1-x} = \ln(1+x) - \ln(1-x)$$

$$= \left( x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots \right) - \left( -x - \frac{x^3}{3} - \frac{x^5}{5} - \cdots \right)$$

$$= 2\left( x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots \right)$$

$$\Delta T_{1m} = \frac{\Delta T_i - \Delta T_e}{2\left[ \dfrac{\Delta T_i - \Delta T_e}{\Delta T_i + \Delta T_e} + \dfrac{1}{3}\left( \dfrac{\Delta T_i - \Delta T_e}{\Delta T_i + \Delta T_e} \right)^3 + \dfrac{1}{5}\left( \dfrac{\Delta T_i - \Delta T_e}{\Delta T_i + \Delta T_e} \right)^5 + \cdots \right]}$$

$$= \frac{\Delta T_i - \Delta T_e}{2\dfrac{\Delta T_i - \Delta T_e}{\Delta T_i + \Delta T_e}\left[ 1 + \dfrac{1}{3}\left( \dfrac{\Delta T_i - \Delta T_e}{\Delta T_i + \Delta T_e} \right)^2 + \cdots \right]}$$

$$= \frac{\Delta T_{am}}{1 + \dfrac{1}{3}\left( \dfrac{\Delta T_i - \Delta T_e}{\Delta T_i + \Delta T_e} \right)^2 + \cdots} \tag{8.19}$$

If $\Delta T_i \lessgtr \Delta T_e$,

$$\mathbf{\Delta T_{1m} < \Delta \mathit{T}_{am}} \tag{8.20}$$

LMTD is thus less than the arithmetic mean temperature difference. It is always safer for the designer to use LMTD so as to provide larger heating surfaces for a certain amount of heat transfer.

If $\Delta T_i = \Delta T_e$, $(\Delta T)_{lm} = (\Delta T)_{am}$

## 8.7 CROSS-FLOW HEAT EXCHANGER

A cross-flow single pass heat exchanger with plate fins and both the fluids unmixed is shown in Fig. 8.13. Since the outlet temperatures of the two fluids are not uniform over the entire cross-section (Fig. 8.14), the calculation of the mean temperature difference is considerably more difficult.

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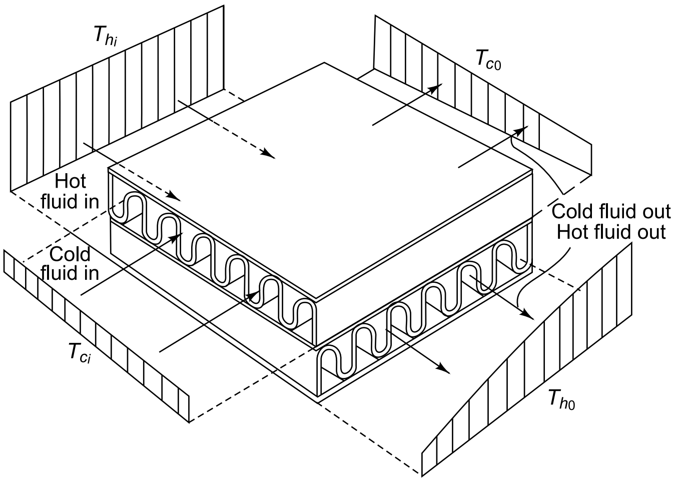


**Fig. 8.13** *Cross-flow heat exchanger*

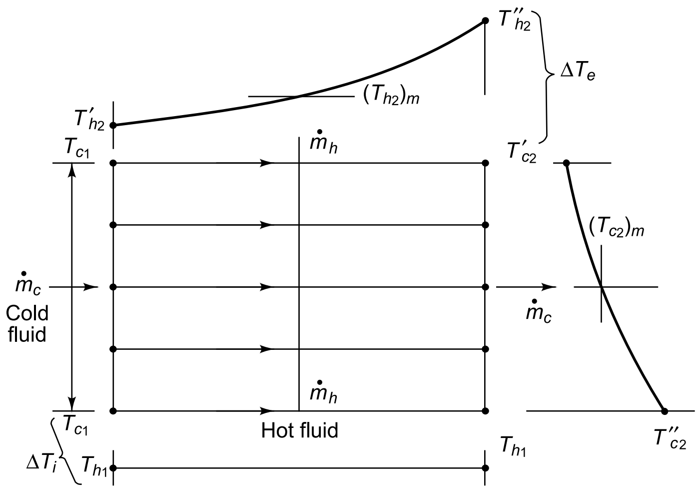


**Fig. 8.14** *Overall heat transfer in cross flow*

Such a calculation was carried out by Nusselt [1]. If $(T_{h2})_m$ and $(T_{c2})_m$ represent the average hot fluid and cold fluid temperatures at exit respectively, then

$$\Delta T_e = (T_{h_2})_m - (T_{c_2})_m$$

whereas

$$\Delta T_i = T_{h_1} - T_{c_1}$$

as shown in Fig. 8.14. Then,

$$\Delta T_{1m} = \frac{\Delta T_i - \Delta T_e}{\ln \Delta T_i / \Delta T_e}$$

and

$$Q = U_0 A_0 \Delta T_{1m}. = \dot{m}_h c_h [T_{h_1} - (T_{h_2})_m]$$
$$= \dot{m}_c c_c [(T_{c_2})_m - T_{c_1}] \tag{8.21}$$

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## 8.8 COMPARISON OF PARALLEL FLOW AND COUNTER FLOW HEAT EXCHANGERS

The temperature profiles of parallel and counterflow heat exchangers are shown again in Fig. 8.15 for comparison.

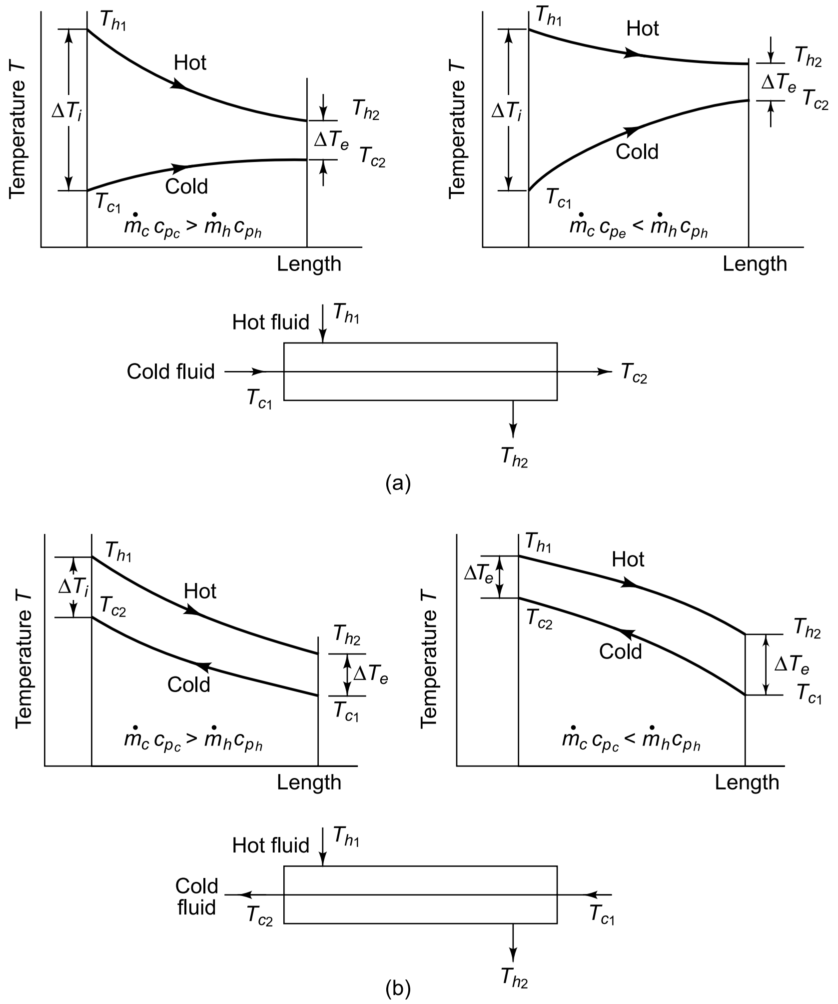


**Fig. 8.15** *Comparison of parallel flow and counterflow heat exchangers*

For the same inlet and exit temperatures of the two fluids, it is found that $(\Delta T)_{1m}$ for counterflow is always greater than that for parallel flow. Since

$$Q = U_0 A_0 (\Delta T)_{1m}$$

for the same heat transfer $Q$ and the same overall heat transfer coefficient $U_0$, the surface area required for counterflow operation is always less than that for parallel flow operation.

In a parallel flow heat exchanger, $T_{h_2} > T_{c_2}$, i.e. the hot fluid cannot be cooled to a temperature lower than the cold fluid temperature. In a counterflow heat exchanger, $T_{h_2}$ can be less than $T_{c_2}$ or $T_{c_2}$ can be higher than $T_{h_2}$, i.e. the hot fluid can be cooled below $T_{c_2}$ or the cold fluid can be heated above $T_{h_2}$.

Counterflow heat exchangers are, therefore, normally more common in industrial practice. For a given rate of heat flow and at given initial and final temperatures, while the smallest heating surface is required

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for counter flow system and the largest for parallel flow system, the cross-flow surface requirement lies between these two.

## 8.9 HEAT TRANSFER WITH PHASE CHANGE

When one of the fluids undergoes phase change (condensation, evaporation, sublimation), the direction of the two fluids is immaterial, and $(\Delta T)_{lm}$ remains the same whether the flow arrangement be parallel, counter or cross (Fig. 8.16). For evaporation of a saturated liquid in case (a), if the hot fluid flows from *a* to *b*,

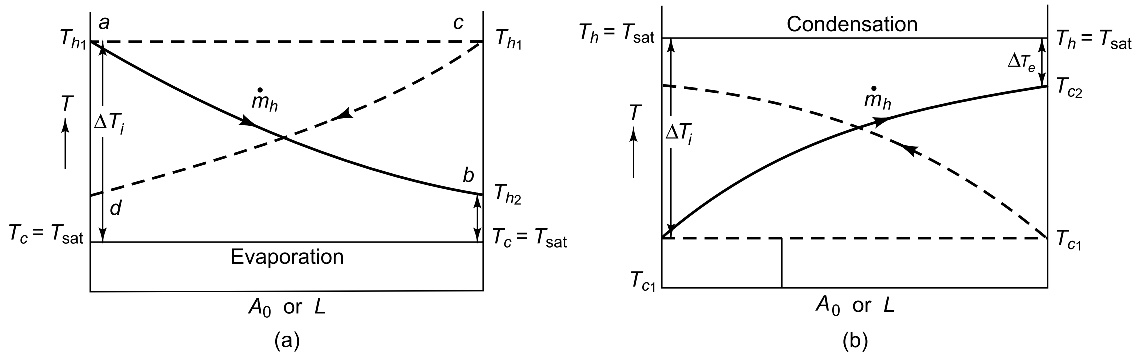


**Fig. 8.16** *Heat transfer with phase change*

$$\Delta T_{1m} = \frac{(T_{h_1} - T_c) - (T_{h_2} - T_c)}{\ln \dfrac{(T_{h_1} - T_c)}{(T_{h_2} - T_c)}} = \frac{T_{h_1} - T_{h_2}}{\ln \dfrac{(T_{h_1} - T_c)}{(T_{h_2} - T_c)}}$$

If the flow is reversed and it takes place from *c* to *d*,

$$\Delta T_{1m} = \frac{(T_{h_2} - T_c) - (T_{h_1} - T_c)}{\ln \dfrac{(T_{h_2} - T_c)}{(T_{h_1} - T_c)}} = \frac{T_{h_2} - T_{h_1}}{\ln \dfrac{(T_{h_2} - T_c)}{(T_{h_1} - T_c)}}$$

$$= \frac{T_{h_1} - T_{h_2}}{\ln \dfrac{(T_{h_1} - T_c)}{(T_{h_1} - T_c)}}$$

which is the same as for the original flows.

A similar situation prevails when a saturated vapour condenses, and shown in case (b).

## 8.10 MULTIPASS HEAT EXCHANGERS

*Single-pass* heat exchangers are those in which the two fluids flow only once in particular directions exchanging heat with each other and the LMTD expression is valid for them. In many forms of heat exchangers, the direction of one or both fluids may change during the travel through the exchanger. The flow paths are so arranged that one or both fluids may reverse direction one or more times in passing through the exchanger. Heat exchangers having several passes are termed *multipass* heat exchangers. A common example is the shell and tube heat exchanger in which one fluid flows over the shell (mixed) and the other through the tubes (unmixed). There may be more than one tube or shell passes so that the two fluids flow alternately in parallel flow and counterflow, and sometimes it approaches cross flow.

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The determination of the mean temperature difference is quite complex. The computations for multipass flow as well as for cross-flow arrangements have been made by Bowman *et al.* [2] Use of these charts is more convenient, yet of sufficient accuracy, than the use of the mathematical expressions. A heat exchanger having one well-baffled shell pass (mixed fluid) and two tube passes (unmixed fluid) along with the temperature profiles is shown in Fig. 8.17. The mean temperature difference $\Delta T_m$ is obtained by multiplying the LMTD for counterflow single-pass arrangement by a *correction factor F* as given below,

$$\Delta T_m = (\Delta T_{1m})_{\text{counterflow}} \times F$$

The rate of heat transfer is given by

$$Q = U_0 A_0 (\Delta T)_m = U_0 A_0 F (\Delta T_{1m})_{\text{counterflow}} \quad (8.22)$$

where $$\Delta T_{lm} = \frac{(T_{h_2} - T_{c_1}) - (T_{h_1} - T_{c_2})}{\ln (T_{h_2} - T_{c_1})/(T_{h_1} - T_{c_2})}$$

The correction factors $F$ for different flow arrangements have been published in chart form by Bowman *et al.* [2] and by TEMA (Tubular Exchanger Manufacturers Association) [3] as given in Figs 8.18–8.21. The factor $F$ has been given as a function of two dimensionless parameters:

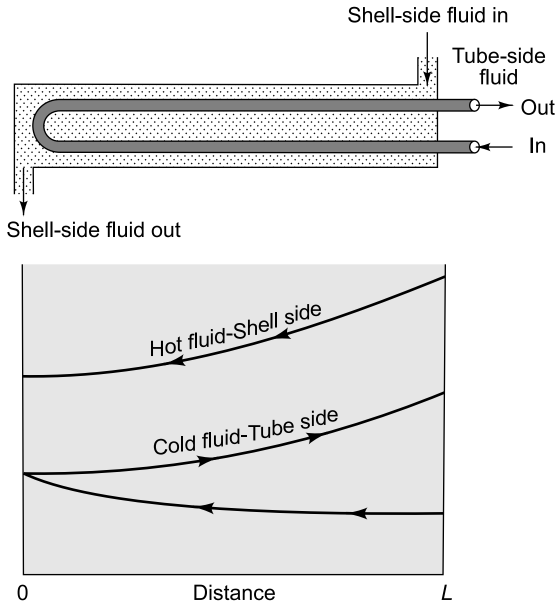


**Fig. 8.17** *Axial temperature distribution in a one-shell pass, two tube-pass heat exchanger*

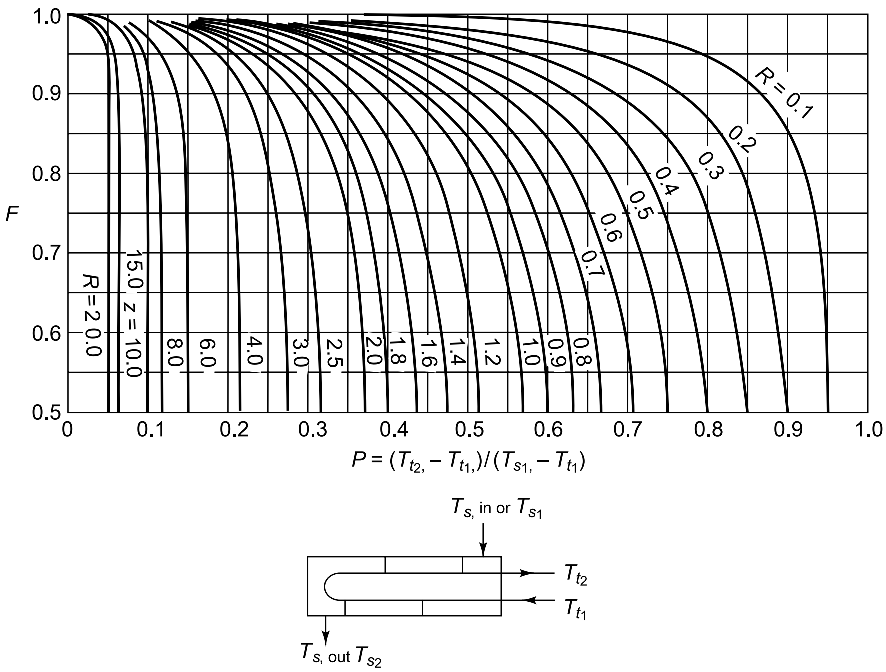


**Fig. 8.18** *Correction-factor to counter flow LMTD for heat exchanger with one shell pass and 2, 4, 6 or any multiple of 2 tube passes (TEMA)*

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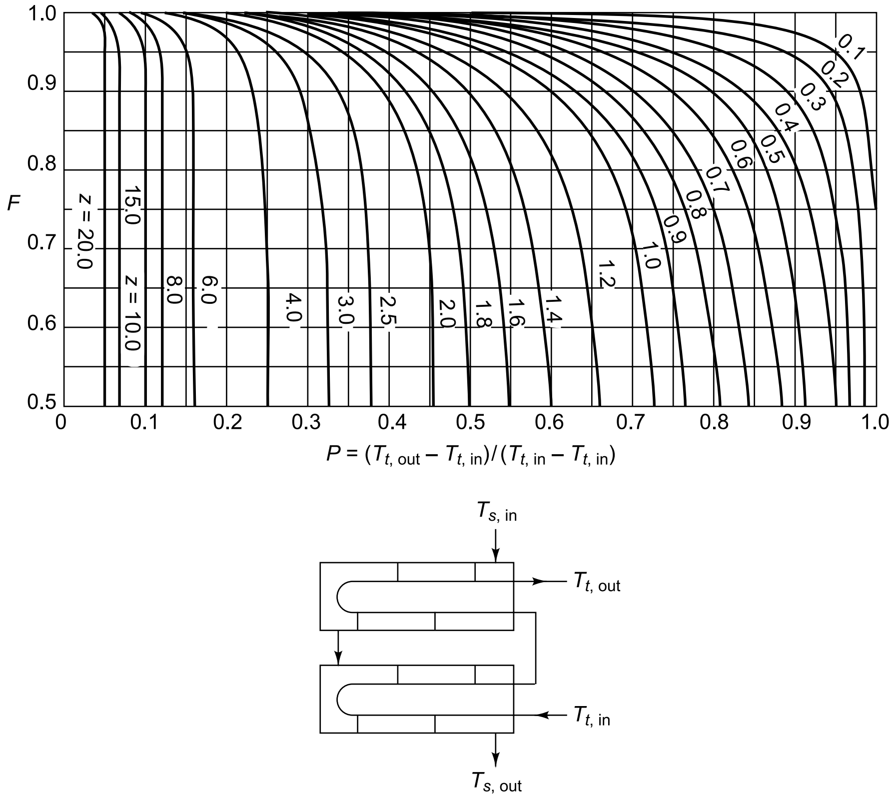


**Fig. 8.19** *Correction factor to counter flow LMTD for heat exchanger with two shell passes and 4, 8, 12, ... tube passes (TEMA)*

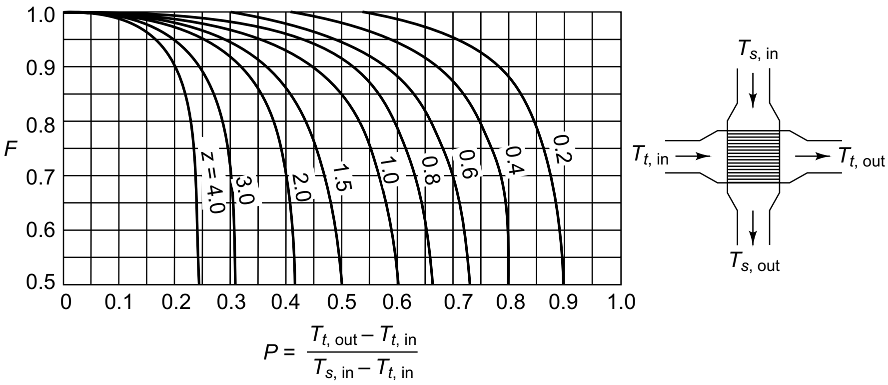


**Fig. 8.20** *Correction factor to counter flow LMTD for cross flow heat exchanger with shell side fluid mixed and tube fluid unmixed, having one tube pass (Bowmen et al.)*

Capacity ratio $$\mathbf{R} = \frac{(\dot{m}c)_t}{(\dot{m}c)_s} = \frac{T_{s_1} - T_{s_2}}{T_{t_2} - T_{t_1}} \tag{8.23}$$

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Temperature ratio $\boldsymbol{P} = \dfrac{\mathbf{T_{t_2} - T_{t_1}}}{\mathbf{T_{s_1} - T_{t_1}}}$ (8.24)

where subscripts *t* and *s* refer to tube and shell and subscripts 1 and 2 refer to inlet and outlet conditions, respectively. The correction factor is applicable whether the hot fluid is in the shell side or tube side. The correction factor will be unity when either one of the fluids undergoes phase change.

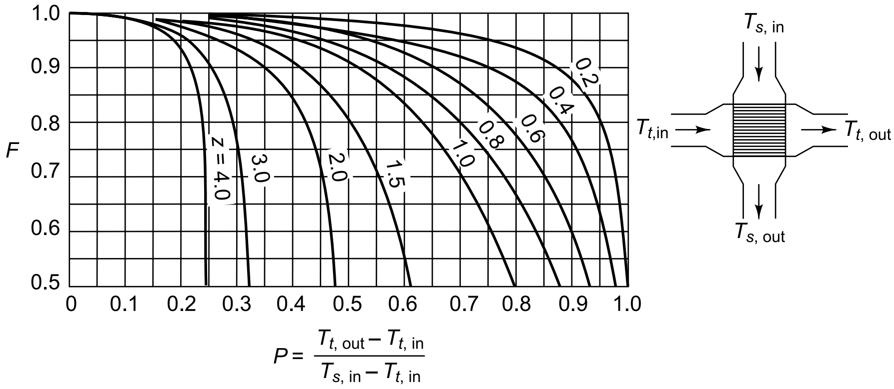


**Fig. 8.21** *Correction factor to counter flow LMTD for a cross flow heat exchanger with both fluids unmixed and one tube pass (Bowmen et al.)*

## 8.11 VARIATION OF $U_0$ ALONG THE HEATING SURFACE

In the preceding calculations, the overall heat transfer coefficient was assumed constant along the heating surface. When $U$ varies appreciably from one end of the heat exchanger to the other end, and if it is assumed that $U_0$ varies linearly with the temperature difference, then

$$\mathbf{U_0 = a + b\,\Delta T} \tag{8.25}$$

where $a$ and $b$ are constant.

For a counterflow heat exchanger, from Eqs (8.14) and (8.16),

$$d(\Delta T) = -\,\mathrm{d}Q\,\mu_c$$
$$\Delta T_i - \Delta T_e = \mu_c Q$$

Also, $$\frac{d\,(\Delta T)}{\mu_c} = -\,U_0\,\mathrm{d}A_0\,\Delta T = -\,\Delta T\,(a + b\Delta T)\,\mathrm{d}A_0$$

$$\int_i^e \frac{d\,(\Delta T)}{\Delta T\,(a + b\Delta T)} = \frac{\Delta T_e - \Delta T_i}{Q}\int_i^e \mathrm{d}A_0$$

$$= \frac{\Delta T_e - \Delta T_i}{Q}A_0 \tag{8.26}$$

Let $\Delta T = x$, when $\Delta T = \Delta T_i$, $x = x_1$ and $\Delta T = \Delta T_e$, $x = x_2$.

Then, $$\text{L.H.S.} = \int_{x_1}^{x_2} \frac{\mathrm{d}x}{x\,(a + bx)} = \int_{x_1}^{x_2} \frac{1}{a}\left(\frac{1}{x} - \frac{b}{a + bx}\right)\mathrm{d}x$$

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$$= \frac{1}{a}\left[\ln x - b\frac{\ln(a+bx)}{b}\right]_{x_1}^{x_2}$$

$$= \frac{1}{a}\left[\ln\frac{x}{a+bx}\right]_{x_1}^{x_2} = \frac{1}{a}\left[\ln\frac{\Delta T}{a+b\Delta T}\right]_{\Delta T_i}^{\Delta T_e}$$

$$= \frac{1}{a}\ln\frac{\Delta T_e\,(a+b\Delta T_i)}{(a+b\Delta T_e)\Delta T_i} \tag{8.27}$$

Let $U_i = a + b\Delta T_i$ and $U_e = a + b\Delta T_e$, so that,

$$a = \frac{U_i\Delta T_e - U_e\Delta T_i}{\Delta T_e - \Delta T_i}$$

$$b = \frac{U_i - U_e}{\Delta T_i - \Delta T_e}$$

From Eqs (8.26) and (8.27),

$$\frac{\Delta T_e - \Delta T_i}{U_i\Delta T_e - U_e\Delta T_i}\ln\frac{\Delta T_e U_i}{\Delta T_i U_e} = \frac{\Delta T_e - \Delta T_i}{Q}A_0$$

$$\mathbf{Q = A_0\,\frac{U_i\Delta T_e - U_e\Delta T_i}{\ln\dfrac{U_i\Delta T_e}{U_e\Delta T_i}}} \tag{8.28}$$

This equation would then replace Eqs (8.18) and (8.18a).

## 8.12 EFFECTIVENESS—NTU METHOD

In heat exchanger calculations, 10 quantities are involved, viz. $\dot{m}_h$, $c_h$, $\dot{m}_c$, $c_c$, $T_{h_1}$, $T_{h_2}$, $T_{c_1}$, $T_{c_2}$, $U_0$ and $A_0$.

For designing the heat exchanger, $\dot{m}_h$, $c_h$, $\dot{m}_c$, $c_c$, $T_{h_1}$, $T_{c_1}$, $T_{h_2}$ (or $T_{c_2}$) and $U_0$ are given, and from the equations

$$Q = \dot{m}_h c_h (T_{h_1} - T_{h_1}) = \dot{m}_c\, c_c (T_{c_2} - T_{c_1})$$
$$= U_0 A_0 (\Delta T)_{1m} \tag{8.18}$$

$T_{c_2}$ (or $T_{h_2}$), $\Delta T_{1m}$ and then $A_0$ are estimated.

But for a given heat exchanger with specified flow rates and inlet fluid temperatures, both of which may be different from the values with which the heat exchanger was designed, one is often required to know the exit temperatures of the fluids. In this case, $\dot{m}_h$, $c_h$, $\dot{m}_c$, $c_c$, $T_{h_1}$, $T_{c_1}$, $U_0$ and $A_0$ are known, and $T_{c_2}$ and $T_{h_2}$ have to be estimated. To obtain the solution of this problem by the LMTD method, a *trial-and-error approach* has to be attempted.

Let us first assume a value for $T_{h2}$. Using Eq. (8.18), we find $Q$, $T_{c_2}$ and $\Delta T_{1m}$. Then we estimate $Q' = U_0 A_0\,\Delta T_{1m}$. If $Q = Q'$, the assumption of $T_{h2}$ was correct. If $Q'$ is different from $Q$, a fresh value of $T_{h2}$ is assumed and the calculations are repeated till the condition $Q = Q'$ is achieved.

The method is quite tedious and can be avoided by following an alternative direct method called the *effectiveness-NTU method*, as discussed below.

The term effectiveness $\varepsilon$ of a heat exchanger is defined as

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Effectiveness, $$\varepsilon = \frac{\text{Actual rate of heat transfer}}{\text{Maximum possible rate of heat transfer}}$$

or, $$\varepsilon = \frac{Q}{Q_{\max}} = \frac{\dot{m}_h c_h (T_{h_1} - T_{h_2})}{(\dot{m}c)_s (T_{h_1} - T_{c_1})}$$

$$= \frac{\dot{\mathbf{m}}_c \mathbf{c}_c (\mathbf{T}_{c_2} - \mathbf{T}_{c_1})}{(\dot{\mathbf{m}}\mathbf{c})_s (\mathbf{T}_{h_1} - \mathbf{T}_{c_1})} \tag{8.29}$$

where subscript "s" denotes the smaller of the two heat capacity rates $\dot{m}_h c_h$ and $\dot{m}_c c_c$, or $C_{\min}$. The maximum possible heat transfer depends on one of the fluids undergoing the maximum possible change in temperature and that will be the fluid which will have the minimum value of heat capacity rate.

If we allow the fluid with the larger value of $\dot{m}_c$ (or $C_{\max}$) go through the maximum temperature difference, then by energy balance,

$$(\dot{m}c)_{\max} (T_{h_1} - T_{c_1}) = \dot{m}_c c_c (T_{c_2} - T_{c_1})$$

Therefore $(T_{c_2} - T_{c_1})$ becomes greater than $(T_{h_1} - T_{C_1})$, which is impossible.

Since, $Q = \dot{m}_h c_h (T_{h_1} - T_{h_2}) = \dot{m} c_c (T_{c_2} - T_{c_1})$,

(i) if $\dot{m}_h c_h < \dot{m}_c c_c$ (Fig. 8.22)

$$(T_{h_1} - T_{h_2}) > (T_{c_2} - T_{c_1})$$

$$\varepsilon = \frac{\dot{m}_h c_h (T_{h_1} - T_{h_2})}{\dot{m}_h c_h (T_{h_1} - T_{c_1})} = \frac{T_{h_1} - T_{h_2}}{T_{h_1} - T_{c_1}} \tag{8.30}$$

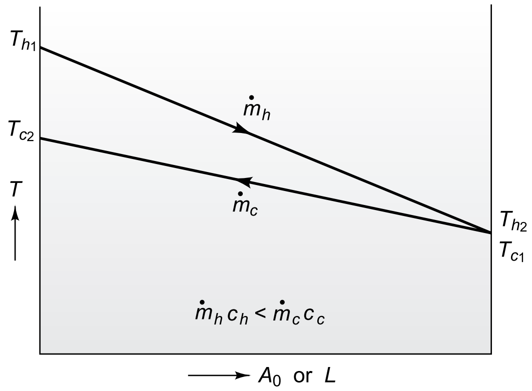


**Fig. 8.22** *For $\dot{m}_h c_h < \dot{m}_c c_c$, the hot fluid is the reference fluid for heat exchanger effectiveness*

(ii) if $\dot{m}_c c_c < \dot{m}_h c_h$ (Fig. 8.23)

$$(T_{c_2} - T_{c_1}) > (T_{h_1} - T_{h_2})$$

$$\varepsilon = \frac{\dot{\mathbf{m}}_c \mathbf{c}_c (\mathbf{T}_{c_2} - \mathbf{T}_{c_1})}{\dot{\mathbf{m}}_c \mathbf{c}_c (\mathbf{T}_{h_1} - \mathbf{T}_{c_1})} = \frac{\mathbf{T}_{c_2} - \mathbf{T}_{c_1}}{\mathbf{T}_{h_1} - \mathbf{T}_{c_1}} \tag{8.31}$$

Thus, the effectiveness can also be defined as

$$\varepsilon = \frac{(\Delta \mathbf{T})_1}{(\Delta \mathbf{T})_{\text{maximum}}} \tag{8.32}$$

where $(\Delta T)_1$ is the larger of the two temperature differences $(T_{h_1} - T_{h_2})$ and $(T_{c_2} - T_{c_1})$, and $(\Delta T)_{\max}$ is the maximum possible temperature rise or fall, which is $(T_{h_1} - T_{c_1})$.

The *heat capacity ratio R* is defined as

$$\mathbf{R} = \frac{(\dot{\mathbf{m}}\mathbf{c})_s}{(\dot{\mathbf{m}}\mathbf{c})_1} = \frac{\mathbf{C}_{\min}}{\mathbf{C}_{\max}} \tag{8.33}$$

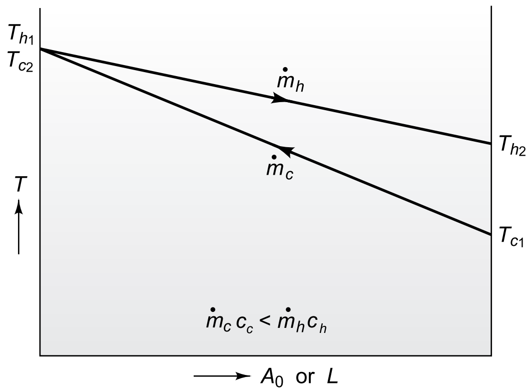


**Fig. 8.23** *For $\dot{m}_c c_c < \dot{m}_h c_h$, the cold fluid is the reference fluid for heat exchanger effectiveness*

where the subscripts "s" and "1" refer to the smaller and the larger of the two values of $\dot{m}_h c_h$ and $\dot{m}_c c_c$. If $\dot{m}_h c_h < \dot{m}_c c_c$,

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$$R = \frac{\dot{m}_h c_h}{\dot{m}_c c_c} = \frac{T_{c_2} - T_{c_1}}{T_{h_1} - T_{h_2}} \tag{8.33a}$$

The values of both $\varepsilon$ and $R$ vary between 0 and 1.

### 8.12.1 Parallel Flow Arrangement

From Eq. (8.11),

$$\ln \frac{\Delta T_i}{\Delta T_e} = U_0 A_0 \mu_p$$

$$\frac{\Delta T_e}{\Delta T_i} = e^{-U_0 A_0 \mu_p}$$

or,
$$1 - \frac{\Delta T_e}{\Delta T_i} = 1 - e^{-U_0 A_0 \mu_p} \tag{8.34}$$

For a parallel flow heat exchanger (Fig. 8.24),

$$1 - \frac{T_{h_2} - T_{c_2}}{T_{h_1} - T_{c_1}} = 1 - e^{-U_0 A_0 \mu_p}$$

or,
$$\frac{T_{h_1} - T_{c_1} - T_{h_2} + T_{c_2}}{T_{h_1} - T_{c_1}} = 1 - e^{-U_0 A_0 \mu_p} \tag{8.35}$$

Let $\dot{m}_h c_h < \dot{m}_c c_c$,

or, $(T_{h_1} - T_{h_2}) > (T_{c_2} - T_{c_1})$

From Eqs (8.30) and (8.32),

$$\varepsilon = \frac{T_{h_1} - T_{h_2}}{T_{h_1} - T_{c_1}}$$

and from Eq. 8.33,

$$R = \frac{T_{c_2} - T_{c_1}}{T_{h_1} - T_{h_2}} = \frac{\dot{m}_h c_h}{\dot{m}_c c_c}$$

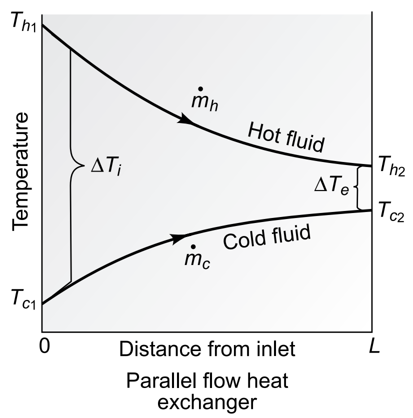


**Fig. 8.24** *Temperature profiles in a parallel flow heat exchanger*

Equation (8.35) can be written in the following form,

$$\frac{(T_{h_1} - T_{h_2}) + (T_{c_2} - T_{c_1})}{(T_{h_1} - T_{h_2})/\varepsilon} = 1 - e^{-U_0 A_0 \mu_p}$$

or,
$$\varepsilon \left(1 + \frac{T_{c_2} - T_{c_1}}{T_{h_1} - T_{h_2}}\right) = 1 - \exp\left[-U_0 A_0 \left(\frac{1}{\dot{m}_h c_h} + \frac{1}{\dot{m}_c c_c}\right)\right]$$

or,
$$\varepsilon(1 + R) = 1 - \exp\left[-\frac{U_0 A_0}{\dot{m}_h c_h}\left(1 + \frac{\dot{m}_h c_h}{\dot{m}_c c_c}\right)\right]$$

$$\boldsymbol{\varepsilon_{pf} = \frac{1 - \exp\left[-\mathrm{NTU}\,(1 + R)\right]}{1 + R}} \tag{8.36}$$

where NTU denotes the *number of transfer units* defined by

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$$\mathbf{NTU} = \frac{\mathbf{U_0 A_0}}{(\dot{\mathbf{m}}\mathbf{c})_s} = \frac{\mathbf{U_0 A_0}}{\mathbf{C_{min}}} \tag{8.37}$$

$(\dot{m}c)_s$ or $C_{\min}$ being equal to $\dot{m}_h c_h$ in this case. NTU gives a measure of the size of the heat exchanger.

### 8.12.2 Counterflow Arrangement

From Eq. (8.17),

$$\ln \frac{\Delta T_i}{\Delta T_e} = U_0 A_0 \mu_c$$

or,

$$\frac{\Delta T_e}{\Delta T_i} = e^{-U_0 A_0 \mu_c}$$

For a counterflow heat exchanger, (Fig. 8.25)

$$\frac{T_{h_2} - T_{c_1}}{T_{h_1} - T_{c_2}} = e^{-U_0 A_0 \mu_c} \tag{8.38}$$

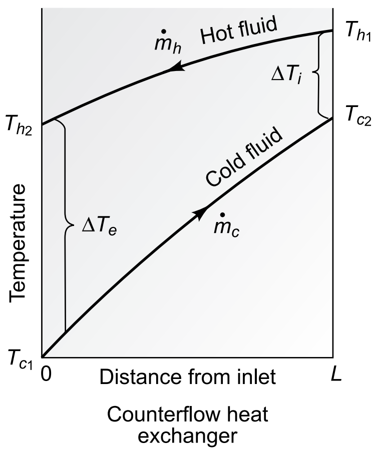


**Fig. 8.25** *Temperature profiles in a counter flow heat exchanger*

Let, $\dot{m}_h c_h < \dot{m}_c c_c$,

then

$$R = \frac{\dot{m}_h c_h}{\dot{m}_c c_c} = \frac{T_{c_2} - T_{c_1}}{T_{h_1} - T_{h_2}}$$

and

$$\varepsilon = \frac{T_{h_1} - T_{h_2}}{T_{h_1} - T_{c_1}}$$

$$\varepsilon R = \frac{T_{c_2} - T_{c_1}}{T_{h_1} - T_{c_1}} \tag{8.39}$$

Equation (8.38) can be written in the form

$$\frac{(T_{h_1} - T_{c_1}) - (T_{h_1} - T_{h_2})}{(T_{h_1} - T_{c_1}) - (T_{c_2} - T_{c_1})} = \exp\left[-U_0 A_0 \left(\frac{1}{\dot{m}_h c_h} - \frac{1}{\dot{m}_c c_c}\right)\right]$$

or,

$$\frac{1 - \dfrac{(T_{h_1} - T_{h_2})}{(T_{h_1} - T_{c_1})}}{1 - \dfrac{(T_{c_2} - T_{c_1})}{(T_{h_1} - T_{c_1})}} = \exp\left[-\frac{U_0 A_0}{\dot{m}_h c_h}\left(1 - \frac{\dot{m}_h c_h}{\dot{m}_c c_c}\right)\right]$$

or,

$$\frac{1 - \varepsilon}{1 - \varepsilon R} = \exp\left[-\mathrm{NTU}\,(1 - R)\right] \tag{8.40}$$

Let the RHS of Eq. (8.40) be equal to $K$.

$$\frac{1 - \varepsilon}{1 - \varepsilon R} = K$$

or,

$$K - \varepsilon R K = 1 - \varepsilon$$

$$\varepsilon (1 - RK) = 1 - K$$

$$\varepsilon = \frac{1 - K}{1 - RK}$$

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$$\varepsilon_{cf} = \frac{1 - \exp\left[-\text{NTU}\,(1 - R)\right]}{1 - R\exp\left[-\text{NTU}\,(1 - R)\right]} \tag{8.41}$$

(1) To find the exit fluid temperature:

The quantities $(\dot{m}_h c_h)$ and $(\dot{m}_c c_c)$ are first computed.

If $\dot{m}_c c_c < \dot{m}_h c_h$, then,

$$R = \frac{\dot{m}_c c_c}{\dot{m}_h c_h} \text{ and NTU} = \frac{U_0 A_0}{\dot{m}_c c_c} \text{ are calculated.}$$

From Eq. (8.36) or (8.41) $\varepsilon_{pf}$ or $\varepsilon_{cf}$ is obtained according to the flow arrangement given in the problem.

Since, $$\varepsilon = \frac{(\Delta T)_1}{T_{h_1} - T_{c_1}} = \frac{T_{c_2} - T_{c_1}}{T_{h_1} - T_{c_1}}$$

$T_{c_2}$ can be computed, and from the equation

$$Q = \dot{m}_h\, c_h\, (T_{h_1} - T_{h_2}) = \dot{m}_c\, c_c - (T_{c_1} - T_{c_1})$$

$T_{h_2}$ is determined.

(2) To determine the surface area required:

If $\dot{m}_h$, $c_h$, $\dot{m}_c$, $c_c$, $T_{h_1}$, $T_{c_1}$, $U_0$ and $T_{h_2}$ are known, $T_{h_2}$ is first determined from energy balance. If $\dot{m}_h c_h < \dot{m}_h c_c$ or $(T_{h_1} - T_{h_2}) > (T_{c2} - T_{c_1})$

$$R = \frac{\dot{m}_h c_h}{\dot{m}_c c_c} \text{ and } \varepsilon = \frac{T_{h_1} - T_{h_2}}{T_{h_1} - T_{c_1}}$$

are computed. From Eq. (8.36) or Eq. (8.41), NTU is determined, and then $A_0$ can be estimated.

Thus, for both types of problems normally encountered with heat exchangers, the effectiveness-NTU approach provides a direct solution. However, designers still prefer the LMTD method because $F$ gives them a direct measure of the penalty for deviation from counter flows. If $F$ is low (say < 0.8), the *HX* will not be economical, hence they will look for other choices.

### 8.12.3 Limiting Cases

Two limiting cases are of interest.

1. For $R = 0$ (phase change heat exchanger)

When one of the fluids undergoes phase change, as in a condenser or evaporator, $(\dot{m}c)_1 = \infty$

$$R = \frac{(\dot{m}c)_s}{(\dot{m}c)_1} = \frac{C_{\min}}{C_{\max}} = \frac{(\Delta T)_s}{(\Delta T)_1} = 0$$

From Eq. (8.36),

$$\varepsilon_{pf} = 1 - \exp\,(-\text{NTU}) \tag{8.42}$$

and from Eq. (8.41),

$$\varepsilon_{cf} = 1 - \exp\,(-\text{NTU}) \tag{8.43}$$

Thus, the same expression of effectiveness holds good both for parallel flow and counterflow heat exchangers.

(2) For $R = 1$ (balanced heat exchanger)

When heat capacity rates of the two fluids are equal,

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$$R = \frac{(\dot{m}c)_s}{(\dot{m}c)_l} = 1$$

For parallel flow heat exchanger, from Eq. (8.36), putting $R = 1$,

$$\boldsymbol{\varepsilon_{\text{pf}} = \frac{1 - \exp(-2\,\text{NTU})}{2}} \tag{8.44}$$

For counterflow heat exchanger, from Eq. (8.41), putting $R = 1$, $\varepsilon$ becomes indeterminate.

Since,
$$\dot{m}_h c_h = \dot{m}_c c_c$$
$$T_{h_1} - T_{h_2} = T_{c_2} - T_{c1}$$

or,
$$T_{h_1} - T_{c_2} = T_{h_2} - T_{c_2} = \Delta T_{1m} \text{ (as shown in Ex. 8.2)}$$
$$Q = U_0 A_0 (T_{h_1} - T_{c_2}) = \dot{m}_h c_h (T_{h_1} - T_{h_2})$$

$$\text{NTU} = \frac{U_0 A_0}{\dot{m}_h c_h} = \frac{T_{h_1} - T_{h_2}}{T_{h_1} - T_{c_2}}$$

$$\varepsilon = \frac{T_{h_1} - T_{h_2}}{T_{h_1} - T_{c_1}} = \frac{T_{h_1} - T_{h_2}}{(T_{h_1} - T_{h_2}) + (T_{h_2} - T_{c_1})}$$

$$= \frac{\text{NTU}\,(T_{h_1} - T_{c_2})}{\text{NTU}(T_{h_1} - T_{c_2}) + (T_{h_1} - T_{c_2})} = \frac{\text{NTU}}{\text{NTU} + 1}$$

or,
$$\boldsymbol{\varepsilon_{\text{cf}} = \frac{\text{NTU}}{\text{NTU} + 1}}$$

This expression can also be obtained from Eq. (8.41) by L'Hospital rule,

$$\lim_{R \to 1} \varepsilon = \lim_{R \to 1} \frac{d/dR\,[1 - e^{-\text{NTU}(1-R)}]}{d/dR\,[1 - Re^{-\text{NTU}(1-R)}]}$$

or,
$$\varepsilon_{\text{cf}} = \frac{\text{NTU}}{\text{NTU} + 1} \tag{8.45}$$

### 8.12.4 Charts of Kays and London

Kays and London [4] presented effectiveness charts for various types of heat exchangers with NTU and $R$ as the variable parameters (Figs 8.26–8.30). In the case of an evaporator or a condenser, when a fluid remains at a constant temperature throughout the heat exchanger, its heat capacity rate is considered infinite, hence $c_{\max} \to \infty$ and $c_{\min}/c_{\max} = 0$.

The charts can be conveniently used in the heat exchanger calculations.

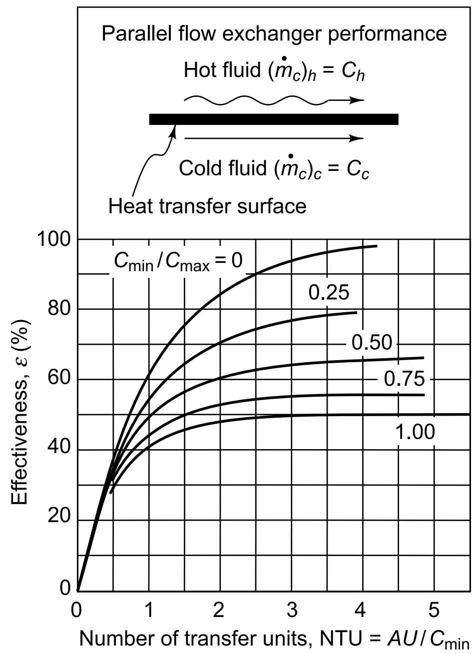


**Fig. 8.26** *Heat exchanger effectiveness of parallel flow*

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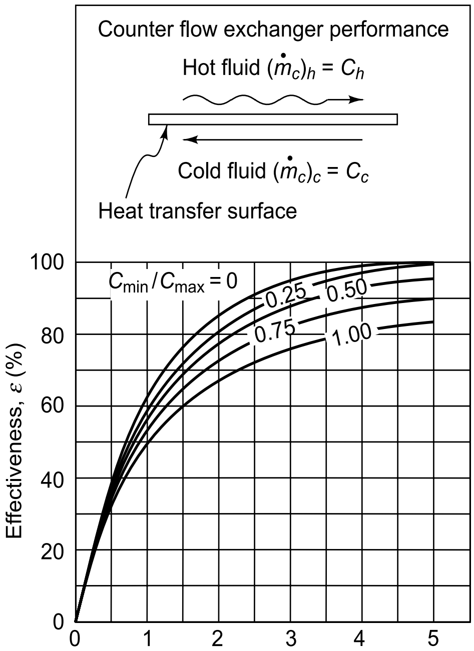


**Fig. 8.27** *Heat exchanger effectiveness for counterflow (4)*

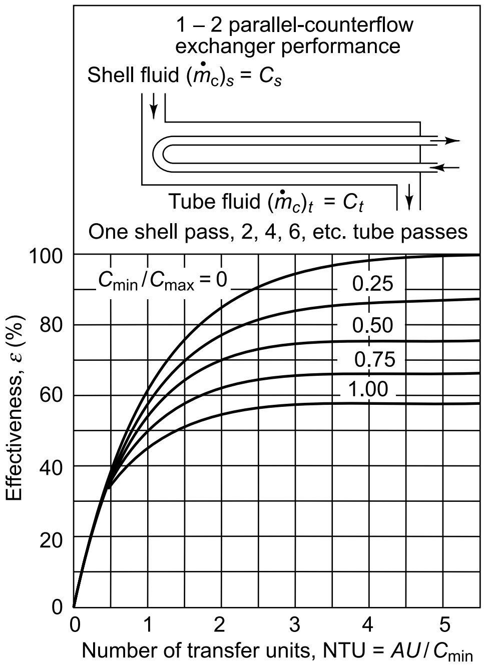


**Fig. 8.28** *Heat exchanger effectiveness for shell-and-tube heat exchanger with one well-baffled shell pass and two (for a multiple of two) tube passes (4)*

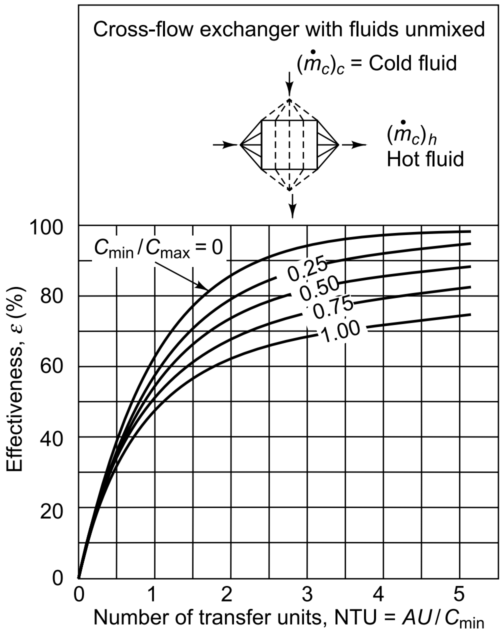


**Fig. 8.29** *Heat exchanger effectiveness for cross-flow with both fluids unmixed (4)*

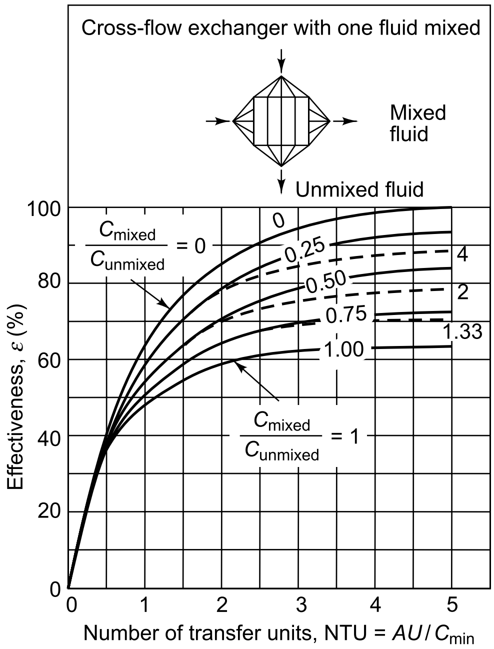


**Fig. 8.30** *Heat exchanger effectiveness for cross-flow with one fluid mixed and the other unmixed. When $C_{mixed}/C_{unmixed} > 1$, NTU is based on $C_{unmixed}$ [4]*

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## 8.13 PLATE HEAT EXCHANGER

A plate heat exchanger (Fig. 8.10) consists of a number of corrugated metal sheets [Fig. 8.32 (a)] provided with gaskets and corner portals, each fluid passing through alternate channels [Fig. 8.32(b)]. The fluids are at all times separated by two gaskets, each open to the atmosphere. The plates are clamped together in a frame that includes connections for the fluids. Greater compactness and accessibility of heat transfer surfaces compared to shell-and-tube heat exchangers and the ease of adding or removing plates for different heat transfer requirements have made the plate heat exchangers widely popular, particularly in dairy and food industries. The plates cannot, however, withstand high operating pressures, the maximum pressure being about 20 atm. A typical expression for heat transfer under turbulent conditions is [5]

$$\text{Nu} = 0.2536\, (\text{Re})^{0.65}\, (\text{Pr})^{0.4} \tag{8.46}$$

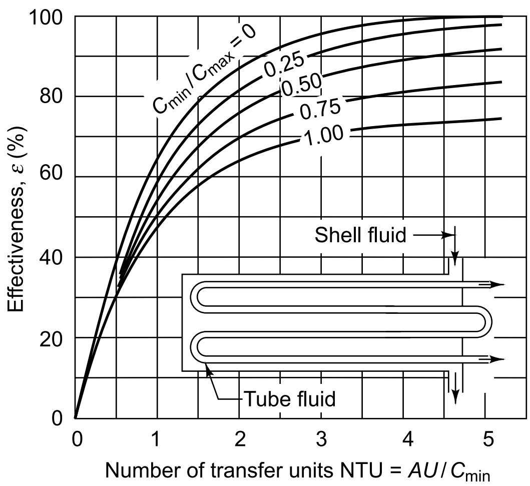


**Fig. 8.31** *Heat exchanger effectiveness for shell-and-tube heat exchanger with two shell passes and 4, 8, 12, ... tube passes*

## 8.14 HEAT TRANSFER ENHANCEMENT

Fins are provided to increase heat transfer surface area and to compensate for low heat transfer coefficient, particularly in natural convection heat transfer. Heat transfer coefficient is proportional to fluid velocity:

$$\left.\begin{aligned} &\text{for laminar flow, } h \propto V^{0.5} \\ &\text{for turbulent flow, } h \propto V^{0.8} \end{aligned}\right\} \text{on a plate}$$

Natural convection is most often laminar and heat transfer coefficient is low, particularly if it is a gas. For liquid-to-gas heat transfer, fins are provided on the gas side.

The various augmentative techniques to enhance heat transfer have been vividly discussed by Bergles [6]. There are three methods: (1) Passive, (2) Active and (3) Combined active and passive. Passive methods operate without much increase of power consumption, like finned or roughened surfaces. Active methods employ supplementary external power for, say, vibration or rotation of the surface, mechanical stirrer, incorporating pulsation or swirl to the fluid, sound vibrations in the fluid from 1 Hz to ultrasonic, electrostatic or magnetic field and so on. Combined methods employ both passive and active techniques like rough surface tubes with swirlers, vibrating finned tubes, etc.

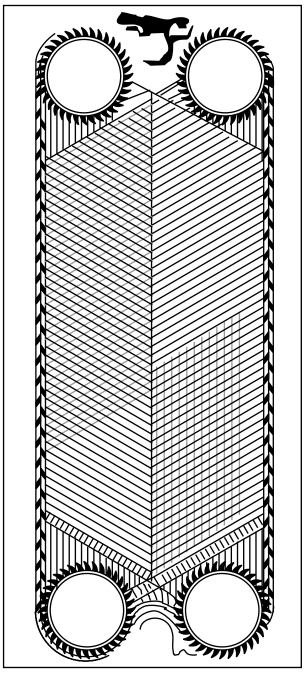

**Fig. 8.32(a)** *Plate heat exchanger built up of corrugated plates assembled to optimize heat transfer*

Increase in heat transfer due to surface treatment is brought about by increased turbulence, increased surface area, improved mixing, or flow swirl. These effects cause an increase in pressure drop and

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hence pumping power ($\dot{V}\ \Delta p$ watts, where $V$ is the volume flow of fluid, m$^3$/s, and $\Delta p$ is the pressure drop, N/m$^2$). Heat transfer enhancement is gaining industrial importance because it gives one an opportunity to (1) reduce the heat transfer surface area required for a given application and thus reduce its size and cost, (2) increase the heat duty of the exchanger and (3) permit closer approach temperatures. Since $Q = UA$ LMTD, any enhancement technique which increases heat transfer coefficient increases $U$. Therefore, one can either reduce surface area $A$, increase the heat duty $Q$ or decrease the temperature difference LMTD.

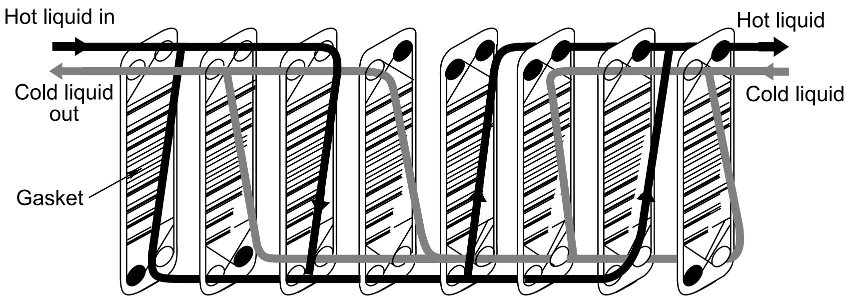


**Fig. 8.32(b)** *Corrugated metal sheet*

The entrance region of a pipe is often roughened to hasten turbulent flow for higher heat transfer in forced convection. Sometimes, a twisted tape swirl flow device is inserted into a tube (Fig. 8.33) to increase flow velocity, generate secondary flow or increase flow path length. Twisted tape insert is often used in high pressure boiler riser tubes to carry away the bubbles formed on the inside wall so as to prevent a stable film from being formed. Figure 8.34 compares the performance of four enhancement techniques for single-phase forced convection in a tube with that for a smooth tube [6].

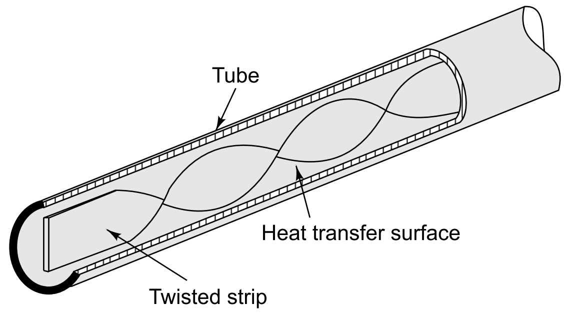


**Fig. 8.33** *Twisted tape insert to unhance internal heat transfer*

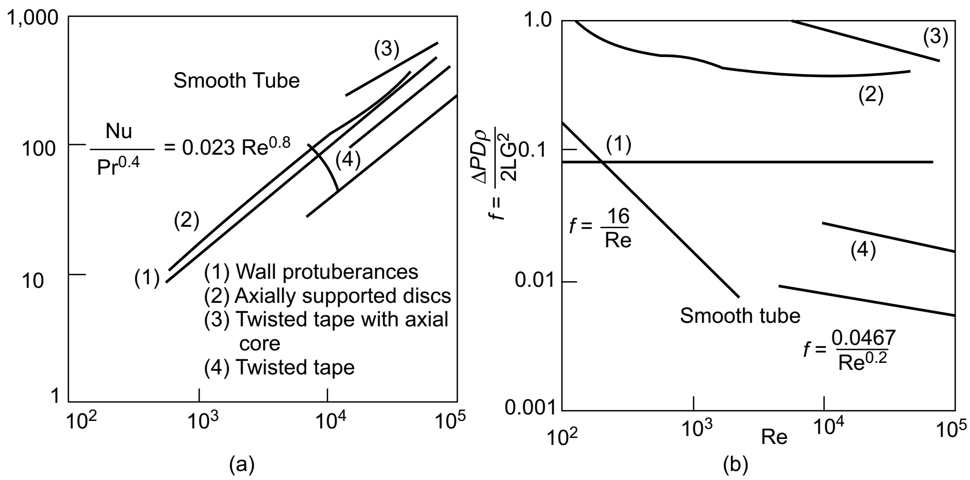


**Fig. 8.34** *Typical data for turbulence promoters inserted inside tubes, (a) Heat transfer data; (b) friction data [6]*

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## 8.15 HEAT PIPES

A heat pipe is basically a sealed container, normally in the form of a tube, containing a wick lining the inside wall (Fig. 8.35). It is used to transfer heat from the source to the sink by means of evaporation and condensation of a fluid in the sealed system. The purpose of the wick is to transport the working fluid in liquid form from one end to the other by capillary action. The heat pipe can transfer heat much more effectively than a solid conductor of the same cross-section. The thermal conductance may be 500 times the best available metallic conductor. The sequence of operation is as follows: 1. Heat in, 2. Evaporation, 3. Vapour flow, 4. Condensation, 5. Heat out and 6. Liquid return in wick.

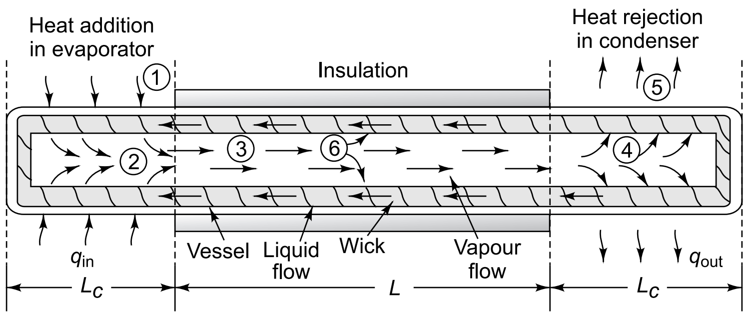


**Fig. 8.35** *Schematic diagram of a heat pipe and the associated heat flow mechanisms*

The heat pipe has (a) no moving parts, (b) requires no external energy other than the heat it transits, (c) is reversible in operation, (d) completely silent and (e) is rugged like any piece of tube or pipe and can withstand a lot of abuse.

The heat pipe is particularly sensitive to the effects of gravity, and hence its inclination to the horizontal (Fig. 8.36). Figure 8.36(a) is much more effective than Fig. 8.36(b). When gravity aids return of the condensate to the evaporator section, the wick may be omitted. Then the device is known as *thermosiphon*.

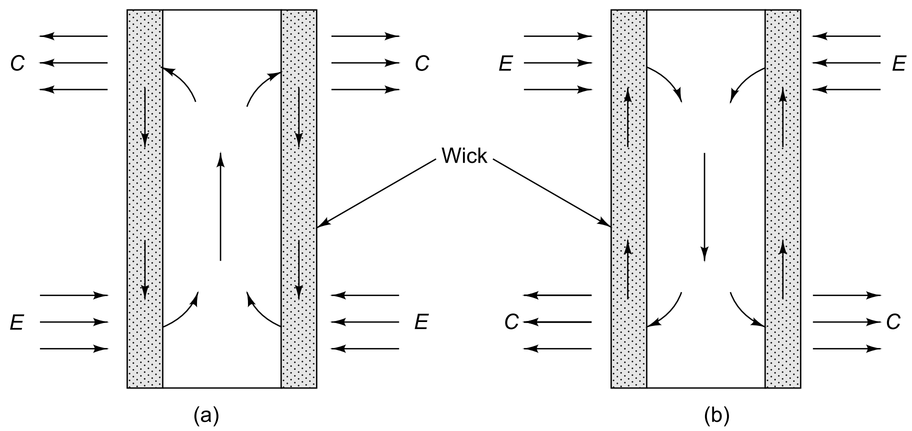


**Fig. 8.36** *(a) Return of condensate aided by gravity, (b) Return of condensate against gravity, (a) is superior to (b)*

Then why is wick needed if (a) is better than (b). (1) The wick provides axial pumping. (2) It distributes the liquid circumferentially and ensures that in the evaporator section, the available surface is wetted so

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that it can support all the radial heat flux. (3) The wick itself, particularly if it is integral with the wall of the heat pipe in the form of grooves, may increase $h_i$ and promote heat transfer. (4) The wick can inhibit or resist entrainment. Entrainment occurs when the shear force between the countercurrent liquid and vapour flows along the heat pipe is sufficient for the vapour to entrain droplets of liquid and carry them back to the condenser. It restricts the heat transport capability of the heat pipe.

Heat pipes are extensively used for waste heat recovery, particularly in gas-to-gas heat recovery. Finned heat pipes are mostly used, since gas-convective coefficient is low (Fig. 8.37). A splitter plate (Fig. 8.38) supports the heat pipes, which can be several meters long, and prevents cross-flow between the streams, effectively sealing them from one another.

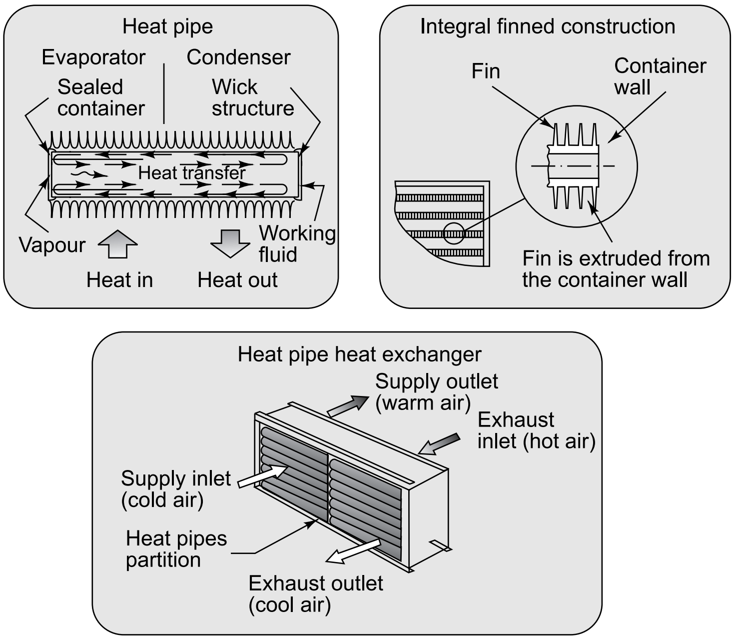


**Fig. 8.37** *Finned heat pipe heat exchanger*

The heat pipe is rated by its "axial power rating" (APR), which is the energy moving axially along the pipe. Heat pipes of different APR like electric heaters are available in standard sizes (1 kW, 10 kW and so on).

The performance of heat pipe depends on the angle of operation. The tube bundle may be horizontal or tilted with the evaporator section below the condenser. Because of this sensitivity, the angle of heat pipes may be adjusted in situ as a means of controlling the heat transport

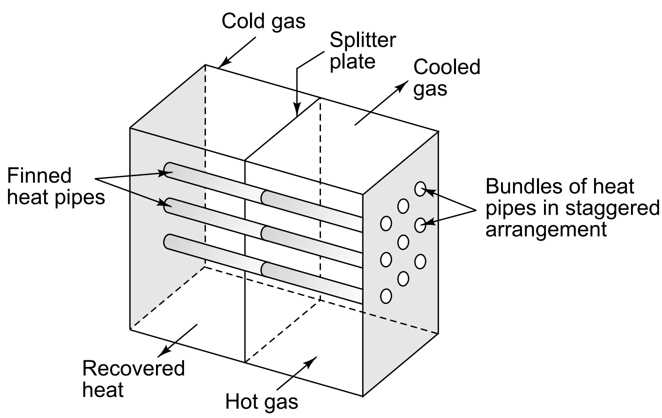


**Fig. 8.38** *Splitter plate in heat pipes for gas-to-gas heat recovery*

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(Fig. 8.39). A number of proprietary units incorporate tilt control mechanisms which can be adjusted either manually or automatically to cater for changes in heat transfer requirements.

A large variety of fluid and pipe material combinations have been used for heat pipes, and some typical working fluid and material combinations, as well as the temperature ranges over which they can operate are presented in Table 8.3 [7, 8].

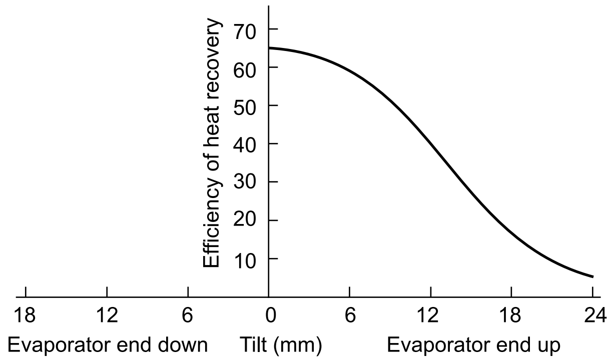


**Fig. 8.39** *Effect of heat pipe tilt on heat recovery performance*

**Table 8.3** *Some typical operating characteristics of heat pipes*

| *Temperature range (K)* | *Working fluid* | *Vessel material* | *Measured axial heat flux (W/cm²)* | *Measured surface heat flux (W/cm²)* |
|---|---|---|---|---|
| 230–400 | Methanol | Copper nickel stainless-steel | 0.45 at 373 K | 75.5 at 373 K |
| 280–500 | Water | Copper nickel | 0.67 at 473 K | 146 at 443 K |
| 360–850 | Mercury | Stainless-steel | 25.1 at 533 K | 181 at 533 K |
| 673–1073 | Potassium | Nickel stainless-steel | 5.6 at 1023 K | 181 at 1023 K |
| 773–1173 | Sodium | Nickel stainless-steel | 9.3 at 1123 K | 224 at 1033 K |

In order for a heat pipe to operate, the maximum capillary pumping head, $(\Delta p_c)_{max}$, must be able to overcome the total pressure drop in the heat pipe. This pressure drop consists of three parts.

1. The pressure drop required to return the liquid from the condenser to the evaporator, $\Delta p_1$.
2. The pressure drop required to move the vapour from the evaporator to the condenser, $\Delta p_v$.
3. The potential head due to the difference in elevation between the evaporator and the condenser, $\Delta p_g$.

The pressure equilibrium can thus be expressed as

$$(\Delta p_c)_{max} > \Delta p_1 + \Delta p_v + \Delta p_g \tag{8.47}$$

If this condition is not satisfied, the wick will dry out in the evaporator region and the heat pipe will cease to operate.

The liquid pressure drop in a flow through a homogeneous wick can be calculated from the empirical relations

$$\Delta p_1 = \frac{\mu_1 L_{eff} \dot{m}}{\rho_1 K_w A_w} \tag{8.48}$$

where $\mu_1$ = liquid viscosity, $\dot{m}$ = mass flow rate, $\rho_1$ = liquid density, $A_w$ = wick cross-sectional area, $K_w$ = wick permeability or wick factor and $L$ = effective length of the evaporator and condenser, given by

$$L_{eff} = L + \frac{L_e + L_c}{2} \tag{8.49}$$

The vapour pressure drop is generally small if vapour velocity is less than 30% of sound velocity (i.e. Mach number < 0.3), the compressibility effect can be neglected. For laminar incompressible flow,

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$$\Delta p_v = f\frac{L_{\text{eff}}}{D}\rho\bar{u}^2 = \frac{64\mu_v \dot{m} L_{\text{eff}}}{\rho_v \pi D_v^4} \tag{8.50}$$

where $D_v$ is the inside wick diameter in contact with the vapour.

The pressure difference ($\Delta p_g$) due to hydrostatic or potential head of the liquid may be positive, negative or zero, depending on the relative positions of the evaporator and condenser. Therefore,

$$\Delta p_g = \rho_1 gL \sin\phi \tag{8.51}$$

where $\phi$ is the angle between the axis of the heat pipe and the horizontal (positive when the evaporator is above the condenser).

The driving force in the wick is the result of surface tension. A molecule near the surface of a liquid will experience a force directed inward due to the attraction of neighbouring molecules below. The pressure on a concave surface is less than that on a convex surface. The resulting pressure difference $\Delta p$ is related to the surface tension $\sigma_1$ and the radius of curvature $r_c$. For a hemispherical surface like a half bubble (Fig. 8.40), the force balance gives

$$\Delta p \cdot \pi r_c^2 = 2\pi r_c \sigma_1$$

$$\Delta p = \frac{2\sigma_1}{r_c} \tag{8.52}$$

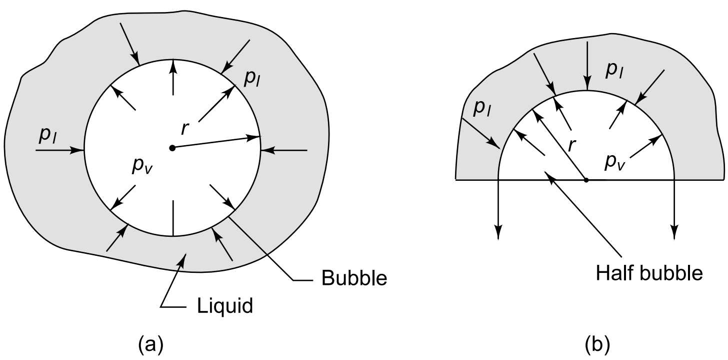


**Fig. 8.40** *A vapour bubble in equilibrium with liquid*

Another illustration of surface tension is observed when a capillary tube is placed vertically in a wetting fluid. The fluid will rise in the tube due to the capillary action (Fig. 8.41). A pressure balance gives

$$\Delta p_c = \rho_1 gh = \frac{2\sigma_1}{r_c}\cos\theta \tag{8.53}$$

where $\theta$ is the contact angle, which varies between 0 and $\pi/2$ for wetting fluids. For a non-wetting fluid, $\theta$ is larger than $\pi/2$, and the liquid level in the capillary tube is depressed below the surface. Hence, to obtain a capillary driving force *only wetting fluids can be used in heat pipes*.

Substituting Eqs (8.48) – (8.51) in Eq. (8.47), one of the key design criteria for heat pipes is yielded:

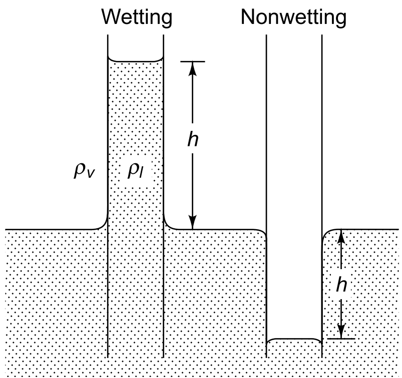


**Fig. 8.41** *Capillary rise in a tube*

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$$\frac{2\sigma_1 \cos\theta}{r_c} = \frac{\mu_1 L_{\text{eff}}\dot{m}}{\rho_1 K_w A_w} + \frac{64\mu_v \dot{m} L_{\text{eff}}}{\rho_v \pi D_v^4} + \rho_1 g L_{\text{eff}} \sin\phi \tag{8.54}$$

If $(64\,\mu_v/\rho_v \pi \Delta_v^4) << (\mu_1/\rho_1 K_w\, A_w)$ the pressure drop of the vapour ($\Delta p_v$) is negligible and the second term in Eq. (8.54) can be deleted.

The maximum heat transport capability of a heat pipe due to wicking limitations is given by

$$Q_{\max} = \dot{m}_{\max}\, h_{fg} \tag{8.55}$$

where $\dot{m}_{\max}$ can be obtained from Eq. (8.54). Assuming perfect wettability $\theta = 0$ or $\cos\theta = 1$, the following expression can be obtained,

$$Q_{\max} = \left(\frac{\rho_1 \sigma_1 h_{fg}}{\mu_1}\right)\left(\frac{A_w K_w}{L_{\text{eff}}}\right)\left(\frac{2}{r_c} - \frac{\rho_1 g L_{\text{eff}} \sin\phi}{\sigma_1}\right) \tag{8.56}$$

In the above equation all the terms in the first parentheses are properties of the working fluid, and the group is called the *figure of merit M*:

$$M = \frac{\rho_1 \sigma_1 h_{fg}}{\mu_1} \tag{8.57}$$

It is plotted in Fig. 8.42 as a function of temperature for a number of heat pipe fluids.

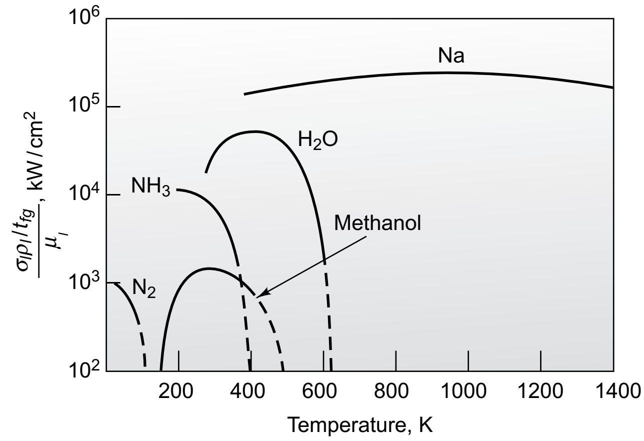


**Fig. 8.42** *Figure of merit for several heat pipe working fluids as a function of temperature*

The wick geometric properties are functions of $A_w$, $K_w$ and $r_c$. Table 8.4 presents data for pore size and permeability for a few wick materials and mesh sizes.

**Table 8.4** *Heat pipe wick pore size and permeability*

| *Material and mesh size* | *Capillary height (cm) (obtained with water)* | *Pore radius $r_c$ (cm)* | *Permeability (m²)* | *Porosity (%)* |
|---|---|---|---|---|
| Glass fibre | 25.4 | — | $0.061 \times 10^{-11}$ | — |
| Monel beads | | | | |
| 30–40 | 14.6 | 0.052 | $4.15 \times 10^{-10}$ | 40 |
| 70–80 | 39.5 | 0.019 | $0.78 \times 10^{-10}$ | 40 |

(*Contd*)

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| *Material and mesh size* | *Capillary height (cm) (obtained with water)* | *Pore radius $r_c$ (cm)* | *Permeability ($m^2$)* | *Porosity (%)* |
|---|---|---|---|---|
| Felt metal | 10.0 | 0.004 | $1.55 \times 10^{-10}$ | — |
| Nickel powder | | | | |
| 200 m | 24.6 | 0.038 | $0.027 \times 10^{-10}$ | — |
| Nickel fibre | | | | |
| 0.01 mm diameter | 40.0 | 0.001 | $0.015 \times 10^{-11}$ | 68.9 |
| Copper powder (sintered) | | | | |
| 45–56 m | 156.8 | 0.0009 | $1.74 \times 10^{-12}$ | 52.0 |
| Phosphor bronze | — | 0.0021 | $0.296 \times 10^{-10}$ | — |

A widely used correlation between the maximum achievable power transfer by a heat pipe and its dominant dimensions and operating parameters is

$$Q_{\max} = \frac{A_w h_{fg} g \rho_1^2}{\mu_1}\left(\frac{l_w K_w}{L_{\text{eff}}}\right) \tag{8.58}$$

where $l_w$ is the wicking height of fluid in wick given by

$$l_w = \frac{2\sigma_1}{r_c \rho_1 g} \tag{8.59}$$

The maximum wicking height with sodium as the working fluid is about 38.5 cm, which is estimated by assuming an effective pore diameter of $8.6 \times 10^{-3}$ cm. This is typical of a screen made with eight $4.1 \times 10^{-3}$ cm diameter wires per square millimeters.

The dominant parameters affecting the total power transfer capacity are the wick area, effective wicking height and the heat pipe length.

Although a heat pipe behaves like a structure of very high thermal conductivity, it has heat transfer limitations which are governed by certain principles of fluid mechanics, as shown in Fig. 8.43, for a liquid metal working fluid, viz. (a) Sonic limit, (b) Entrainment limit, (c) Wicking limit and (d) Boiling limit [8].

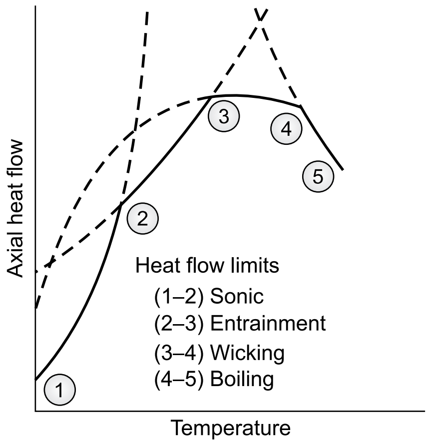


**Fig. 8.43** *Limitations to heat transport in a heat pipe*

### 8.15.1 Sonic Limitation

Since $Q = \dot{m}_v h_{fg}$, and for high density low velocity vapour, only small gradients are necessary to move the vapour, the heat transfer is nearly isothermal. Substituting $m_v = \rho_v \overline{u} A_v$,

$$\frac{Q}{A_v} = \rho_v \overline{u} h_{fg} \tag{8.60}$$

= Axial heat flux based on the cross-sectional area of the vapour passage

The axial heat flux $Q/A_v$ can be held constant and the condenser environment adjusted to lower the pressure, temperature and density of the vapour until the vapour flow at the evaporator exit becomes sonic.

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Once this occurs, pressure changes in the condenser will not be transferred to the evaporator. Although heat pipes are normally not operated at sonic flow, during start-up, it may occur when the temperatures of the evaporator inlet are higher than those at the evaporator exit. It is represented by the curve 1–2 in Fig. 8.43.

### 8.15.2 Entrainment Limitation

If the vapour density is allowed to increase without an accompanying decrease in velocity, some liquid from the wick return may be entrained, the onset of which can be expressed by *Weber number* (Wb) given by

$$\text{Wb} = \frac{\text{Inertia force}}{\text{Surface tension force}} = \frac{\rho_v \bar{u}^2 L_c}{2\pi\sigma_1} = 1$$

or,
$$\bar{u} = \left(\frac{2\pi\sigma_1}{\rho_v L_c}\right)^{1/2} \tag{8.61}$$

where $L_c$ is the characteristic length describing the pore size.

If Wb > 1, fluid circulation, increases until the liquid return path cannot accommodate the increased flow. This causes draught and overheating of the evaporator.

The entrainment limit can be estimated from

$$\frac{Q}{A_v} = \rho_v \bar{u} h_{fg} = \rho_v h_{fg}\left(\frac{2\pi\sigma_1}{\rho_v L_c}\right)^{1/2}$$

$$= h_{fg}\left(\frac{2\pi\sigma_1 \rho_v}{L_c}\right)^{1/2} \tag{8.62}$$

which is represented by the curve between 2 and 3 in Fig. 8.43.

### 8.15.3 Wicking Limitation

Fluid circulation in a heat pipe is maintained by capillary forces that develop in the wick structure at the liquid–vapour interface. When a typical meniscus is characterized by two principal radii of curvature $r_1$ and $r_2$, the pressure drop across the liquid surface is

$$\Delta p_c = \sigma\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$$

If the liquid wets the wick perfectly, the radii will be defined by the pore size of the wick, which fixes the limit of heat transfer. Any further increase in heat transfer will cause the liquid to retreat into the wick, and dryout and overheating will occur at the evaporator end. The capillary force can be increased by decreasing the pore size of the wick, as shown by Poiseuille equation

$$\Delta p_1 = \frac{8\pi \dot{m}_1 L}{\pi r^4 \rho}$$

The wicking limitation is represented by the solid line 3–4 in Fig. 8.43.

### 8.15.4 Boiling Limitation

Formation of vapour bubbles is undesirable because they could cause hot spots and destroy the action of the wick. Therefore, heat pipes are heated isothermally before being used to allow the liquid to wet the inner heat pipe wall and to fill all but the smallest nucleation sites.

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Boiling may occur at high input heat fluxes, and high operating temperatures. The curve between points 4 and 5 in Fig. 8.43 is based on the equations

$$p_v - p_1 = \frac{2\sigma}{r} \tag{8.63}$$

$$\frac{Q}{A} = \frac{k(T_w - T_v)}{t} \tag{8.64}$$

where $p_v$ = vapour pressure inside the bubble, $p_1$ = liquid pressure outside the bubble, $r$ = radius of the largest nucleation site, $A$ = heat input area, $k$ = thermal conductivity of the saturated wick, $T_w$ = temperature at the inside wall, $T_v$ = temperature at the liquid–vapour interface and $t$ = thickness of the first layer in the wick. If $r$ is small at the nucleation site, $\Delta p$ has to be large for bubbles to grow. The above two equations show the various factors which influence boiling. Boiling is, however, not a limitation with liquid metals, but when water is used as the working fluid, boiling may be a major heat transfer limitation because $k$ of water is low and it does not readily fill the nucleation sites.

### 8.15.5 Heat Pipe Components

Shell or vessel, wick and working fluid are the most important components of a heat pipe. Temperature, corrosion, erosion and environmental factors affect the selection of tube and fin material. Fin pitch and fin shape depend on allowable pressure drop and fouling. Air velocity of about 2–4 m/s is generally accepted to keep the pressure drop through the heat pipe bundle to a reasonable value. The length of heat pipe may be as high as 5 m, but more common are the tubes of 1–2 m. Wick design and performance become much more critical as heat pipe length increases [9].

The working fluids with compatible vessel materials are (a) Water in 50°C –200°C range in copper vessel, (b) Freons upto 50°C in aluminium shell, (c) Diphenyl oxide and other organic fluids upto 300°C in steel container, (d) Liquid metals like sodium upto 600°C in stainless steel and (e) Niobium or tantalum upto 1500°C in inconel or refractory vessel. An alternate to refractory metals is ceramic tubing. Ceramics such as silicon carbide and alumina have excellent corrosion and erosion resistance. Ceramic heat pipes with sodium as working fluid are used for waste heat recovery in industrial furnaces.

Glycerine, methanol, acetone, etc. can also be used as working fluids. Cryogenic heat pipes use water, liquid oxygen, liquid nitrogen, etc. as working fluids from –200°C onward. The working fluids should be (1) nontoxic, (2) noncorrosive, (3) less viscous, (4) of high surface tension, (5) of high latent heat and (6) chemically compatible with the vessel.

The wick can be a woven cloth, glass fibres, porous materials, sintered powder, wire mesh, grooved structure and so on. Screen wick consists of a few layers of wire. The wick must provide adequate capillary force for the liquid to return.

### 8.15.6 Applications of Heat Pipe

Heat pipes are versatile in removing localised heat and for waste heat recovery.

1. Electrical and electronic systems: Heat pipe can reduce the size of most magnetic components by 30% and more. Transformers are big in size because they need a lot of surface area to dissipate the heat generated. Heat pipes can be inserted into the transformer core to dissipate more heat and reduce the transformer size substantially. They can be used to remove heat from high voltage TV circuits, motor stators and armatures.
2. Airconditioning system: Exhaust air leaving the room can be used to preheat or precool the incoming air by heat pipe (Fig. 8.44).

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3. I.C. engines and gas turbines: One recent application in car engines is the VAPIPE. Heat pipes can be used to vaporise gasoline by the exhaust gases before it enters the engine via carburettor. The vaporised fuel makes a homogeneous mixture of fuel and air, and improves combustion. In a gas turbine, the exhaust is used to preheat the compressed air by heat pipes before it enters the combustion chamber *CC* (Fig. 8.45).

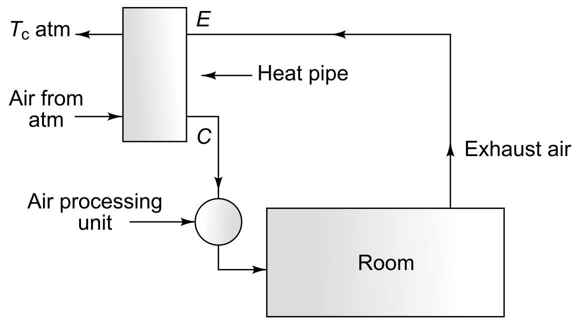


**Fig. 8.44** *Precooling or preheating of incoming air by heat pipe*

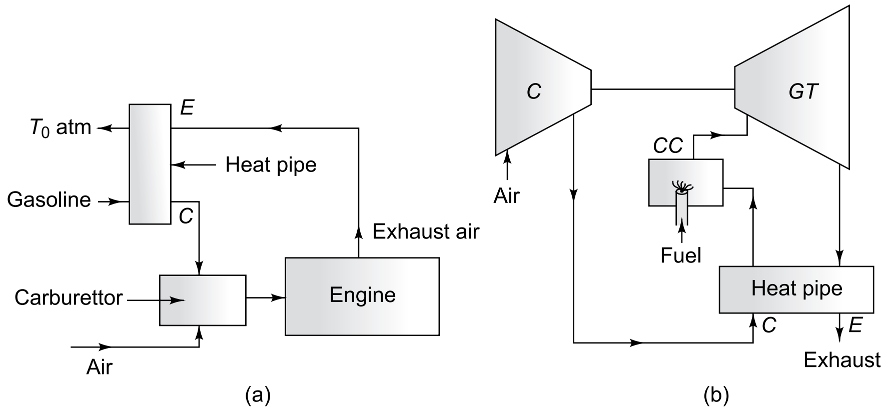


**Fig. 8.45** *Applications of heat pipe in (a) I.C. engine and (b) gas turbine plant*

4. Manufacturing systems: In extrusion of plastic materials the temperature gradient can be levelled by the use of heat pipe. The temperature uniformity can be maintained in texturising fibres.
5. Heat pipes are also used in solar collector, space application, snow melting, kitchen cooking (heat pipe inserted into the meat speeds up the heating process), spray drying, welding booths, pollution control, pharmaceutical, laundries, biscuit and bread ovens, brick kilns and so on.

## 8.16 RUN-AROUND COIL SYSTEMS

A run-around coil is a heat recovery system which connects two recuperative heat exchangers by a third fluid exchanging heat with each fluid in turn, as shown diagrammatically in Fig. 8.46.

A run-around coil can be used in cases where the two fluids which are required to exchange heat are too far apart to use a direct recuperative heat exchanger. It can also be used when there is a risk of cross-contamination between the two primary fluids. The intermediate secondary fluid can be suitably chosen to meet the desired heat transfer duty. One of the common examples of a run-around coil system is heat recovery from a fluid at one stage in a process to the same fluid at a different stage. Figure 8.47 shows such a system with temperature variations. By energy balance,

$$Q = (\dot{m}c)_h (T_{h_1} - T_{h_2}) = (\dot{m}c)_c (T_{c_1} - T_{c_2})$$
$$= (\dot{m}c)_s (T_{s1} - T_{s_2})$$

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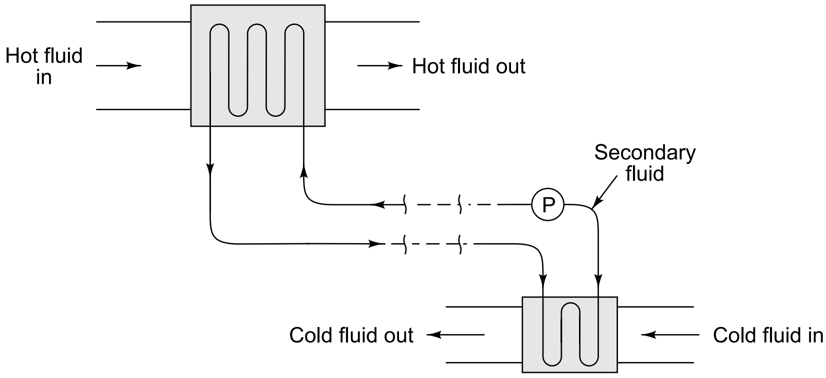


**Fig. 8.46** *Run-around coil system of heat recovery*

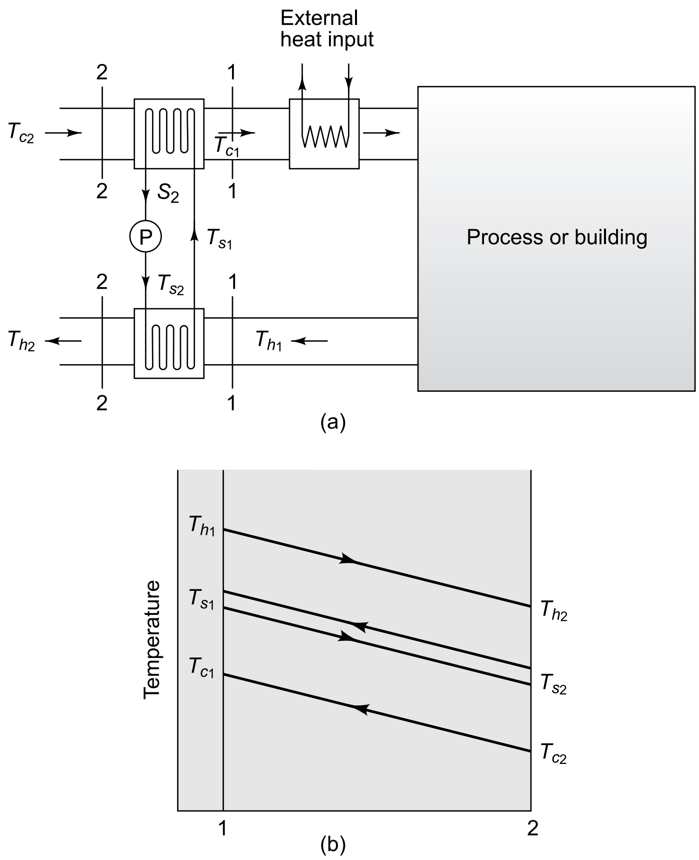


**Fig. 8.47** *Run-around coil heat recovery between fluids with the same thermal capacity*

Since the thermal capacity of the cold fluid ($\dot{m}c$) is the same as that of the hot fluid,

$$(\dot{m}c)_h = (\dot{m}c)_c = (\dot{m}c)_s$$

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Therefore, $T_{h_1} - T_{h_2} = T_{c_1} - T_{c_2} = T_{s_1} - T_{s_2}$

or, $T_{h_1} - T_{s_1} = T_{h_2} - T_{s_2}$

and $T_{s_1} - T_{c_1} = T_{s_2} - T_{c_2}$

The three temperature lines are therefore straight and parallel (Fig. 8.45) and $\Delta T_{1m} = \Delta T_i = \Delta T_e$

$$(UA)_h = (UA)_c = \frac{Q}{T_{h_1} - T_{s_1}} = \frac{Q}{T_{s_1} - T_{c_1}}$$

The temperature line for the secondary fluid must be midway between the temperature lines for the hot and cold fluids.

$$T_{s_t} = \frac{T_{h_1} + T_{c_1}}{2} \text{ and } T_{s_2} = \frac{T_{h_2} + T_{c_2}}{2}$$

and
$$Q = (UA)_h (T_{h_1} - T_{s_1}) = (UA)_c (T_{s_1} - T_{c_1})$$
$$= (UA)_0 \ (T_{s_1} - T_{c_1})$$

$\therefore$
$$(UA)_h = (UA)_c = 2\,(UA)_0 \tag{8.65}$$

When the inlet temperature of the hot and cold fluids are known, the total heat recovery is

$$Q = (UA)_h \frac{T_{h_1} - T_{c_1}}{2}$$

and since
$$T_{c_1} = T_{c_2} + \frac{Q}{(\dot{m}c)_c}$$

Thus,
$$Q = \frac{(UA)_h (T_{h_1} - T_{c_2})}{2 + [(UA)_h / (\dot{m}c)_c]} \tag{8.66}$$

When the thermal capacities of the hot and cold fluids are not equal (Fig. 8.48), it can be shown

$$(\text{LMTD})_0 = (\text{LMTD})_{h-s} + (\text{LMTD})_{s-c}$$

and therefore,
$$\frac{1}{(UA)_0} = \frac{1}{(UA)_h} + \frac{1}{(UA)_c} \tag{8.67}$$

At any cross-section,

$$\frac{T_s - T_c}{T_h - T_s} = \text{constant} = \frac{(UA)_h}{(UA)_c} = Y, \text{ say}$$

$\therefore$
$$Y = \frac{T_{s_1} - T_{c_1}}{T_{h_1} - T_{s_1}} = \frac{T_{s_2} - T_{c_2}}{T_{h_2} - T_{s_2}}$$

$\therefore$
$$T_{s_1} = (T_{c_1} + YT_{h_2})/(1 + Y)$$
$$T_{s_1} = (T_{c_1} + YT_{h_2})/(1 + Y)$$
$$T_{s_1} - T_{s_2} = \frac{(T_{c_1} - T_{c_2}) + Y(T_{h_1} - T_{h_2})}{1 + Y}$$
$$\frac{1}{(\dot{m}c)_s} = \frac{1/(\dot{m}c)_c + Y/(\dot{m}c)_h}{1 + Y}$$

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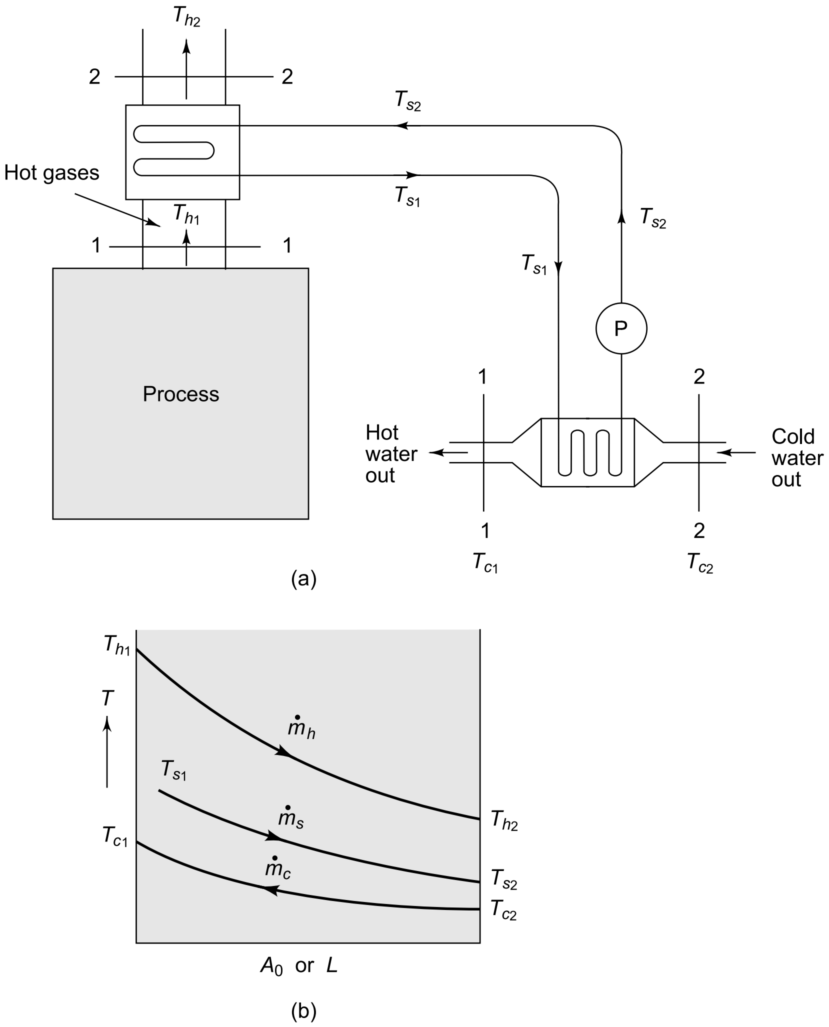


**Fig. 8.48** *Run-around coil heat recovery between fluids having different thermal capacities*

$$\dot{m}_s = \frac{(\dot{m}c)_h(\dot{m}c)_c[(UA)_h + (UA)_c]}{c_s[(\dot{m}c)_h(UA)_c + (\dot{m}c)_c(UA)_h]} \tag{8.68}$$

Mass flow rate of the secondary fluid is independent of the temperatures of the fluids and the heat transfer characteristics of the two heat exchangers.

The effectiveness and NTU are given by [11]

$$\varepsilon = \frac{1 - e^{-\mathrm{NTU}(1-R)}}{1 - R^{-\mathrm{NTU}(1-R)}} \tag{8.69}$$

and

$$\mathrm{NTU} = \frac{(UA)_h(UA)_c}{(\dot{m}c)_{\min}\,[(UA)_h + (UA)_c]} \tag{8.70}$$

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## 8.17 HEAT EXCHANGER DESIGN CONSIDERATIONS

For most small simple units operating at moderate pressures and temperatures, standard heat exchanger designs may be used. However, for special applications, individually designed units may be required. For a large variety of applications, the criteria for optimization depend on the minimum weight, minimum volume of heat transfer surface, minimum initial cost, minimum operating cost, maximum heat transfer rate, minimum pressure drop for a specified heat transfer rate, minimum mean temperature difference, and so on.

The first step in the optimization process is the solution of the rating and the sizing problems. The rating problem is concerned with the determination of the heat transfer rate, the outlet temperatures and the pressure drop on each side. The following quantities are generally specified in the rating problem: type of heat exchanger, surface geometries, flow arrangement, flow rates, inlet temperatures, and the overall dimensions of the matrix.

The sizing problem is concerned with the determination of the matrix dimensions to meet the specified heat transfer and pressure drop requirements. The designer's task is to select the type of construction, flow arrangement, and surface geometries on both sides. the following quantities are generally specified: fluid inlet and outlet temperatures, fluid pressure drops, and heat transfer rate.

Apart from the heat transfer requirements an important consideration in heat exchanger design, as cited earlier, is the pressure drop or pumping power. The size of the heat exchanger can be reduced by forcing the fluids through it at higher velocities thereby increasing the overall heat transfer coefficient. But higher velocities will result in large pressure drops ($\propto u^2$) and so larger pumping costs. The selection of optimum pump size also has an effect on the pumping costs. For a given flow rate, the smaller diameter pipe may involve less initial cost but higher operating or pumping cost. For an incompressible fluid, $\Delta P \propto \dot{m}^2$ and pumping power $\propto \dot{m}^3$, where $\dot{m}$ is the mass flow of the fluid. Since the pumping cost increases considerably with higher velocities, a compromise between the larger overall heat transfer coefficient and the corresponding velocities will have to be made. The cost of materials and the floor space occupied by the heat exchanger may impose limitations on the physical size of the heat exchanger.

## 8.18 SELECTION OF HEAT EXCHANGERS

An engineer going through the catalogs of heat exchanger manufacturers is often overwhelmed by the type and number of readily available off-the-shelf heat exchangers. The proper selection depends on several factors as explained below.

### (a) Heat Transfer Rate

This is the most important quantity. A heat exchanger should be capable of transferring heat at the specified rate in order to achieve the desired temperature change of the fluid at the specified mass flow rate.

### (b) Cost

Budgetary limitation often restricts the selection of the heat exchanger. An off-the-shelf heat exchanger has a definite cost advantage over those made to order. However, in many cases, the standard available heat exchanger is not satisfactory. It is then needed to undertake the expensive and time-consuming task of designing and manufacturing a heat exchanger from scratch to suit the needs. The operation and maintenance costs of the heat exchanger are also required to consider for assessing the overall cost.

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### (c) Pumping Power

In a heat exchanger, both fluids are usually forced to flow by pumps or fans that consume electrical power. The annual cost of electricity associated with the operation of the pumps and fans can be determined from Operating cost = [Pumping power, kW × Hours of operation, *h* × Price of electricity, Rs./kWh] where the pumping power is the total electricity consumed by the motors of the pumps and fans.

Minimizing the pressure drop and the mass flow rate of the fluids will *minimize* the operating cost of the heat exchanger, but it will *maximize* the size of the heat exchanger and thus the initial cost.

### (d) Size and Weight

Normally, the *smaller* and *lighter* the heat exchanger, the better it is. This is particularly important in automotive and aerospace industries. Also a larger heat exchanger, not only carries a higher price tag, but also requires more space.

### (e) Type

The type of heat exchanger to be selected depends primarily on the type of fluids involved, the size and weight limitations, and the presence of phase-change processes. A heat exchanger is suitable to cool a liquid by a gas. On the other hand, a plate or shell-and-tube heat exchanger is very suitable for cooling a liquid by another liquid.

### (f) Materials

The materials used in the construction of the heat exchanger have an important effect on the selection. The thermal and structural stress effects need not be considered at pressures below 15 atm or temperatures below 150° C. Differential thermal expansion problems need be considered be it a temperature difference of 50° C or more exists the tubes and the shell.

### (g) Other Considerations

Heat exchanger should be leak-tight particularly for toxic or expensive fluids. There should be ease of servicing, low maintenance cost, safety, reliability and silence in operation.

## SOLVED EXAMPLES

**Example 8.1** Water ($c_p$ = 4.187 kJ/kg K) is heated at the rate of 1.4 kg/s from 40°C to 70°C by an oil ($c_p$ = 1.9 kJ/kg K) entering at 110°C and leaving at 60°C in a counterflow heat exchanger. If $U_o$ = 350 W/m$^2$ K, calculate the surface area required.

Using the same entering fluid temperatures and the same oil flow rate, calculate the exit temperatures of oil and water and the rate of heat transfer, when the water flow rate is halved.

**Solution** Given (Fig. Ex. 8.1):

$$T_{h_1} = 110°\text{C},\ T_{h_2} = 60°\text{C},\ T_{c_1} = 40°\text{C},\ T_{c_2} = 70°\text{C}$$

$$\Delta T_i = 110 - 70 = 40°\text{C}$$

$$\Delta T_e = 60 - 40 = 20°\text{C}$$

$$\Delta T_{1m} = \frac{\Delta T_i - \Delta T_e}{\ln \Delta T_i/\Delta T_e} = \frac{40-20}{\ln 40/20} = \frac{20}{\ln 2} = 28.85°\text{C}$$

$$Q = \dot{m}_c c_c = (T_{c_2} - T_{c_1}) = \dot{m}_h c_h (T_{h_1} - T_{h_2}) = U_0 A_0 \Delta T_{1\text{m}}$$

$$A_0 = \frac{1.4 \times 4.187 \times 30}{0.35 \times 28.85} = 17.42 \text{ m}^2 \quad \textit{Ans.}$$

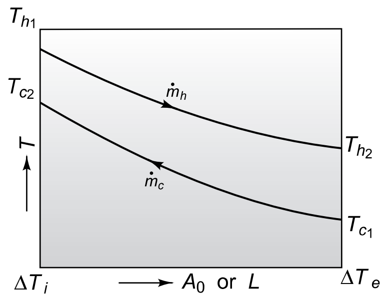


**Fig.** Ex. 8.1

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$$1.4 \times 4.187 \times 30 = \dot{m}_h \times 1.9 \times 50$$
$$\dot{m}_h = 1.851 \text{ kg/s}$$
$$\dot{m}_h c_h = 1.851 \times 1.9 = 3.52 \text{ kW/K}$$
$$\dot{m}_c c_c = 0.7 \times 4.187 = 2.93 \text{ kW/K}$$

(with water flow halved)

$$\dot{m}_c c_c = C_{\min} \text{ and } \dot{m}_h\, c_h = C_{\max}$$
$$R = \frac{C_{\min}}{C_{\max}} = \frac{2.93}{3.52} = 0.832$$
$$\text{NTU} = \frac{U_0 A_0}{C_{\min}} = \frac{0.35 \times 17.42}{2.93} = 2.08$$
$$\exp[-\text{NTU}(1 - R)] = \exp[-2.08(1 - 0.832)]$$
$$= e^{-0.34944} = 0.705$$

The effectiveness of the heat exchanger is

$$\varepsilon = \frac{1 - \exp[-\text{NTU}(1-R)]}{1 - R\exp[-\text{NTU}(1-R)]} = \frac{1 - 0.705}{1 - 0.832 \times 0.705}$$
$$= \frac{0.295}{0.413} = 0.714$$

Now,
$$\varepsilon = \frac{(\Delta T)_1}{T_{h_1} - T_{c_1}} = \frac{T_{c_2} - T_{c_1}}{T_{h_1} - T_{c_1}} = \frac{T_{c_2} - 40}{110 - 40} = 0.714$$

∴
$$T_{c_2} = \text{exit water temperature}$$
$$= 90°\text{C} \quad \textit{Ans.}$$
$$\dot{m}_h c_h (T_{h_1} - T_{h_2}) = \dot{m}_c c_c (T_{c_2} - T_{c_1})$$
$$1.851 \times 1.9\,(110 - T_{h_2}) = 0.7 \times 4.187\,(90 - 40)$$
$$T_{h_2} = 68.38°\text{C} = \text{exit temperature of oil} \quad \textit{Ans.}$$
$$Q = 1.851 \times 1.9\,(110 - 68.38) = 146.5 \text{ kW} \quad \textit{Ans.}$$

**Example 8.2** In a counter flow heat exchanger if $\Delta T_i = \Delta T_e$, show that $\Delta T_{lm} = \Delta T_i = \Delta T_e$.

**Solution**

$$\Delta T_{1m} = \frac{\Delta T_i - \Delta T_e}{\ln \Delta T_i / \Delta T_e}$$

Let
$$\Delta T_i = a \text{ and } \Delta T_i - \Delta T_e = x$$

∴
$$\Delta T_e = a - x$$

On substitution,

$$\Delta T_{1m} = \frac{x}{\ln a/(a-x)}$$
$$\lim_{x \to 0} \Delta T_{1m} = \lim_{x \to 0} \frac{x}{\ln a/(a-x)}$$

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$$= \lim_{x \to 0} \frac{1}{\frac{(a-x)}{a}(-a)(-1)\frac{1}{(a-x)^2}}$$

$$= \lim_{x \to 0} (a - x) = a = \Delta T_i$$

$\therefore \qquad \Delta T_{1m} = \Delta T_i = \Delta T_e \quad$ Proved.

**Example 8.3** In a balanced counter flow heat exchanger where $\dot{m}_h c_h = \dot{m}_c c_c$ show that $\Delta T_i = \Delta T_e = \Delta T$, at any section, and the temperature profiles of the two fluids are linear and parallel.

**Solution** As shown in Fig. Ex. 8.3, for a differential length d$L$ of the heat exchanger.

$$dQ = -\dot{m}_h c_h \, dT_h = -\dot{m}_c c_c \, dT_c$$
$$= U_0 \, dA_0 \, \Delta T$$

Since $\qquad \dot{m}_h c_h = \dot{m}_c c_c$

$$dT_h = dT_c$$

or, $\qquad d(T_h - T_c) = 0$

or, $\qquad \Delta T = T_h - T_c = \text{constant}$

$\therefore \qquad \Delta T_i = \Delta T_e = \Delta T$, at any section. $\quad$ Proved.

From energy balance,

$$-\dot{m}_h c_h dT_h = -\dot{m}_c c_c \, dT_c = U_0 \, \pi D_0 \, dL \, \Delta T$$

$$\frac{dT_h}{dL} = -\frac{U_0 \pi D_0 \Delta T}{\dot{m}_h c_h} = \text{constant}$$

$$\frac{dT_c}{dL} = -\frac{U_0 \pi D_0 \Delta T}{\dot{m}_c c_c} = \text{constant}$$

Thus temperature profiles of both hot and cold fluids are linear and parallel. $\quad$ Proved.

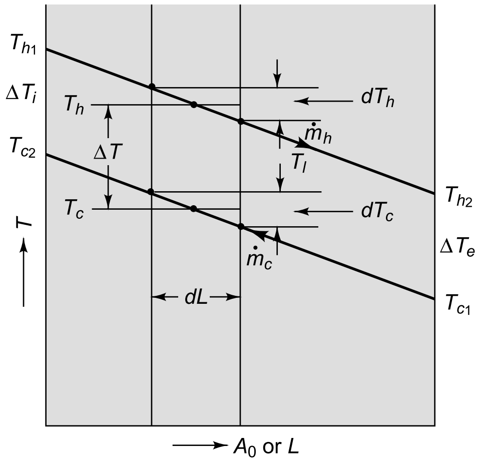


**Fig.** Ex. 8.3

**Example 8.4** In an oil cooler, oil enters at 160°C. If water entering at 35°C flows parallel to oil, the exit temperatures of oil and water are 90°C and 70°C respectively. Determine the exit temperatures of oil and water if the two fluids flow in opposite directions. Assume that the flow rates of the two fluids and $U_0$ remain unaltered. What would be the minimum temperatures to which oil could be cooled in parallel flow and counterflow operations?

**Solution** Heat capacity ratio $R = \dfrac{C_{\min}}{C_{\max}} = \dfrac{(\Delta T)_s}{(\Delta T)_1}$

In Fig. Ex. 8.4,

$$T_{h_1} - T_{h_2} = 160 - 90 = 70°\text{C}$$

$$T_{c_2} - T_{c_1} = 70 - 35 = 35°\text{C}$$

$$R = \frac{T_{c_2} - T_{c_1}}{T_{h_1} - T_{h_2}} = \frac{35}{70} = 0.5$$

For parallel flow operation,

$$\varepsilon_p = \frac{(\Delta T)_1}{T_{h_1} - T_{c_1}} = \frac{70}{160 - 35} = 0.56$$

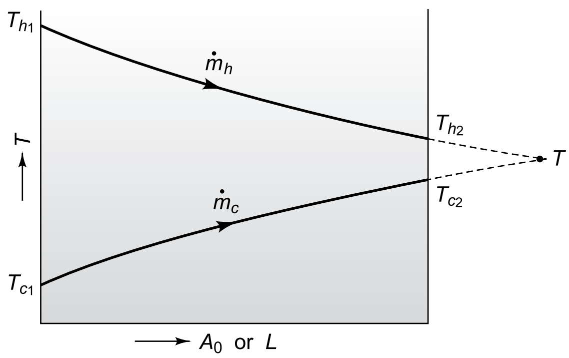


**Fig.** Ex. 8.4

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Again, $$\varepsilon_p = \frac{1 - \exp[-\text{NTU}(1+R)]}{1+R}$$

$$= \frac{1 - \exp[-1.5\,\text{NTU}]}{1.5} = 0.56$$

$$\exp(-1.5\,\text{NTU}) = 0.16$$

$\therefore$ $$\text{NTU} = 1.222$$

Now, $$\text{NTU} = U_0 A_0 / C_{min}$$

Since in counterflow operation, $U_0$ and $A_0$ remain the same and the flow rates also do not change, NTU will remain the same as in parallel flow operation. Again, $R = 0.5$. Therefore,

$$\varepsilon_c = \frac{1 - \exp[-\text{NTU}(1-R)]}{1 - R\exp[-\text{NTU}(1-R)]}$$

$$= \frac{1 - \exp[-1.222\,(1-0.5)]}{1 - 0.5\,\exp[-1.222\,(1-0.5)]}$$

$$= \frac{1 - 0.5428}{1 - 0.2714} = 0.6275$$

Now, $$\varepsilon_c = \frac{(\Delta T)_1}{T_{h_1} - T_{c_1}} = \frac{T_{h_1} - T_{c_2}}{T_{h_1} - T_{c_1}} = \frac{160 - T_{h_2}}{160 - 35} = 0.6275$$

$\therefore$ $$T_{h_2} = 160 - 78.44 = 81.56°\text{C} \quad \textit{Ans.}$$

$$R = \frac{C_{min}}{C_{max}} = \frac{(\Delta T)_s}{(\Delta T)_1} = \frac{T_{c_2} - T_{c_1}}{T_{h_1} - T_{h_2}} = \frac{T_{c_2} - 35}{160 - 81.56} = 0.5$$

$\therefore$ $$= 74.22°\text{C} \quad \textit{Ans.}$$

**Minimum oil temperature:** For parallel flow operation, the minimum oil temperature will correspond to the common temperature $T$ when the hot and cold temperature profiles meet (Fig. Ex. 8.4). Therefore,

$$\dot{m}_h c_h (T_{h_1} - T) = \dot{m}_c c_c (T - T_{c_1})$$

$$0.5(160 - T) = (T - 35)$$

$$(T)_{min} = 76.7°\text{C} \quad \textit{Ans.}$$

The minimum oil temperature can also be estimated for parallel flow operation by assuming that it would occur if the heat exchanger were infinitely large, or $A_0 = \infty$, i.e. NTU $= \infty$

$$\varepsilon_p = \frac{1 - \exp[-\text{NTU}\,(1+R)]}{1+R} = \frac{1}{1+R} = \frac{1}{1+0.5} = \frac{2}{3}$$

$$\frac{T_{h_1} - T_{h_2}}{T_{h_1} - T_{c_1}} = \frac{60 - T_{h_2}}{160 - 35} = \frac{2}{3}$$

$$(T_{h2})_{min} = 76.7°\text{C} \quad \textit{Ans.}$$

For counterflow operation, the minimum exit temperature of oil would be 35°C, when $T_{h2} = T_{c_1}$ and $\varepsilon_c = 1$. *Ans.*

**Example 8.5** In a counterflow waste heat recuperator, a stream of hot waste gas ($w_g$) gives up heat to a stream of cold air ($w_a$) which is flowing in the opposite direction. Show that the efficiency of the recuperator, defined as the ratio of the heat actually gained by the air to the heat that would be gained if the air attained the inlet temperature of waste gas, is given by

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$$\eta = 1 - \frac{1-(w_g c_g/w_a c_a)}{1-(w_g c_g/w_a c_a)e^{-\alpha}}$$

where
$$\alpha = U_0 A_0 \left(\frac{1}{w_g c_g} - \frac{1}{w_a c_a}\right)$$

A recuperator is designed to achieve an efficiency of 60% when heating 30 kg/s of air ($c_p$ = 1.005 kJ/kg K) with a stream of 25 kg/s of hot waste gas ($c_p$ = 1.1 kJ/kg K). By what factor would the heating surface have to be increased to raise the efficiency to 70%? Assume that $U_0$ remains unchanged.

**Solution** Energy balance of a differential element (Fig. Ex. 8.5) gives

$$dQ = -w_g c_g\, dT_g = -w_a c_a\, dT_a = U_0 \pi D\, dx\, \Delta T$$

$$d(\Delta T) = dT_g - dT_a$$

$$= -\frac{dQ}{w_g c_g} + \frac{dQ}{w_a c_a} = -dQ\left(\frac{1}{w_g c_g} - \frac{1}{w_a c_a}\right)$$

$$= -U_0 \pi D\, dx\, \Delta T \left(\frac{1}{w_g c_g} - \frac{1}{w_a c_a}\right)$$

$$\int_{T_i}^{T_e} \frac{d(\Delta T)}{\Delta T} = -\int_0^L U_0 \pi D \left(\frac{1}{w_g c_g} - \frac{1}{w_a c_a}\right) dx$$

$$\ln \frac{\Delta T_e}{\Delta T_i} = -U_0 \pi D L \left(\frac{1}{w_g c_g} - \frac{1}{w_a c_a}\right) = -\alpha$$

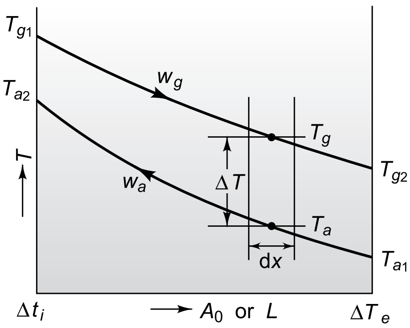


**Fig.** *Ex. 8.5*

$$\frac{\Delta T_e}{\Delta T_i} = e^{-\alpha}$$

$$\frac{T_{g_2} - T_{a_1}}{T_{g_1} - T_{a_2}} = e^{-\alpha}$$

$$\frac{T_{g_1} - T_{a_1} - (T_{g_1} - T_{g_2})}{(T_{g_1} - T_{a_1}) - (T_{a_2} - T_{a_1})} = e^{-\alpha}$$

or,
$$\frac{1 - \left(\dfrac{T_{g_1} - T_{g_2}}{T_{g_1} - T_{a_1}}\right)}{1 - \left(\dfrac{T_{a_2} - T_{a_1}}{T_{g_1} - T_{a_1}}\right)} = e^{-\alpha} = \frac{1 - \left(\dfrac{T_{g_1} - T_{g_2}}{T_{a_2} - T_{a_1}}\right) \cdot \eta}{1 - \eta}$$

where
$$\eta = \frac{T_{a_2} - T_{a_1}}{T_{g_1} - T_{a_1}}$$

$$e^{-\alpha} = \frac{1 - (w_a c_a / w_g c_g)\,\eta}{1 - \eta} = \frac{1 - (\eta/x)}{1 - \eta}$$

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where, $$x = \frac{w_g c_g}{w_a c_a}$$

$$xe^{-\alpha} = \frac{x-\eta}{1-\eta}$$

or, $$1 - xe^{-\alpha} = 1 - \frac{x-\eta}{1-\eta} = \frac{1-x}{1-\eta}$$

$$1 - \eta = \frac{1-x}{1-xe^{-\alpha}}$$

$\therefore$ $$\eta = 1 - \frac{1-(w_g c_c / w_a c_a)}{1-(w_g c_g / w_a c_a)e^{-\alpha}} \quad \text{Proved.}$$

Given: $\eta_1 = 0.60$, $w_a = 30$ kg/s, $c_a = 1.005$ kJ/kg K

$w_g = 25$ kg/s, $c_g = 1.1$ kJ/kg K, $\eta_2 = 0.7$

Substituting

$$x = \frac{w_g c_g}{w_a c_a} = \frac{25 \times 1.1}{30 \times 1.005} = 0.912$$

$$\eta_1 = 1 - \frac{1-0.912}{1-0.912e^{-\alpha_1}} = 0.6$$

$$\frac{0.088}{1-0.912e^{-\alpha_1}} = 0.4$$

$$0.912\, e^{-\alpha_1} = 1 - \frac{0.088}{0.4} = 0.78$$

$$e^{\alpha_1} = \frac{0.912}{0.78} = 1.169$$

$$\alpha_1 = 0.156$$

When, $$\eta_2 = 0.7, \quad \frac{0.088}{1-0.912\, e^{\alpha_2}} = 0.3$$

$\therefore$ $$\alpha_2 = 0.2547$$

$$\frac{\alpha_2}{\alpha_1} = \frac{A_{02}}{A_{01}} = \frac{0.2547}{0.156} = 1.633 \quad \textit{Ans.}$$

Area has to be increased by 63.3% *Ans.*

**Example 8.6** In an open heart surgery under hypothermic conditions, the patient's blood is cooled before the surgery and rewarmed afterwards. It is proposed that a concentric tube counterflow heat exchanger of length 0.5 m is to be used for this purpose, with a thin-walled inner tube having a diameter of 55 mm. If water at 60°C and 0.1 kg/s is used to heat blood entering the exchanger at 18°C and 0.05 kg/s, what is the temperature of the blood leaving the exchanger and the heat flow rate. Take $U_0$ = 500 W/m$^2$ K, $c_p$ of blood = 3.5 kJ/kg K and $c_p$ of water = 4.183 kJ/kg K.

**Solution**

$$C_c = \dot{m}_c c_c = 0.05 \times 3.5 = 0.175 \text{ kW/K}$$

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$$C_h = \dot{m}_h c_h = 0.1 \times 4.183 = 0.4183 \text{ kW/K}$$

$\therefore$ $$C_{min} = (\dot{m} c)_s = C_c$$

$$\text{NTU} = \frac{U_0 A_0}{C_{min}} = \frac{500 \times \pi \times 0.055 \times 0.5}{0.175} = 0.247$$

$$R = \frac{C_{min}}{C_{max}} = \frac{0.175}{0.4183} = 0.4184$$

$$\varepsilon_{cf} = \frac{1 - \exp[-\text{NTU}\ (1 - R)]}{1 - R \exp[-\text{NTU}\ (1 - R)]}$$

$$= \frac{1 - \exp[-0.247(1 - 0.4184)]}{1 - 0.4184 \exp[-0.247(1 - 0.4184)]} = 0.21$$

Now, $$\varepsilon_{cf} = \frac{\Delta T_1}{T_{h_1} - T_{c_1}} = \frac{T_{c_2} - T_{c_1}}{T_{h_1} - T_{c_1}}$$

$$= \frac{T_{c_2} - 18}{60 - 18} = 0.21$$

$\therefore$ $$T_{c_2} = 26.82°\text{C} \quad \textit{Ans.}$$

$$Q = C_c\,(T_{c_2} - T_{c_1}) = 0.175\ (26.82 - 18)$$

$$= 1.543 \text{ kW} \quad \textit{Ans.}$$

**Example 8.7** A 4 kg/s product stream from a distillation column is to be cooled by a 3 kg/s water stream in a counterflow heat exchanger. The hot and cold stream inlet temperatures are 400 K and 300 K respectively, and the area of the exchanger is 30 m$^2$. If the overall heat transfer coefficient is estimated to be 820 W/m$^2$K, determine the product stream outlet temperature, if its specific heat is 2500 J/kgK and the coolant outlet temperature.

**Solution** The effectiveness of counterflow heat exchanger is given by

$$\varepsilon = \frac{1 - \exp[-\text{NTU}\,(1 - R)]}{1 - R \exp[-\text{NTU}\,(1 - R)]}$$

where $$R = \frac{C_{min}}{C_{max}}, \ \text{NTU} = \frac{U_0 A_0}{C_{min}}$$

$$C_h = (\dot{m} c_p)_h = 4 \times 2500 = 10{,}000 \text{ W/K}$$

$$C_c = (\dot{m} c_p)_c = 3 \times 4180 = 12{,}540 \text{ W/K}$$

$\therefore$ $$C_{min} = 10{,}000 \text{ W/K}$$

$$R = \frac{10{,}000}{12{,}540} = 0.797$$

$$\text{NTU} = \frac{U_0 A_0}{C_{min}} = \frac{820 \times 30}{10{,}000} = 2.46$$

$\therefore$ $$\varepsilon = \frac{1 - \exp[-2.46\,(1 - 0.97)]}{1 - 0.797 \exp[-2.46\,(1 - 0.797)]} = 0.761$$

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Also,

$$\varepsilon = \frac{C_h(T_{h_1} - T_{h_2})}{C_{\min}(T_{h_1} - T_{c_1})}$$

or,
$$0.761 = \frac{10{,}000(400 - T_{h_2})}{10{,}000(400 - 300)}$$

∴ $T_{h_2} = 323.9$ K *Ans.*

By energy balance,

$$C_h(T_{h_1} - T_{h_2}) = C_c(T_{c_2} - T_{c_1})$$
$$10{,}000\ (400 - 323.9) = 12{,}540\ (T_{c_2} - 300)$$

∴ $T_{c_2} = 360.7$ K *Ans.*

**Example 8.8** In a solar-assisted air-conditioning system, 0.5 kg/s of ambient air at 270 K is to be preheated by the same amount of air leaving the system at 295 K. If a counterflow heat exchanger has an area of 30 m$^2$, and the overall heat transfer coefficient is estimated to be 25 W/m$^2$K, determine the outlet temperature of the preheated air. Take $c_p$ for air as 1000 J/kgK.

**Solution** It is a balanced heat exchanger where $C_h = C_c$.

∴
$$\text{NTU} = \frac{UA}{C} = \frac{25 \times 30}{0.5 \times 1000} = 1.5$$

For a balanced counterflow heat exchange,

$$\varepsilon = \frac{\text{NTU}}{\text{NTU} + 1} = \frac{1.5}{2.5} = 0.6$$

$$\varepsilon = \frac{T_{c_2} - T_{c_1}}{T_{h_1} - T_{c_1}}$$

$$0.6 = \frac{T_{c_2} - 270}{295 - 270}$$

∴ $T_{c_2} = 285$ K *Ans.*

**Example 8.9** A flow of 0.1 kg/s of exhaust gases at 700 K from a gas turbine is used to preheat the incoming air, which is at the ambient temperature of 300 K. It is desired to cool the exhaust to 400 K, and it is estimated that an overall heat transfer coefficient of 30 W/m$^2$ K can be achieved in an appropriate exchanger. Determine the area required for a counterflow heat exchanger. Take the specific heat of exhaust gases the same as for air, which is 1000 J/kgK.

**Solution** It is also a balanced heat exchanger

where $C_h = C_c = C$. The effectiveness is

$$\varepsilon = \frac{T_{h_1} - T_{h_2}}{T_{h_1} - T_{c_1}} = \frac{700 - 400}{700 - 300} = 0.75$$

Again,
$$\varepsilon = \frac{\text{NTU}}{\text{NTU} + 1}$$

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$$\therefore \qquad \text{NTU} = \frac{\varepsilon}{1-\varepsilon} = \frac{0.75}{1-0.75} = 3.0$$

$$\text{NTU} = \frac{UA}{C} = 3$$

$$\therefore \qquad A = \frac{3 \times 0.1 \times 1000}{30} = 10 \text{ m}^2 \quad \textit{Ans.}$$

**Example 8.10** After a long time in service, a counterflow oil cooler is checked to ascertain if its performance has deteriorated due to fouling. In the test a standard oil flowing at 2.0 kg/s is cooled from 420 K to 380 K by a water supply of 1.0 kg/s at 300 K. If the heat transfer surface is 3.33 m$^2$ and the design value of the overall heat transfer coefficient is 930 W/m$^2$K, how much has it been reduced by fouling? Take $c_p$ of oil as 2330 J/kgK and $c_p$ of water as 4174 J/kgK.

**Solution** By energy balance of the heat exchange (Fig. Ex. 8.10),

$$Q = \dot{m}_h c_h (T_{h_1} - T_{h_2}) = m_c c_c (T_{c_2} - T_{c_1})$$

$$\therefore \qquad Q = 2 \times 2330 \times (420 - 380)$$

$$= 186{,}400 \text{ W}$$

To find $T_{c_2}$,

$$186{,}400 = 1 \times 4174 (T_{c_2} - 300)$$

$$\therefore \qquad T_{c_2} = 344.7 \text{ K.}$$

$$\Delta T_{\text{l.m.}} = \frac{(420-344.7)-(380-300)}{\ln \dfrac{420-344.7}{380-300}}$$

$$= \frac{75.3-80}{\ln \dfrac{75.3}{80}} = 77.6 \text{ K.}$$

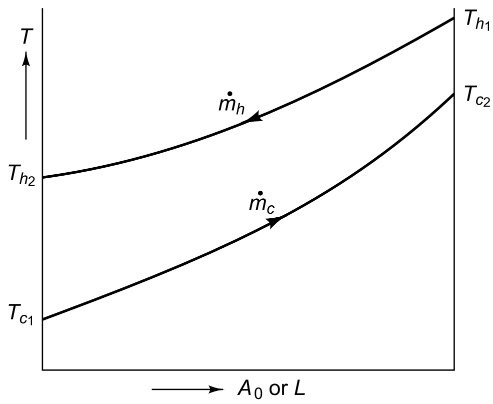


**Fig.** *Ex. 8.10*

Now, $$Q = U_0 A_0 \Delta T_{\text{l.m.}}$$

$$186{,}400 = U_0 \times 3.33 \times 77.6$$

$$\therefore \qquad U_0 = 721 \text{ W/m}^2\text{K.}$$

Reduction in $U_0$ due to fouling is

$$= (930 - 721)/930 = 0.225 \text{ or } 22.5\% \quad \textit{Ans.}$$

**Example 8.11** A coaxial tube counterflow heat exchanger is to cool 0.03 kg/s of benzene from 360 K to 310 K with a counterflow of 0.02 kg/s of water at 290 K. If the inner tube outside diameter is 2 cm and the overall heat transfer coefficient based on outside area is 650 W/m$^2$K, determine the required length of the exchanger. Take the specific heats of benzene and water as 1880 and 4175 J/kgK, respectively.

**Solution** By energy balance,

$$Q = \dot{m}_h c_h (T_{h_1} - T_{h_2}) = \dot{m}_c c_c (T_{c_2} - T_{c_1})$$

$$Q = 0.03 \times 1880 \,(360 - 310) = 0.02 \times 4175 \,(T_{c_2} - 290)$$

$$Q = 2820 \text{ W}, \; T_{c_2} = 323.8 \text{ K}$$

$$\Delta T_{\text{l.m.}} = \frac{36.2-20}{\ln \dfrac{36.2}{20}} = 27.3 \text{ K}$$

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$$Q = U_0 A_0 \Delta T_{\text{l.m.}} = U_0 \pi d_0 L \Delta T_{\text{l.m.}}$$

$\therefore$ $$L = \frac{2820}{650 \times \pi \times 0.02 \times 27.3} = 2.53 \text{ m} \quad \textit{Ans.}$$

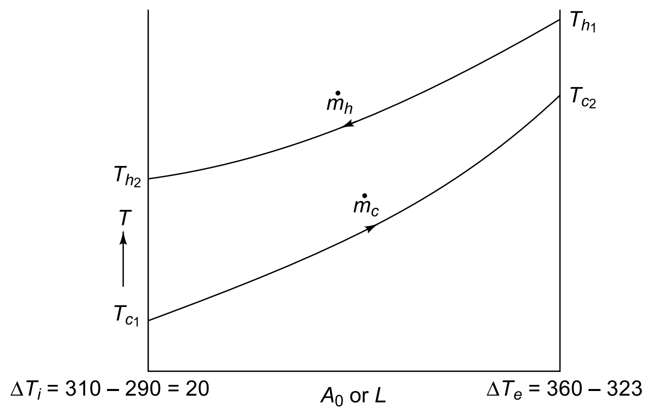


**Fig.** *Ex. 8.11*

**Example 8.12** A brass ($k$ = 111 W/mK) condenser tube has a 30 mm outer diameter and 2 mm wall thickness. Sea water enters the tube at 290 K, and saturated low pressure steam condenses on the outer side of the tube. The inside and outside heat transfer coefficients are estimated to be 4000 and 8000 W/m$^2$K, respectively, and a fouling resistance of $10^{-4}$ (W/m$^2$K) on the water side is expected. Estimate the overall heat transfer coefficient based on inside area.

**Solution** Total resistance to heat transfer

$$\frac{1}{UP} = \frac{1}{h_i 2\pi r_i} + \frac{\ln r_o/r_i}{2\pi k} + \frac{1}{h_o 2\pi r_o},$$

where $P$ is the perimeter.

$$\frac{1}{UP} = \frac{1}{4000(2\pi)(0.013)} + \frac{\ln \dfrac{0.015}{0.013}}{2\pi \times 111} + \frac{1}{8000(2\pi)(0.015)}$$

$$= 10^{-3}(3.06 + 0.21 + 1.33) = 4.6 \times 10^{-3} \text{ (W/mK)}^{-1}$$

The inside perimeter, $P = 2\pi r_i = 2\pi \times 0.013 = 0.0817$ m

$\therefore$ $$\frac{1}{U} = 0.0817 \times 4.6 \times 10^{-3} = 3.76 \times 10^{-4} \text{ (W/m}^2\text{ K)}^{-1}$$

$\therefore$ $$U = 2660 \text{ W/m}^2\text{ K} \quad \textit{Ans.}$$

For the fouled tube,

$$\frac{1}{U_f} = \frac{1}{U} + R_f = 3.76 \times 10^{-4} + 10^{-4}$$

$$= 4.76 \times 10^{-4} \text{ (W/m}^2\text{ K)}^{-1}$$

$\therefore$ $$U_f = 2100 \text{ W/m}^2\text{ K} \quad \textit{Ans.}$$

The fouling reduces the overall heat transfer coefficient by

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$$\frac{2660-2100}{2660}\times 100 \text{ or } 21\%. \quad \textit{Ans.}$$

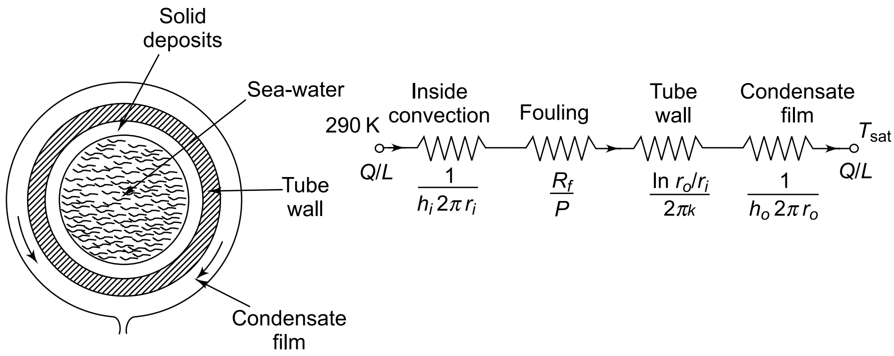


**Fig.** *Ex. 8.12*

**Example 8.13** A counterflow heat exchanger is employed to cool 0.55 kg/s ($c_p$ = 2.45 kJ/kg°C) of oil from 115°C to 40°C by the use of water. The inlet and outlet temperatures of cooling water are 15°C and 75°C, respectively. The overall heat transfer coefficient is expected to be 1450 W/m$^2$K. Using the NTU method, calculate the following: (a) The mass flow rate of water, (b) the effectiveness of the heat exchanger and (c) the surface area required.

**Solution**

$$Q = \dot m_h c_h (T_{h_1} - T_{h_2}) = c_c \dot m_c (T_{c_2} - T_{c_1})$$

$$= 0.55 \times 2.45\,(115 - 40) = \dot m_c \times 4.18\,(75 - 15)$$

$$\therefore \quad \dot m_c = 0.403 \text{ kg/s} \quad \textit{Ans. (a)}$$

$$C_c = \dot m_c c_c = 0.403 \times 4.18 = 1.675 \text{ kW/K}$$

$$C_h = \dot m_h c_h = 0.55 \times 2.45 = 1.347 \text{ kW/K}$$

Since $C_c > C_h$, $\therefore (T_{h_1} - T_{h_2}) > (T_{c_2} - T_{c_1})$

$$\varepsilon = \frac{(\Delta T)_1}{T_{h_1} - T_{c_1}} = \frac{115-40}{115-15} = \frac{75}{100} = 0.75 \quad \textit{Ans. (b)}$$

$$R = \frac{C_{\min}}{C_{\max}} = \frac{1.347}{1.672} = 0.806$$

$$\varepsilon_{cf} = \frac{1-\exp[-\text{NTU}\,(1-R)]}{1-R\exp[-\text{NTU}\,(1-R)]}$$

$$0.75 = \frac{1-\exp[-\text{NTU}\,(1-0.806)]}{1-0.806\exp[-\text{NTU}\,(1-0.806)]}$$

By rearrangement,

$$\frac{\varepsilon-1}{\varepsilon R-1} = \exp\,[-\text{NTU}(1-R)]$$

$$\frac{0.75-1}{0.75\times 0.806-1} = \exp[-\text{NTU}\,(1-0.806)]$$

$$0.632 = \exp\,[-\text{NTU}\times 0.194]$$

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$$\therefore \qquad \text{NTU} = 2.365 = \frac{UA}{C_{\min}}$$

$$A = \frac{2.365 \times 1347}{1450} = 2.197 \text{ m}^2 \quad \textit{Ans.} \text{ (c)}$$

**Example 8.14** Two fluids *A* and *B* exchange heat in a counterflow heat exchanger. Fluid *A* enters at 420°C and has a mass flow rate of 1 kg/s. Fluid *B* enters at 20°C and has a mass flow rate of 1 kg/s. The effectiveness of heat exchanger is 75%. Determine (i) the heat transfer rate, (ii) the exit temperature of fluid *B*. Specific heat of fluid *A* is 1 kJ/kgK and that of fluid B is 4 kJ/kgK.

**Solution**

$$C_{\min} = C_A = \dot{m}_h c_h = 1 \times 1 = 1 \text{ kW/K}$$

$$C_{\max} = C_B = \dot{m}_c C_c = 1 \times 4 = 4 \text{ kW/K}$$

$$\therefore \qquad (\Delta T)_1 = T_{h_1} - T_{h_2}$$

$$\varepsilon = \frac{(\Delta T)_1}{T_{h_1} - T_{c_1}} = \frac{T_{h_1} - T_{h_2}}{T_{h_1} - T_{c_1}} = \frac{420 - T_{h_2}}{420 - 20} = 0.75$$

$$\therefore \qquad T_{h_2} = 420 - 0.75 \times 400 = 120°\text{C} \quad \textit{Ans.}$$

$$Q = \dot{m}_h C_h (T_{h_1} - T_{h_2}) = 1 \times 1 \times (420 - 120)$$

$$= 300 \text{ kW} \quad \textit{Ans.} \text{ (i)}$$

$$Q = \dot{m}_c C_c (T_{c_2} - T_{c_1}) = 1 \times 4 (T_{c_2} - 20) = 300$$

$$T_{c_2} = 95°\text{C} \quad \textit{Ans.} \text{ (ii)}$$

**Example 8.15** A chemical having specific heat of 3.3 kJ/kgK flowing at the rate of 20,000 kg/h enters a parallel flow heat exchanger at 120°C. The flow rate of cooling water is 50,000 kg/h with an inlet temperature of 20°C. The heat transfer area is 10 m$^2$ and the overall heat transfer coefficient is 1050 W/m$^2$K. Find (i) the effectiveness of the heat exchanger, (ii) the outlet temperatures of water and chemical. Take for water $c_p$ = 4.186 kJ/kgK.

**Solution**

$$\dot{m}_h = \frac{20,000}{3600} = 5.56 \text{ kg/s}, \ c_h = 3.3 \text{ kJ/kgK}$$

$$\therefore \qquad C_h = 5.56 \times 3.3 = 18.35 \text{ kW/K}$$

$$\dot{m}_c = \frac{50,000}{3600} = 13.89 \text{ kg/s}, \ c_c = 4.186 \text{ kJ/kgK}$$

$$\therefore \qquad C_c = 13.89 \times 4.186 = 58.14 \text{ kW/K}$$

$$\therefore \qquad C_h < C_c$$

$$\therefore \qquad C_{\min} = 18.35 \text{ kW/K}$$

$$\text{NTU} = \frac{UA}{C_{\min}} = \frac{1050 \times 10}{18350} = 0.572$$

$$R = \frac{C_{\min}}{C_{\max}} = \frac{18.35}{58.14} = 0.3156$$

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$$\therefore \qquad \varepsilon_p = \frac{1-\exp[-\text{NTU}(1+R)]}{1+R}$$

$$= \frac{1-\exp[-0.572\times1.3156]}{1.3156}$$

$$= \frac{1-0.471}{1.3156} = 0.402 \quad \textit{Ans.}$$

$$\varepsilon = \frac{(\Delta T)_1}{T_{h_1}-T_{c_1}} = \frac{T_{h_1}-T_{h_2}}{T_{h_1}-T_{c_1}} = 0.402 = \frac{120-T_{h_2}}{120-20}$$

$$\therefore \qquad T_{h_2} = 79.8°\text{C} \quad \textit{Ans.}$$

$$\dot{m}_h c_h (T_{h_1}-T_{h_2}) = \dot{m}_c c_c (T_{c2}-T_{c_1})$$

$$T_{c_2}-T_{c_1} = 0.3156\times40.2 = 12.69$$

$$\therefore \qquad T_{c_2} = 32.69°\text{C} \quad \textit{Ans.}$$

**Example 8.16** A counterflow heat exchanger is to heat air entering at 400°C with a flow rate of 6 kg/s by the exhaust gas entering at 800°C with a flow rate of 4 kg/s. The overall heat transfer coefficient is 100 W/m$^2$K and the outlet temperature of air is 551.5°C. Specific heat of air, $c_p$, for both air and exhaust gas can be taken as 1100 J/kgK. Calculate (i) the heat transfer area needed and (ii) the number of transfer units.

**Solution**

$$C_h = \dot{m}_h c_h = 4\times1100 = 4400 \text{ W/K}$$

$$C_c = \dot{m}_c c_c = 6\times1100 = 6600 \text{ W/K}$$

$$C_h < C_c, \quad \therefore C_{\min} = \dot{m}_h c_h = 4400 \text{ W/K}$$

$$Q = \dot{m}_h c_h (T_{h_1}-T_{h_2}) = \dot{m}_c c_c (T_{c_2}-T_{c_1})$$

$$4400\,(800-T_{h_2}) = 6600\,(551.5-400)$$

$$\therefore \qquad T_{h_2} = 572.75°\text{C}$$

$$Q = 999900 \text{ W} = 999.9 \text{ kW}$$

$$\Delta T_{\text{l.m.}} = \frac{248.5-172.5}{\ln\dfrac{248.5}{172.5}} = \frac{76.0}{0.365} = 208.19°\text{C}$$

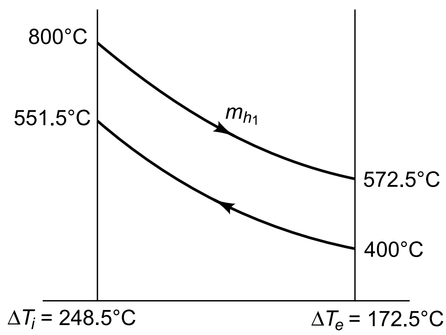


**Fig.** *Ex. 8.13*

$$Q = U_0 A_0 \Delta T_{\text{l.m.}}$$

$$999900 = 100\times A_0\times208.19$$

$$A_0 = 48.03 \text{ m}^2 \quad \textit{Ans.} \text{ (i)}$$

$$\text{NTU} = \frac{U_0 A_0}{C_{\min}} = \frac{100\times48.03}{4400}$$

$$= 1.092 \quad \textit{Ans.} \text{ (ii)}$$

**Example 8.17** Hot engine oil available at 150°C flowing through the shell side is used to heat 2.4 kg/s of water from 20°C to 80°C in a shell-and-tube heat exchanger. Water flows through eight tubes of 25 mm diameter. Each tube makes six passes through the shell. The exit oil temperature is 90°C. Neglecting the tube wall resistance, find the oil flow rate and the length of the tubes. Take the oil side heat transfer coefficient as 400 W/m$^2$. For engine oil at 140°C, $c_p$ = 2.34 kJ/kg K.

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For water at 50°C, $c_p$ = 4.181 kJ/kg K, $\mu = 548 \times 10^{-6}$ N s/m$^2$, $k$ = 0.643 W/m K and Pr = 3.56.

**Solution**

$$Q = \dot{m}_c c_{p_c} (T_{c_2} - T_{c_1})$$
$$= 2.4 \times 4.181\ (80 - 20) = 602.064 \text{ kW}$$

$$\dot{m}_h = \frac{Q}{c_{p_h}(T_{h_1} - T_{h_2})} = \frac{602.064}{2.34(150-90)} = 4.288 \text{ kg/s} \quad \textit{Ans.}$$

Flow rate of water in one tube (Fig. Ex. 8.17)

$$\dot{m}_c = \frac{2.4}{8} = 0.3 \text{ kg/s}$$

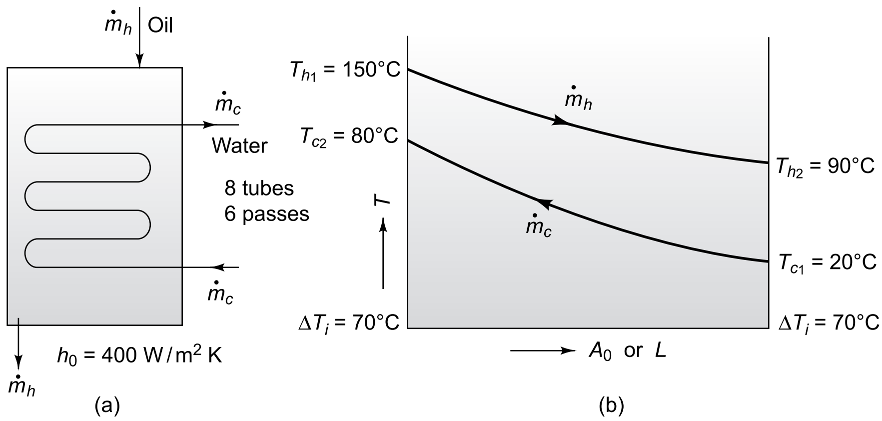


**Fig.** Ex. 8.17

$$\dot{m}_c = \frac{\pi}{4} D^2 \times \rho \times \overline{V}$$

$$\text{Re}_D = \frac{VD\rho}{\mu} = \frac{4\dot{m}_c}{\pi D \mu} = \frac{4 \times 0.3}{\pi \times 0.025 \times 548 \times 10^{-6}}$$
$$= 27{,}881$$

Using Dittus–Boelter equation,

$$\text{Nu}_D = 0.023\ \text{Re}_D^{0.8}\ \text{Pr}^{0.4}$$
$$= 0.023\ (27.881)^{0.8}\ (3.56)^{0.4}$$
$$= 0.023 \times 3599.54 \times 1.658 = 137.27$$

$$h_i = \frac{137.27 \times 0.643}{0.025} = 3531 \text{ W/m}^2 \text{ K}$$

Hence,
$$U_0 = \frac{1}{(1/400 + 1/3531)} = \frac{400 \times 3531}{3931}$$
$$= 359.3 \text{ W/m}^2 \text{ K}$$

The correction factor $F$ can be obtained from Fig. 8.18,

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$$R = \frac{150 - 90}{80 - 20} = 1.0, \quad P = \frac{80 - 20}{150 - 20} = \frac{60}{130} = 0.46$$

Hence, $F = 0.87$

$$(\Delta T_{1m})_{\text{counterflow}} = \Delta T_i = \Delta T_e = 70°C$$
$$Q = \pi NDL\ U_0\ F(\Delta T_{1m})_{cf}$$
$$602.064 \times 10^3 = \pi \times 8 \times 0.025\ L \times 359.3 \times 0.87 \times 70$$
$$L = 43.79 \text{ m} \quad \textit{Ans.}$$

**Example 8.18** In a gas-to-gas heat recovery unit, air is preheated from 30°C to 260°C at the rate of 21.5 kg/s by waste gas available at the rate of 19.6 kg/s at 380°C. The air preheater is a tubular two-pass unit where the gas moves through the tubes ($k_{\text{wall}}$ = 46.5 W/m K) having diameters $d_0/d_i$ = 53/50 mm with a mean velocity of 14 m/s, and air flows across the bank of tubes with a mean velocity of 8 m/s. For cross-flow, the following equation may be used

$$\text{Nu} = 0.41\ \text{Re}^{0.6}\ \text{Pr}^{0.33}$$

Determine the required heating surface, the number of tubes and the height of tubes in one pass. Given: Properties of air at 145°C: $\rho_a$ = 0.844 kg/m$^3$, $c_{pa}$ = 1.01 kJ/kg K, $k_a$ = 3.52 × 10$^{-2}$ W/m K, $V_{a_0}$ = 28.3 × 10$^{-6}$ m$^2$/s and Pr$_a$ = 0.684. Properties of gas at 265°C: $\rho_g$ = 0.622 kg/m$^3$, $C_{pg}$ = 1.11 kJ/kg K, $k_g$ = 0.0454 W/m K, $v_g$ = 41.2 × 10$^{-6}$ m$^2$/s and Pr$_g$ = 0.66.

**Solution** By making energy balance [Fig. Ex. 8.18 (a)], the rate of heat transfer

$$Q = \dot{m}_g c_{p_g} (T_{g_1} - T_{g_2}) = \dot{m}_a c_{p_a} (T_{a_2} - T_{a_1})$$

or, $$19.6 \times 1.11\ (380 - T_{g_2}) = 21.5 \times 1.01\ (260 - 30)$$

∴ $$T_{g_2} = 150°C$$

Mean gas temperature = $\dfrac{380 + 150}{2}$ = 265°C and

mean air temperature = $\dfrac{30 + 260}{2}$ = 145°C,

at which properties have been given.

Gases are flowing through the tubes.

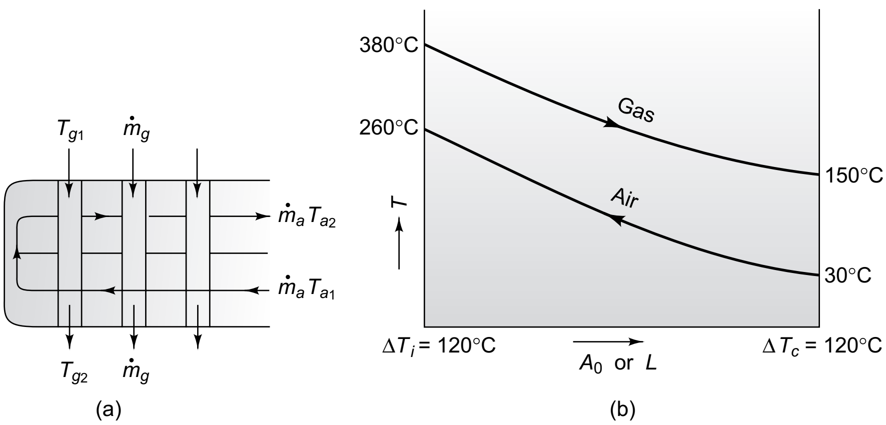


**Fig.** *Ex. 8.18*

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$$\text{Re}_g = \left(\frac{u_m d_i}{\nu}\right)_g = \frac{14 \times 0.050}{41.2 \times 10^{-6}} = 17{,}000$$

$$\text{Nu}_g = 0.023\ \text{Re}^{0.8}\ \text{Pr}^{0.33} = 0.023\ (17000)^{0.8}\ (0.66)^{0.33}$$

$$= 49.2 = \frac{h_i d_i}{k_g}$$

$$h_i = \frac{49.2 \times 0.0454}{0.050} = 44.67\ \text{W/m}^2\ \text{K}$$

Air is flowing outside the tubes.

$$\text{Re}_a = \left(\frac{u_m d_0}{\nu}\right)_a = \frac{8 \times 0.053}{28.3 \times 10^{-6}} = 14{,}982$$

$$\text{Nu}_a = 0.41\ (14{,}982)^{0.6}\ (0.684)^{0.33} = 115.8 = h_0 d_0 / k_a$$

$$h_0 = 115.8 \times \frac{0.0352}{0.053} = 76.91\ \text{W/m}^2\ \text{K}$$

$$\frac{1}{U_0} = \frac{1}{h_i} + \frac{x_w}{k_w} + \frac{1}{h_0}$$

$$\frac{1}{U_0} = \frac{1}{44.67} + \frac{1.5 \times 10^{-3}}{46.5} + \frac{1}{76.91}$$

$$U_0 = 28.21\ \text{W/m}^2\ \text{K}.$$

From Fig. 8.18(b)

$$\Delta T_{1\text{m}} = \Delta T_i = \Delta T_e = 120°\text{C}$$

One shell pass and two tube passes for which

$$P = \frac{T_{t_2} - T_{t_1}}{T_{s_1} - T_{t_1}} = \frac{150 - 380}{30 - 380} = \frac{-230}{-350} = 0.66$$

$$R = \frac{\Delta T_s}{\Delta T_t} = \frac{230}{230} = 1.0$$

From Fig. 8.18, the correction factor $F$ is found to be 0.88.

$$Q = U_0 A_0\ (\Delta T_{1\text{m}})_{cf} F$$

$$21.5 \times 1.01 \times 230 = 28.21 \times A_0 \times 120 \times 0.88 \times 10^{-3}$$

$$A_0 = 1677\ \text{m}^2 \quad \textit{Ans.}$$

$$\dot{m}_g = \left(n \frac{\pi}{4} d_i^2\right) \times \rho_g \overline{V}_g$$

$$n = \frac{4 \times 19.6}{\pi \times (0.05)^2 \times 0.622 \times 14} = 1077 \quad \textit{Ans.}$$

$$l_1 = \frac{A_0}{2 n \pi d_0} = \frac{1677}{2 \times 1077 \times \pi \times 0.053} = 4.68\ \text{m} \quad \textit{Ans.}$$

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**Example 8.19** In a double-pipe counterflow heat exchanger the inner tube has a diameter of 20 mm and very little thickness. The inner diameter of the outer tube is 30 mm. Water flows through the inner tube at a rate of 0.5 kg/s, and oil flows through the shell at a rate of 0.8 kg/s. Take the average temperatures of the water and the oil as 47°C and 80°C, respectively and assume fully developed flow. Determine the overall heat transfer coefficient. Given: For water at 47°C, $\rho$ = 989 kg/m$^3$, $k$ = 0.637 W/m K, $\nu$ = 0.59 × 10$^{-6}$ m$^2$/s and Pr = 3.79. For oil at 80°C, $\rho$ = 852 kg/m$^3$, $k$ = 0.138 W/m K, $\nu$ = 37.5 × 10$^{-6}$ m$^2$/s and Pr = 490.

**Solution**

**Water-side:**

$$V = \frac{\dot{m}}{\rho A} = \frac{0.5}{989 \times \pi/4 \times (0.02)^2} = 1.61 \text{ m/s}$$

$$\text{Re}_d = \frac{VD}{\nu} = \frac{1.61 \times 0.02}{0.59 \times 10^{-6}} = 54{,}576$$

The flow of water is thus turbulent. Using the Dittus–Boelter equation,

$$\text{Nu}_d = 0.023\, \text{Re}_d^{0.8}\, \text{Pr}^{0.4} = 0.023\,(54{,}576)^{0.8}\,(3.79)^{0.4}$$

$$= 241.4 = h_i D/k$$

$$h_i = \frac{241.4 \times 0.637}{0.02} = 7690 \text{ W/m}^2 \text{ K}$$

**Oil-side:** Hydraulic diameter of the annulus,

$$D_h = D_0 - D_i = 0.03 - 0.02 = 0.01 \text{ m}$$

$$V = \frac{\dot{m}}{\rho A} = \frac{0.8}{852 \times \pi/4\,(0.03^2 - 0.02)^2} = 2.39 \text{ m/s}$$

$$\text{Re}_d = \frac{VD_h}{\nu} = \frac{2.39 \times 0.01}{37.5 \times 10^{-6}} = 637$$

The flow of oil is laminar. Assuming fully developed flow, corresponding to $D_i/D_0$ = 0.02/0.03 = 0.667, from Table 8.5.

**Table 8.5** *Nusselt number of fully developed laminar flow in a circular annulus with one surface insulated and the other isothermal [3]*

| $D_i/D_0$ | $Nu_i$ | $Nu_0$ |
|---|---|---|
| 0.00 | — | 3.66 |
| 0.05 | 17.46 | 4.06 |
| 0.10 | 11.56 | 4.11 |
| 0.25 | 7.37 | 4.23 |
| 0.50 | 5.74 | 4.43 |
| 1.00 | 4.86 | 4.8 |

$$\text{Nu}_i = 5.45 = \frac{h_0 D_h}{k}$$

$$h_0 = \frac{5.45 \times 0.138}{0.01} = 75.2 \text{ W/m}^2 \text{ K}$$

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$$\frac{1}{U_0} = \frac{1}{h_i} + \frac{1}{h_0} = \frac{1}{7690} + \frac{1}{75.2}$$

$$U_0 = 75.1 \text{ W/m}^2 \text{ K} \quad \textit{Ans.}$$

It may be noted that $h_i >> h_0$,

$$U_0 \cong h_0$$

The oil-side heat transfer coefficient offers more thermal resistance and controls the rate of heat transfer. Some enhancement technique (like the use of fins) is employed on the oil side to compensate for low $h_0$.

**Example 8.20** The condenser of a large steam power plant is a shell-and-tube heat exchanger having a single shell and 30,000 tubes, with each tube making two passes. The tubes are thin-walled with 25 mm diameter and steam condenses on the outside of the tubes with $h_0$ = 11 kW/m$^2$ K. The cooling water flowing through the tubes is 30,000 kg/s and the heat transfer rate is 2 GW. Water enters at 20°C while steam condenses at 50°C. Find the length of the tubes in one pass. Properties of water at 27°C are $c_p$ = 4.18 kJ/kg K, m = 855 × 10$^{-6}$ N s/m$^2$, $k$ = 0.613 W/m K and Pr = 5.83.

**Solution**

$$Q = \dot{m}_c c_{p_c} (T_{c_2} - T_{c_1}) = 2 \times 10^9 \text{ W}$$

$$T_{c_2} - 20 = \frac{2 \times 10^9}{30{,}000 \times 4.18} = 16$$

$\therefore$ $\quad T_{c_2} = 36°\text{C}$

Also, $\quad Q = \text{UAF} (\Delta T_{1\text{m}})_{cf}$

where $\quad A = N \times \pi D(2L)$, $L$ = length of one tube pass

$$\text{Re}_D = \frac{\bar{V} D \rho}{\mu} = \frac{4\dot{m}}{\pi D \mu} = \frac{4 \times 1 \text{ kg/s}}{\pi \times 0.025 \text{ m} \times 855 \times 10^{-6} \text{ Ns/m}^2}$$

$$= 59{,}567$$

The flow is turbulent. Dittus–Boelter equation is used to evaluate $h_i$,

$$\text{Nu}_D = 0.023 \text{ Re}_D^{0.8} \text{ Pr}^{0.4}$$

$$= 0.023\, (59{,}567)^{0.8}\, (5.83)^{0.4} = 308$$

$$\frac{h_i D}{k} = 308$$

$$h_i = \frac{308 \times 0.613}{0.025} = 7552 \text{ W/m}^2 \text{ K}$$

$$\frac{1}{U} = \frac{1}{h_i} + \frac{1}{h_0} = \frac{1}{7552} + \frac{1}{11{,}000}$$

$$U = 4478 \text{ W/m}^2 \text{ K}$$

$$(\Delta T_{1\text{m}})_{cf} = \frac{30 - 14}{\ln 30/14} = 21°\text{C}$$

The correction factor $F$ is obtained from Fig. 8.18,

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$$P = \frac{36-20}{50-20} = 0.53,\ R = \frac{50-50}{36-20} = 0$$

$$F = 1$$

$$\therefore \qquad L = \frac{Q}{UN2\pi DF(\Delta T_{\mathrm{lm}})_{cf}}$$

$$= \frac{2\times 10^9}{4478\times 30,000\times 2\times \pi\times 0.025\times 1\times 21}$$

$$= 4.51 \text{ m} \quad \textit{Ans.}$$

**Example 8.21** A water heat pipe operating at 100°C and atmospheric pressure has an inner diameter of 20 mm and is 300 mm long. The heat pipe is inclined at 60° with the evaporator above the condenser. The wick consists of four layers of wire-screen with the wire diameter of 0.025 mm on the inner surface of the pipe. The pore radius is $10^{-5}$ m and the permeability is $4 \times 10^{-11}$ m$^2$. Determine the number of heat pipes needed to remove 1 kW of heat from a system. Properties of water at 100°C are $h_{fg}$ = 2260 kJ/kg, $\rho_l$ = 958 kg/m$^3$, $\mu_l$ = 279 × $10^{-6}$ N s/m$^2$ and $\sigma_l$ = 58.9 × $10^{-3}$ N/m. Assume perfect wetting.

**Solution** The pressure balance relation to prevent dryout is

$$(\Delta p_c)_{\max} > \Delta p_1 + \Delta p_v + \Delta p_g$$

Neglecting vapour pressure drop,

$$\frac{2\sigma_1 \cos\theta}{r_c} = \frac{\mu_1 Q L_{\mathrm{eff}}}{\rho_1 h_{fg} A_w k_w} + \rho_1 g L_{\mathrm{eff}} \sin\phi$$

$$A_w = \text{Area of the wick}$$

$$= \pi D t$$

$$= \pi \times 20 \text{ mm} \times 0.025 \text{ mm} \times 4$$

$$= 6.28 \text{ mm}^2$$

Let $\qquad L_{\mathrm{eff}} = L = 300$ mm

For perfect wetting the contact angle $\theta = 0$

$$\cos\theta = 1$$

$$\phi = 60°$$

Since $\qquad \dot{m}_{\max} h_{fg} = Q_{\max}$,

$$\dot{m}_{\max} = \left(\frac{2\sigma_1}{r_c} - \rho_1 g L_{\mathrm{eff}} \sin\phi\right)\frac{\rho_1 A_w K_w}{\mu_1 L_{\mathrm{eff}}}$$

$$= \left(\frac{2\times 58.9\times 10^{-3}\,\text{N/m}}{10^{-5}\,\text{m}} - 958\,\frac{\text{kg}}{\text{m}^3}\times 9.81\,\frac{\text{m}}{\text{s}^2}\times 0.3\,\text{m}\times 0.866\right)$$

$$\times \frac{958 \text{ kg/m}^3 \times 6.28\times 10^{-6}\,\text{m}^2 \times 4\times 10^{-11}\,\text{m}^2}{279\times 10^{-6}\,\text{Ns/m}^2 \times 0.3\,\text{m}}$$

$$= (117.8 \times 100 - 2441.6)\ \frac{\text{N}}{\text{m}^2} \times 287.5 \times 10^{-11} \text{ ms}$$

$$= 2.6848 \times 10^{-5} \text{ kg/s.}$$

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$$Q_{max} = 2.6848 \times 10^{-5} \times 2260 \times 10^3 \text{ W}$$
$$= 60.67 \text{ W}$$

This is the heat removal by one heat pipe.
To remove 1 kW, the number of heat pipes required

$$= \frac{1000}{60.67} = 16.48 \text{ or } 17 \quad \textit{Ans.}$$

**Example 8.22** Water entering at 10°C at the rate of 20 kg/s is heated by a corrosive gas flowing at the rate of 30 kg/s from a process at 300°C using a run-around coil as shown in Fig. 8.46. Calculate (a) the mass flow rate of the secondary fluid required, (b) the effectiveness of the overall heat transfer, (c) the exit temperature of the water and (d) the temperatures of the secondary fluid. Given: $c_p$ of gases = 1.2 kJ/kg K, $c_p$ of water = 4.2 kJ/kg K, $c_p$ of secondary fluid = 3.8 kJ/kg K, (*UA*) for the gas to secondary fluid heat exchanger = 40 W/m$^2$ K and (*UA*) for the secondary fluid to water heat exchanger = 200 W/m$^2$ K.

**Solution** (a) For the hot fluid

$$(\dot{m}c)_h = 30 \times 1.2 = 36 \text{ kW/K}$$

and for the cold fluid

$$(\dot{m}c)_c = 20 \times 4.2 = 84 \text{ kW/K}$$

Using Eq. (8.68),

$$\dot{m}_s = \text{mass flow of secondary fluid}$$
$$= \frac{36 \times 84\,(40 + 200)}{3.8\left[(84 \times 40) + (36 \times 200)\right]}$$
$$= 18.09 \text{ kg/s} \quad \textit{Ans.} \text{ (a)}$$

Now, from Eq. (8.67)

$$\frac{1}{(UA)_0} = \frac{1}{40} + \frac{1}{200} = 0.03$$

$$(UA)_0 = 33.333 \text{ kW/K}$$

$$\text{NTU} = \frac{(UA)_0}{(\dot{m}c)_{min}} = \frac{33.333}{36} = 0.926$$

$$R = \frac{(\dot{m}c)_{min}}{(\dot{m}c)_{max}} = \frac{36}{84} = 0.429$$

Effectiveness,
$$\varepsilon = \frac{1 - e^{-0.926 \times 0.571}}{1 - 0.429 e^{-0.926 \times 0.571}}$$
$$= 0.55 \quad \textit{Ans.} \text{ (b)}$$

Again,
$$\varepsilon = 0.55 = \frac{300 - T_{h_2}}{300 - 10}$$

$$T_{h_2} = 140.5°\text{C}$$

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Water exit temperature,

$$T_{c_2} = T_{c_1} + R\,(T_{h_1} - T_{h_2})$$
$$= 10 + 0.429\,(300 - 140.5)$$
$$= 78.4°\text{C} \quad \textit{Ans.}\ \text{(c)}$$

$$\frac{T_{h_1} - T_{s_1}}{T_{s_1} - T_{c_1}} = \frac{(UC)_c}{(UA)_h} = \frac{200}{40} = 5$$

$$T_{s_1} = \frac{300 + (5 \times 78.4)}{6} = 115.3°\text{C} \quad \textit{Ans.}\ \text{(d)}$$

$$\frac{T_{h_2} - T_{s_2}}{T_{s_2} - T_{c_2}} = 5$$

$$T_{s_2} = \frac{140.5 + 5 \times 10}{6} = 31.8°\text{C} \quad \textit{Ans.}\ \text{(d)}$$

The complete temperature changes are shown in Fig. Ex. 8.22.

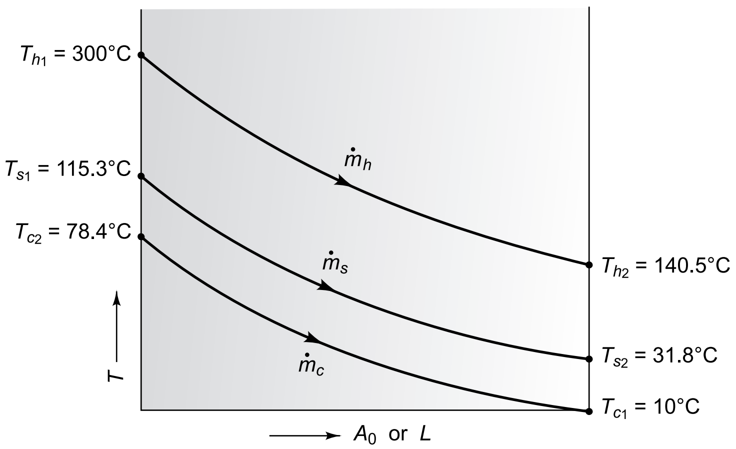


**Fig.** *Ex. 8.22*

## Summary

The chapter begins with the definition of a heat exchanger, followed by its classification and types of applications. The concept of the overall heat transfer coefficient is then re-emphasized and the use of fouling factor is explained. Typical axial temperature distributions in various types of heat exchangers are shown, providing the detailed derivation for temperature profiles in parallel-flow and counter-flow heat exchangers. The concept of LMTD-correction factor approach used in heat exchanger calculations is discussed along with the use of correction factor graphs. The basic idea of effectiveness-NTU method and its advantages over LMTD-correction factor approach in heat exchanger design are explained. Various augmentative techniques for heat transfer enhancement are discussed. The schematic diagram, operating characteristics and applications of heat pipes are explained in fairly elaborate details. Finally, the principle of operation of a run-around coil for waste heat recovery is briefly discussed.

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## Important Formulae and Equations

| *Equation no.* | *Equation* | *Remarks* |
|---|---|---|
| (8.1) | $Q = UA(T_h - T_c)$ <br> where $\dfrac{1}{UA} = \Sigma R = \dfrac{1}{h_1 A} + \dfrac{x_w}{k_w A} + \dfrac{1}{h_2 A}$ | Heat transfer from a hot to a cold fluid through a plane wall |
| (8.3) | $Q = U_0 A_0 (T_h - T_c)$ <br> where $\dfrac{1}{U_0 A_0} = \dfrac{1}{h_i A_i} + \dfrac{x_w}{k_w A_{\text{lm}}} + \dfrac{1}{h_0 A_0}$, <br> $A_0 = \pi D_0 L, A_i = \pi D_i L, A_{\text{lm}} = \dfrac{A_2 - A_1}{\ln(A_2/A_1)}$ | Heat transfer through a cylindrical wall |
| (8.5) | $\dfrac{1}{U_0' A_0} = \dfrac{1}{h_{fi} A_i} + \dfrac{1}{h_i A_i} + \dfrac{x_w}{k_w A_{\text{lm}}} + \dfrac{1}{h_{f0} A_o} + \dfrac{1}{h_0 A_0}$ <br> where $h_{fi}$ and $h_{f0}$ are scaled heat transfer coefficients | Overall heat ransfer coefficient with scaled surfaces of a pipe |
| (8.6) | $R_f = \dfrac{1}{U_0'} - \dfrac{1}{U_0}$ | Fouling factor is the reciprocal of the scale heat transfer coefficient |
| (8.11) | $\ln \dfrac{\Delta T_i}{\Delta T_e} = U_0 A_0 \mu_p$ <br> where $\mu_p = \dfrac{1}{\dot{m}_h c_h} + \dfrac{1}{\dot{m}_c c_c}$ | Ratio of inlet and outlet temperature differences in a parallel flow heat exchanger |
| (8.12) | $Q = U_0 A_0 \Delta T_{\text{lm}} = \dot{m}_h c_h (T_{h1} - T_{h2})$ <br> $= \dot{m}_c c_c (T_{c_2} - T_{c_1})$ <br> where $\Delta T_{\text{lm}} = \dfrac{\Delta T_i - \Delta T_e}{\ln(\Delta T_i/\Delta T_e)}$ = LMTD | Rate of heat transfer in a parallel-flow or counter-flow heat exchanger |
| (8.17) | $\ln \Delta T_i/\Delta T_0 = U_0 A_0 \mu_c$ <br> where $\mu_c = \dfrac{1}{\dot{m}_h c_h} - \dfrac{1}{\dot{m}_c c_c}$ | Ratio of inlet and outlet temperature differences in a counterflow heat exchanger |
| (8.22) | $Q = U_0 A_0 F(\Delta T_{\text{lm}})_{\text{Counterflow}}$ | Correction factor $F$ in a multipass heat exchanger |
| (8.28) | $Q = A_o \dfrac{U_i \Delta T_e - U_e \Delta T_i}{\ln \dfrac{U_i \Delta T_e}{U_e \Delta T_i}}$ <br> where $U_i = a + b\,\Delta T_i$ and $U_e = a + b\,\Delta T_e$ | Heat transfer it $U_0$ varies along the heating surface |

(*Contd*)

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| Equation no. | Equation | Remarks |
|---|---|---|
| (8.29) | $\varepsilon = \dfrac{Q}{Q_{\max}} = \dfrac{\dot{m}_c c_c (T_{c_2} - T_{c_1})}{(m.c)_s (T_{h_1} - T_{c_1})}$ | Effectiveness $\varepsilon$ of a heat exchanger $(\dot{m}c)_s$ is the smaller of the two heat capacities or $C_{\min}$ |
| (8.32) | $\varepsilon = \dfrac{(\Delta T)_l}{T_{h_1} - T_{c_1}}$ | Effectiveness defined alternatively where $(\Delta T)_l$ is the larger of the two temperature differences $(T_{h_1} - T_{h_2})$ and $(T_{c_2} - T_{c_1})$ |
| (8.33) | $R = \dfrac{(\dot{m}c)_s}{(\dot{m}c)_l} = \dfrac{C_{\min}}{C_{\max}}$ | Heat capacity ratio |
| (8.36) | $\varepsilon_{pf} = \dfrac{1 - \exp[-\mathrm{NTU}(1+R)]}{1+R}$ | Effectiveness of a parallel flow heat exchanger |
| (8.37) | $\mathrm{NTU} = \dfrac{U_0 A_0}{(\dot{m}c)_s} = \dfrac{U_0 A_0}{C_{\min}}$ | Number of transfer units. |
| (8.41) | $\varepsilon_{cf} = \dfrac{1 - \exp[-\mathrm{NTU}(1-R)]}{1 - R\exp[-\mathrm{NTU}(1-R)]}$ | Effectiveness of a counterflow heat exchanger |
| (8.42) | $\varepsilon_{pf} = 1 - \exp(-\mathrm{NTU}) = \varepsilon_{cf}$ | Effectiveness of a phase change heat exchanger |
| (8.44) | $\varepsilon_{fp} = \dfrac{1 - \exp(-2\mathrm{NTU})}{2}$ | Effectiveness of a balanced parallel flow heat exchanger when $R = 1$ |
| (8.45) | $\varepsilon_{cf} = \dfrac{\mathrm{NTU}}{\mathrm{NTU}+1}$ | Effectiveness of a balanced counterflow heat exchanger when $R = 1$ |
| (8.56) | $Q_{\max} = \left(\dfrac{\rho_l \sigma_l h_{fg}}{\mu_l}\right)\left(\dfrac{A_w k_w}{L_{\text{eff}}}\right)\left(\dfrac{1}{r_c} - \dfrac{\rho_l g L_{\text{eff}} \sin\phi}{\sigma_f}\right)$ | Maximum heat transport capability of a heat pipe |
| (8.57) | $M = \dfrac{\rho_l \sigma_l h_{fg}}{\mu_l}$ | Figure of merit of a heat pipe |
| (8.58) | $Q_{\max} = \dfrac{A_w h_{fg} \rho_l^2}{\mu_l}\left(\dfrac{l_w k_w}{L_{\text{eff}}}\right)$ | Maximum achievable power transfer by a heat pipe |

## Review Questions

8.1 What is a heat exchanger? What are its applications?
8.2 Give the three broad classes of heat exchangers.
8.3 Explain storage type heat exchangers.
8.4 What is a direct contact heat exchanger? Give some examples.
8.5 What are the different flow arrangements in recuperative heat exchangers?
8.6 What do you mean by fouling factor? What are the causes of fouling?

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8.7 What is a compact heat exchanger? What are its applications?

8.8 What do you understand by mixed flow and unmixed flow?

8.9 What is a shell-and-tube heat exchanger? Why are baffles used? What are headers?

8.10 What are multipass heat exchangers? When are they used?

8.11 Explain the operation of a plate heat exchanger? What are its applications?

8.12 In a gas-to-liquid heat exchanger, why are fins provided on the gas side?

8.13 Give a comparison of parallel-flow and counterflow heat exchangers. Why are counterflow heat exchangers mostly used?

8.14 For a balanced counterflow heat exchanger where $\dot{m}_h c_h = \dot{m}_c c_c$, show that the temperature profiles of the two fluids along the heat exchanger are linear and parallel.

8.15 Show that as $\Delta T_i \to \Delta T_e$, $\Delta T_{\text{lm}} \to \Delta T_{\text{am}}$.

8.16 Explain when one of the fluids undergoes phase change, the directions of the two fluids are immaterial in evaluating $(\Delta T)_{\text{lm}}$.

8.17 How is the mean temperature difference between the two fluids in a multipass heat exchanger estimated? What is the correction factor?

8.18 How are TEMA charts useful in the design of multipass heat exchangers?

8.19 If $U_0$ varies linearly with temperature, show that the rate of heat transfer is given by

$$Q = A_0 \frac{U_e \Delta T_i - U_i \Delta T_e}{\ln (U_e \Delta T_i / U_i \Delta T_e)}$$

where $U_i$ and $U_e$ are the values of $U_0$ at the inlet and exit of the heat exchanger.

8.20 Define effectiveness and NTU of a heat exchanger.

8.21 What is the limitation of the LMTD method? How is $\varepsilon$-NTU method superior to correction factor-LMTD method?

8.22 In the definition of effectiveness, explain why minimum heat capacity value ($C_{\min}$) is used for the maximum possible rate of heat transfer.

8.23 Show that for parallel flow heat exchanger

$$\varepsilon = \frac{1 - \exp[-\text{NTU}(1 + R)]}{1 + R}$$

8.24 Show that for counterflow heat exchanger

$$\varepsilon = \frac{1 - \exp[-\text{NTU}(1 - R)]}{1 - R \exp[-\text{NTU}(1 - R)]}$$

8.25 How are exit fluid temperatures determined with the help of $\varepsilon$-NTU method?

8.26 When one of the two fluids undergoes phase change, show that the effectiveness values for both parallel flow and counterflow heat exchangers are equal and given by

$$\varepsilon = 1 - \exp(-\text{NTU})$$

8.27 For a balanced heat exchanger ($R = 1$), show that

(a) for parallel flow, $\varepsilon = \dfrac{1 - \exp(-2\text{NTU})}{2}$

(b) for counterflow, $\varepsilon = \dfrac{\text{NTU}}{\text{NTU} + 1}$

8.28 Explain how the charts provided by Kays and London are useful in the design of heat exchangers.

8.29 Explain the operation of a heat pipe. Why is it called a superconductor?

8.30 What are the advantages of a heat pipe?

8.31 Why is a wick needed in a heat pipe? State its function.

8.32 What is a thermosiphon? How is it different from a heat pipe?

8.33 How is a heat pipe sensitive to the effects of gravity?

8.34 What is the function of a splitter plate?

8.35 How is a heat pipe rated? Why are fins used in a heat pipe?

8.36 How does the performance of a heat pipe depend on the angle of operation?

8.37 Give examples of a few working fluid–pipe material combinations of heat pipes along with temperature ranges.

8.38 What is the maximum capillary pumping head of a heat pipe? How is it balanced?

8.39 Why only wetting fluids can be used in heat pipes? What is wicking height?

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8.40 What is the maximum heat transport capability of a heat pipe?

8.41 State the four limitations which restrict the performance of a heat pipe.

8.42 What are the applications of heat pipes? Why are heat pipes versatile in removing localised heat and for waste heat recovery?

8.43 Explain a run-around coil system. What are its applications?

## Objective Questions

8.1 In a concentric double-pipe heat exchanger where one of the fluids undergoes phase change
(a) the two fluids should flow opposite to each other.
(b) the two fluids should flow parallel to each other.
(c) the two fluids should flow normal to each other.
(d) the direction of flow of the two fluids are of no consequence.

8.2 In case of a heat exchanger, the value of logarithmic temperature difference should be
(a) as small as possible
(b) as large as possible
(c) constant
(d) none of the above

8.3 A heat exchanger with heat transfer surface area $A_0$ and overall heat transfer coefficient $U_0$ handles two fluids of heat capacities $C_1$ and $C_2$ with $C_1 > C_2$. The parameter NTU (number of transfer units) used in heat exchanger analysis is specified as
(a) $\dfrac{U_0 A_0}{C_1}$ (b) $\dfrac{U_0 A_0}{C_2}$
(c) $\dfrac{U_0}{A_0 C_2}$ (d) $\dfrac{A_0 C_0}{U_0}$

8.4 Compared to parallel flow heat exchanger, the LMTD of a counteflow heat exchanger is
(a) more
(b) less
(c) the same
(d) none of the above

8.5 In a two fluid heat exchanger, the inlet and outlet temperatures of the hot fluid are 65ºC and 40ºC respectively. For the cold fluid these are 15ºC and 43ºC. The heat exchanger is a
(a) parallel flow heat exchanger
(b) counterflow heat exchanger
(c) heat exchanger where both parallel flow and parallel flow operations are possible.
(d) none of the above

8.6 In a counterflow heat exchanger, the hot fluid enters at 100ºC and leaves at 60ºC. The cold fluid enters at 40ºC and leaves at 80ºC. It is a balanced heat exchanger with $\dot{m}_h c_h = \dot{m}_c c_c$. The LMTD of the heat exchanger is
(a) zero (b) indeterminate
(c) 40ºC (d) 20ºC

8.7 For evaporators and condensers under the given conditions, LMTD for counteflow will be
(a) greater than parallel flow
(b) equal to parallel flow
(c) less than parallel flow
(d) very much larger than parallel flow

8.8 For a balanced counterflow heat exchanger with $\dot{m}_h c_h = \dot{m}_c c_c$, the effectiveness is given by
(a) $\varepsilon = \dfrac{1-\exp(-2\text{NTU})}{2}$
(b) $\varepsilon = \dfrac{1+\exp(-2\text{NTU})}{2}$
(c) $\varepsilon = \dfrac{\text{NTU}}{\text{NTU}+1}$
(d) $\varepsilon = \dfrac{\text{NTU}+1}{\text{NTU}}$

8.9 For $C_{min}/C_{max} = 0$, the effectiveness is given by the expression
(a) $\varepsilon = 1 - \exp(\text{NTU})$
(b) $\varepsilon = 1 - \exp(-\text{NTU})$
(c) $\varepsilon = 1 + \exp(-\text{NTU})$
(d) $\varepsilon = \exp(\text{NTU})^{-1}$

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8.10 Match List I with List II and select the correct answers using the codes given below:

| **List I** | **List II** |
|---|---|
| A. Number of transfer units | 1. Regenerators |
| B. Periodic flow heat exchangers | 2. Fouling factor |
| C. Phase change | 3. A measure of heat exchanger size |
| D. Deposition on heat exchanger surface | 4. Condensers |

**Codes:**

| | A | B | C | D |
|---|---|---|---|---|
| (a) | 1 | 2 | 3 | 4 |
| (b) | 2 | 3 | 4 | 1 |
| (c) | 3 | 1 | 4 | 2 |
| (d) | 4 | 3 | 2 | 1 |

8.11 Consider the following statements:
In a shell and tube heat exchanger, baffles are provided on the shell side to
1. prevent the stagnation of the shell side fluid
2. improve heat transfer
3. provide support for tubes
4. prevent fouling of tubes

Of these statements
(a) 1, 2, 3 and 4 are correct
(b) 1, 2 and 3 are correct
(c) 1 and 2 are correct
(d) 2 and 4 are correct

8.12 A counterflow shell and tube heat exchanger is used to heat water with hot exhaust gases. The water ($c$ = 4180 J/kgK) flows at the rate of 2 kg/s and the exhaust gases ($c$ = 1000 J/kgK) flow at the rate of 5 kg/s. If the heat transfer surface area is 32 m$^2$ and the overall heat transfer coefficient is 200 W/m$^2$K. The NTU of the heat exchanger is
(a) 4.5 (b) 2.4
(c) 8.6 (d) 1.28

8.13 For multipass flow and cross-flow shell-and-tube heat exchangers, to determine the mean temperature difference $\Delta T_m$ between the two fluids, a correction factor $F$ is used such that
(a) $\Delta T_m = F(\text{LMTD})_{\text{counterflow}}$
(b) $\Delta T_m = F(\text{LMTD})_{\text{parallel flow}}$
(c) $\Delta T_m = \dfrac{F(\text{LMTD})_{\text{counterflow}}}{F}$
(d) $\Delta T_m = \dfrac{F(\text{LMTD})_{\text{parallel flow}}}{F}$

8.14 For a phase change heat exchanger, the heat capacity ratio $R$ is equal to
(a) 1 (b) 0
(c) $\infty$ (d) indeterminate

8.15 For a balanced counterflow heat exchanger, the temperature profiles of the two fluids along the length of the heat exchanger
(a) linear (b) parallel
(c) linear and parallel (d) parabolic

8.16 An oil cooler for a lubrication system has to cool 1000 kg/h of oil ($c_p$ = 4.18 kJ/kgK) from 80ºC to 40ºC by using a cooling water ($c_p$ = 4.18 kJ/kgK) flow of 1000 kg/h available at 30ºC. The heat exchanger is to be
(a) parallel flow
(b) counterflow
(c) cross-flow
(d) none of the above

8.17 Consider the following statements:
The effect of fouling in a water-cooled steam condenser is that
1. reduces the heat transfer coefficient of water.
2. reduces the overall heat transfer coefficient.
3. reduces the area available for heat transfer.
4. increases the pressure drop of water.

Of these statements:
(a) all of the above are correct
(b) 2, 3 and 4 are correct
(c) 2 and 4 are correct
(d) 1, 2 and 4 are correct

8.18 **Assertion (A):** If $C_{\max} = C_c$, it is not possible that $T_{c_2}$ be equal to $T_{h_1}$
**Reasoning (R):** Because this would require $(T_{h_1} - T_{h_2})$ greater than $(T_{h_1} - T_{c_1})$.

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**Codes:**

(a) Both A and R are true
(b) Both A and R are false
(c) A is true, R is false
(d) A is false, R is true

8.19 If any of the two fluids undergoes phase change, the correction factor $F$ in a multipass heat exchange would be

(a) 0.2 (b) 0.5
(c) 0.8 (d) 1.0

8.20 A heat pipe is used to transfer heat from the source to the sink by a fluid by means of

(a) conduction
(b) evaporation
(c) condensation
(d) evaporation and condensation

8.21 The return of the condensate in a heat pipe takes place by

(a) gravity
(b) buoyancy effect
(c) capillary effect through a wick
(d) entrainment effect

8.22 It is a heat recovery system which connects two recuperative heat exchangers by a third fluid exchanging heat with each fluid

(a) helical coil
(b) run-around coil
(c) spiral tube heat exchanger
(d) heat pipe

## Answers

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 8.1 | (d) | 8.2 | (b) | 8.3 | (b) | 8.4 | (a) | 8.5 | (b) |
| 8.6 | (c) | 8.7 | (b) | 8.8 | (c) | 8.9 | (b) | 8.10 | (c) |
| 8.11 | (b) | 8.12 | (d) | 8.13 | (a) | 8.14 | (b) | 8.15 | (c) |
| 8.16 | (b) | 8.17 | (d) | 8.18 | (a) | 8.19 | (d) | 8.20 | (d) |
| 8.21 | (c) | 8.22 | (b) | | | | | | |

## Open Book Problems

8.1 In a certain double pipe heat exchanger hot water flows at a rate of 5000 kg/h and gets cooled from 95ºC to 65ºC. At the same time 50,000 kg/h of cooling water at 30ºC enters the heat exchanger. The flow conditions are such that the overall heat transfer coefficient remains constant at 2270 W/m²K. Determine the heat transfer area required and the effectiveness, assuming two streams are in parallel flow and have $c_p = 4.2$ kJ/kgK.

*Hints:* $Q = \dot{m}_h c_h (T_{h_1} - T_{h_2}) = m_c c_c (T_{c_2} - T_{c_1}) = U_0 A_0 \Delta T_{\text{lm}}$.

Find $T_{c_2}$ and $(\Delta T_{\text{lm}})_{p.f}$. Given : $U_0 = 2270$ W/m²K. Find $A_0$. Now $\dot{m}_h c_h < \dot{m}_c c_c$,

$$\text{effectiveness } \varepsilon = \frac{\dot{m}_h c_h (T_{h_1} - T_{h_2})}{\dot{m}_h c_h (T_{h_1} - T_{c_1})}.$$

8.2 An oil cooler for a lubrication system has to cool 1000 kg/h of oil ($c_p = 2.09$ kJ/kgK) from 80ºC to 40°C by using a cooling water flow of 1000 kg/h at 30°C. Give your choice for a parallel flow or a counterflow heat exchanger with reasons. Calculate the surface area of the heat exchanger, if the overall heat transfer coefficient is 24 W/m²K.

*Hints:* $Q = \dot{m}_h c_h (T_{h_1} - T_{h_2}) = \dot{m}_c c_c (T_{c_2} - T_{c_1}) = U_0 A_0 \Delta T_{\text{lm}}$. From this equation find $T_{c_2} < T_{h_2}$, counterflow arrangement must be used. Find $(\Delta T_{\text{lm}})_{cf}$ and $A_0$ required.

8.3 A counterflow heat exchanger through which passes 12.5 kg/s of air to be cooled from 540ºC to 140ºC, contains 4200 tubes, each having a diameter of 30 mm. The inlet and outlet temperatures of cooling water are 25ºC and 75ºC respectively. If the water side resistance to flow is regligible, calculate the tube length required for this duty. For turbulent flow inside tubes, you can use: $\text{Nu} = 0.023\ \text{Re}^{0.8}\ \text{Pr}^{0.4}$. Properties of the air at the average temperature are as follows:

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$\rho$ = 1.009 kg/m$^3$, $c_p$ = 1.0082 kJ/kg°C, $\mu$ = 2.075 × 10$^{-5}$ kg/ms and $k$ = 3.003 × 10$^{-2}$ W/m$^2$K.

*Hints:* Find $\text{Re}_d = \dfrac{\rho V d}{\mu}, \dot{m} = NAV\rho, \text{Re} = \dfrac{\dot{m}d}{Na\mu}$

$$\text{Pr} = \frac{\mu c_p}{k}, \text{Nu}_d = 0.023\,\text{Re}_d^{0.8}\,\text{Pr} = \frac{h_i D}{k}.$$

Since water side resistance is negligible, $\dfrac{1}{U_0} = \dfrac{1}{h_i} + \dfrac{1}{h_e} = \dfrac{1}{h_i}$

∴ $U_0 = h_i$. Find $(\Delta T)_{\text{lm}}$.

Then $Q = \dot{m}_h c_h (T_{h_1} - T_{h_2} = U_0 A_0 (\Delta T)_{\text{lm}}$
$= U_0 \times N\,\pi d_0 L \times (\Delta T)_{\text{lm}}$.

$L$ can be determined.

8.4 A counterflow concentric tube heat exchanger is used to cool lubricating oil for a large gas turbine. The flow rate of cooling water through the inner tube ($d_i$ = 20 mm) is 0.18 kg/s, while the flow rate of oil through the outer annulus ($d_0$ = 40 mm) is 0.12 kg/s. The inlet and outlet tempratures of oil are 95ºC and 65ºC respectively. The water enters at 30ºC to the exchanger. Neglecting the tube wall resistance, fouling factors and heat loss to the surroundings, calculate the length of the tube. Properties of engine oil at 80ºC: $c_p$ = 2131 J/kgK, $\mu$ = 0.0325 Ns/m$^2$, $k$ = 0.138 W/mK. For water, $c_p$ = 4174 J/kgK, $\mu$ = 725 × 10$^{-6}$ Ns/m$^2$, $k$ = 0.625 W/mK and Pr = 4.85.

*Hints:* $Q = \dot{m}_h c_h (T_{h_1} - T_{h_2}) = \dot{m}_c c_c (T_{c_2} - T_{c_1}) = U_0 A_0\,(\Delta\text{T})_{\text{lm}}$. Find $T_2$ and $(\Delta T_{\text{lm}})_{cf}$. Now, $\dfrac{1}{U_0} = \dfrac{1}{h_i} + \dfrac{1}{h_0}$. To find $h_i$ use $\text{Nu}_d$ = 0.023 $\text{Re}_d^{0.8}$ $\text{Pr}^{0.4}$ after establishing that the flow is turbulent. For oil flowing through the annulus, find hydraulic diameter, $D_h$ and Re. If Re is less than 2300, and the flow is laminar, you can use $Nu$ = 3.66, assuming uniform temperature along the annular surface. If the flow is turbulent, you have to use appropriate equation to find $h_0$. Find $A_0 = \pi d_0 L$ and then $L$.

8.5 A crossflow heat exchanger is used to heat an oil in the tubes ($c$ = 1.9 kJ/kgK) from 15°C to 85°C. Blowing across the outside of the tubes is steam which enters at 130ºC and leaves at 110ºC with a mass flow of 5.2 kg/s. The overall heat transfer coefficient is 275 W/m$^2$K and $c$ for steam is 1.86 kJ/kgK. Calculate the surface area of the heat exchanger.

*Hints:* By energy balance, $Q = \dot{m}_h c_h (T_{h_1} - T_{h_2})$
We can find the area by using Eq. (8.22), $Q = U_0 A_0 F (\Delta T_{\text{lm}})_{cf}$. Find $(\Delta T_{\text{lm}})_{cf}$, capacity ratio $R = \dfrac{T_{s_1} - T_{s_2}}{T_{t_2} - T_{t_1}}$ and temperature ratio $P = \dfrac{T_{t_2} - T_{t_1}}{T_{s_1} - T_{t_1}}$ where subscript Δ refers to the shell side and subscript $t$ refers to the tube side fluid. Consulting Fig. 8.20 and considering the mixed fluid as mixed and the oil as unmixed, we find the correction factor $F$ and then $A_0$.

8.6 Water enters a counterflow double-pipe heat exchanger at 15ºC flowing at the rate of 1700 kg/h. It is heated by oil ($c_p$ = 2000 J/kgK) flowing at the rate of 550 kg/h from an inlet temperature of 94°C. For an area of 1 m$^2$ and an overall heat transfer coefficient of 1075 W/m$^2$K, determine the total heat transfer and the outlet temperatures of water and oil. ($c_p$ of water = 4186 J/kgK).

*Hints:* Find heat capacity rates of water = $\dot{m}_c c_c = C_c$ and of oil $C_h = \dot{m}_h c_h$ and check which value is bigger. So as to identify $C_{\min}$ and $C_{\max}$ and then find $R = \dfrac{C_{\min}}{C_{\max}}$ and $\text{NTU} = \dfrac{U_0 A_0}{C_{\min}}$. The effectiveness for counterflow heat exchanger,

$$\varepsilon = \frac{1 - \exp[-\text{NTU}(1-R)]}{1 - \text{R}\exp[-\text{NTU}(1-R)]} = \frac{Q}{Q_{\max}}$$
$$= \frac{\dot{m}_h c_h (T_{h_1} - T_{h_2}) \text{ or } \dot{m}_c c_c (T_{c_2} - T_{c_1})}{C_{\min}(T_{h_1} - T_{c_1})}$$

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or $\varepsilon = \dfrac{(\Delta T)_l}{T_{h_1} - T_{c_1}}$, where $(\Delta T)_l$ is the larger of the two values $(T_{h_1} - T_{h_2})$ and $(T_{c_2} - T_{c_1})$. Then, find $Q_{max} = C_{min}(T_{h_1} - T_{c_1})$ and $Q = \varepsilon Q_{max}$. From energy balance, $Q = \dot{m}_h c_h (T_{h_1} - T_{h_2}) = \dot{m}_c c_c (T_{c_2} - T_{c_1})$, find $T_{h_2}$ and $T_{c_2}$.

8.7 A 1-shell 2-tube pass steam condenser of 3000 brass tubes of 20 mm diameter. Cooling water enters the tubes at 20ºC with a mean flow rate of 3000 kg/s. The heat transer coefficient for condensation on the outer surfaces of the tubes is 15,500 W/m$^2$K. If the heat load of the condenser is 2.3 ×10$^8$ W when the steam condenses at 50ºC, determine (a) the outlet temperature of water, (b) the overall heat transfer coefficient, (c) the tube length per pass using the NTU-method, (d) the rate of condensation of steam if $h_{fg}$ = 2380 kJ/kg.

*Hints:* By energy balance, $Q = \dot{m}c\,(T_{c_2} - T_{c_1}) = 2.3 \times 10^8$ W $= \dot{m}_{st}\,\dot{m}_{fg}$. Find $T_{c_2}$. $U_0 = \dfrac{1}{\dfrac{1}{h_i} + \dfrac{1}{h_0}}$

and $\mathrm{Re}_d = \dfrac{4\dot{m}_c}{\pi D\mu} > 10{,}000$. Use Dittus–Boelter equation $\mathrm{Nu}_d = 0.023\,\mathrm{Re}_d^{0.8}\,\mathrm{Pr}^{0.4}$ to find $h_i$. Now, $h_0$ being given, find $U_0$. (c) $C_h = C_{max} = \infty$.

∴ $\varepsilon = 1 - \exp(-\mathrm{NTU})$. $C_{min} = \dot{m}_c c_c (T_{c_2} - T_{c_1})$

$Q_{max} = C_{min}(T_{h_1} - T_{c_1})$.

$$\varepsilon = \frac{Q}{Q_{max}} = 1 - \exp(-\mathrm{NTU})$$

∴ NTU is found out. $A_0 = \dfrac{\mathrm{NTU}.C_{min}}{U_0}$ $= N \times 2L \times \pi D$

Find $L$ in one pass. (d) $\dot{m}_{st} = \dfrac{Q}{h_{fg}}$

8.8 A simple counterflow heat exchanger operates under the following conditions: Fluid $A$: inlet and outlet temperatures 80°C and 40°C, Fluid B: inlet and outlet temperatures 20°C and 40°C. The exchanger is cleaned, causing an increase in the overall heat transfer coefficient by 10% and inlet temperature of fluid B is changed to 30°C. What will be the new outlet temperatures of fluid A and of fluid B. Assume that the capacity rates remain unaltered.

*Hints:* Given: Case I – $T_{h_1}$ = 80°C, $T_{h_2}$ = 40°C, $T_{c_1}$ = 20°C, $T_{c_2}$ = 40°C

Case II – $T_{h_1}$ = 80°C, $T_{h_2}$ = ?, $T_{c_1}$ = 30°C, $T_{c_2}$ = ? $U_2 = 1.1\,U_1$

Case I: Find, $R = \dfrac{C_{min}}{C_{max}} = \dfrac{T_{c_2} - T_{c_1}}{T_{h_1} - T_{h_2}} = \dfrac{\dot{m}_h c_h}{\dot{m}_c c_c}$

$$Q_1 = \dot{m}_h c_h (T_{h_1} - T_{h_2}) = U_1 A (\Delta T_{lm})_{cf}$$

$$T_{h_1} - T_{h_2}\,\mathrm{NTU}_1 \times C_{min}(\Delta T_{\ln})_{cf}$$

Find NTU$_1$. Then $\mathrm{NTU}_2 = \dfrac{U_2 A}{C_{min}} = 1.1\,\mathrm{NTU}_1$

$$\varepsilon = \frac{1 - \exp[-\mathrm{NTU}_2(1-R)]}{1 - R\exp[-\mathrm{NTU}_2(1-R)]}$$

Also, $\varepsilon = \dfrac{\dot{m}_h c_h (T_{h_1} - T_{h_2})}{C_{min}(T_{h_1} - T_{c_1})} = \dfrac{\dot{m}_c c_c (T_{c_2} - T_{c_1})}{C_{min}(T_{h_1} - T_{c_1})}$

Find $T_{h_2}$ and $T_{C_2}$

## Problems for Practice

8.1 A double-pipe heat exchanger is constructed of a stainless steel ($k$ = 15.1 W/m K) inner tube of $D_i$ = 15 mm and $D_0$ = 19 mm, and the outer tube of diameter 32 mm. The convective heat transfer coefficient is given to be $h_i$ = 800 W/m$^2$ K and $h_0$ = 1200 W/m$^2$ K. For a fouling factor of $R_{f,\,i}$ = 0.0004 m$^2$ K/W on the tube side and $R_{f,\,0}$ = 0.0001 m$^2$ K/W on the shell side, determine (a) the total thermal resistance, (b) $U_i$ and (c) $U_0$ of the heat exchanger.

(*Ans.* (a) 0.0532 K/W, (b) 399.1 W/m$^2$K, (c) 314.9 W/m$^2$ K)

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8.2 Steam in the condenser of a steam power plant is to be condensed at a temperature of 30°C with cooling water entering at 14°C and leaving at 22°C. The surface area of the tubes is 45 m$^2$, and the overall heat transfer coefficient is 2100 W/m$^2$ K. Determine the mass flow rate of the cooling water needed and the rate of condensation of steam. Take $h_{fg}$ at 30°C = 2430.5 kJ/kg.

(*Ans.* $\dot{m}_c$ = 32.5 kg/s, $\dot{m}_h$ = 0.45 kg/s)

8.3 A shell-and-tube heat exchanger is to heat 10,000 kg/h of water from 16°C to 84°C by hot engine oil flowing through the shell. The oil makes a single shell pass, entering at 160°C and leaving at 94°C, with an average heat transfer coefficient of 400 W/m$^2$ K. The water flows through 11 brass tubes of 22.9 mm inner diameter and 25.4 mm outer diameter, with each tube making four passes through the shell. Assuming fully developed flow for the water, find the required tube length per pass.

8.4 A one shell pass, two tube pass heat exchanger has a total surface area of 5 m$^2$, and its overall heat transfer coefficient based on that area is found to be 1400 W/m$^2$ K. If 4500 kg/h of water enters the shell side at 315°C while 9000 kg/h of water enters the tube side at 40°C, find the outlet temperatures using (a) the F-LMTD method and (b) the $\varepsilon$-NTU method. Take $c_p$ for both fluid streams as 4.187 kJ/kg K.

(*Ans.* $T_{h_2}$ = 146.5°C, $T_{c_2}$ = 124.2°C)

8.5 Hot oil is to be cooled by water in a one shell pass and eight tube passes heat exchanger. The tubes are thin-walled and made of copper with an internal diameter of 14 mm. The length of each tube pass is 5 m and $U_0$ = 310 W/m$^2$ K. Water flows through the tubes at a rate of 0.2 kg/s, and the oil through the shell at a rate of 0.3 kg/s. The water and the oil enter at temperatures of 20°C and 150°C respectively. Determine the rate of heat transfer and the exit temperatures of the water and the oil.

(*Ans.* 49 kW, 78.6°C, 73.3°C)

8.6 A tubular heater of the counterflow type is used to heat 1.26 kg/s of fuel oil ($c_p$ = 3.14 kJ/kg K) from 10°C to 26.7°C. Heat is supplied by means of 1.51 kg/s of water which enters the heater at 82°C. (a) Derive an equation relating the temperatures of oil and water at any section of the heater. (b) Determine the necessary surface area if the overall heat transfer coefficient is 1.135 kW/m$^2$ K.

(*Ans.* 1.013 m$^2$)

8.7 In a test on a steam condenser the rate of flow of cooling water was varied while the condensation temperature was maintained constant. The following results were obtained:

| Overall heat transfer coefficient, $U$ (kW/m$^2$K) | 2.7 | 2.98 | 3.39 | 3.59 |
|---|---|---|---|---|
| Water velocity, $V$(m/s) | 0.996 | 1.27 | 1.83 | 2.16 |

Assuming the surface coefficient on the water side to be proportional to $V^{0.8}$, determine from an appropriate graph the mean value of the steam side surface coefficient. The thickness of the metal wall is 1.22 mm and thermal conductivity of tube material is 0.111 kW/m K.

(*Ans.* 6.04 kW/m$^2$ K)

8.8 A conterflow heat exchanger consists of a bundle of 20 mm diameter tubes contained in a shell. Oil flowing in the tubes is cooled by water flowing in the shell. The flow area within the tubes is 4.4 × 10$^{-3}$ m$^2$. The flow of oil is 2.5 kg/s. It enters at 65°C and leaves at 48°C. Water enters the shell at the rate of 20 kg/s and at 15°C. Calculate the area of the tube surface and the effectiveness of the exchanger.

For the oil in the tubes, take $\text{Nu}_d = 0.023\, \text{Re}_d^{0.8}\, \text{Pr}^{0.3}$, $c_p$ = 2.15 kJ/kg K, $\mu$ = 2.2 × 10$^{-5}$ Pa.s, $\rho$ = 880 kg/m$^3$, $k$ = 190 × 10$^{-6}$ kW/m K; for water, $h$ = 1.2 kW/m$^2$ K, $c_p$ = 4.19 kJ/kg K.

(*Ans.* 2.23 m$^2$, 34%)

8.9 An oil cooler for a lubrication system has to cool 1000 kg/h of oil ($c_p$ = 2.09 kJ/kg K) from 80°C to 40°C by using a cooling water flow

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of 1000 kg/h available at 30°C. Give your choice for a parallel flow or a counterflow heat exchanger, with reasons. Estimate the surface area of the heat exchanger, if $U_0$ = 24 W/m$^2$ K.

(*Ans.* 53.16 m$^2$)

8.10 Water is evaporated continuously at 100°C in an evaporator by cooling 500 kg of air per hour from 260°C to 150°C. Calculate the heat transfer surface area required and the steam evaporation per hour, if the liquid enters at 100°C. Take $U_0$ = 46 W/m$^2$ K and $c_p$ of air 1.005 kJ/kg K. At 100°, $h_{fg}$ = 2257 kJ/kg.

(*Ans.* 3.53 m$^2$)

8.11 Oil with a specific heat of 2 kJ/kg K is cooled from 110°C to 70°C by a flow of water in a counterflow heat exchanger. Water flows at the rate of 2 kg/s and is heated from 35°C to 65°C. The overall heat transfer coefficient is estimated to be 0.37 kW/m$^2$ K. Determine the exit temperatures of oil and water, if the water flow rate drops to 1.5 kg/s at the same oil flow rate. Take $c_p$ of water as 4.18 kJ/kg K.

(*Ans.* $T_{h_2} = T_{c_2}$ = 72.5°C)

8.12 A tubular counterflow oil cooler is to use a supply of cold water as the cooling fluid. Using the following data calculate the required surface area of the tubes:

| *Data* | *Oil* | *Water* |
|---|---|---|
| Entry temperature, °C | 121 | 15.6 |
| Exit temperature, °C | 82.3 | — |
| Mass flow rate, kg/s | 0.189 | 0.378 |
| Specific heat, kJ/kg K | 2.094 | 4.187 |
| Mean $U_0$ = 0.454 kW/m$^2$ K | | |

(*Ans.* 0.422 m$^2$)

8.13 A tank contains 272 kg of oil which is stirred so that its temperature is uniform. The oil is heated by an immersed coil of pipe 2.54 cm diameter in which steam condenses at 149°C. The oil of specific heat 1.675 kJ/kg K is to be heated from 32.2°C to 121°C in 1 h. Calculate the length of pipe in the coil if the surface coefficient is 0.653 kW/m$^2$ K.

(*Ans.* 3.47 m)

8.14 An oil fraction flowing at the rate of 20.15 kg/s at a temperature of 121°C is to be cooled in a simple counterflow heat exchanger using 5.04 kg/s of water initially at 10°C. The exchanger contains 200 tubes, each 4.87 m long and 19.7 mm outer diameter. If the specific heat of oil is 2.094 kJ/kg K, calculate the exit temperature of the oil. Take $U_0$ = 0.34 kW/m$^2$ K.

(*Ans.* 90.8°C)

8.15 Oil is cooled by water is a counterflow heat exchanger. The oil flow rate is 2000 kg/h, entering at 107°C and leaving at 30°C. Its mean $c_p$ is 2.51 kJ/kg K. Water enters at 15°C and its exit temperature is not to exceed 80°C. The overall heat transfer coefficient is expected to be 1.5 kW/m$^2$ K. Determine the water flow rate, the surface area required and the effectiveness of the exchanger.

(*Ans.* 1425 kg/h, 3.5 m$^2$, 0.837)

8.16 A liquid ($c_p$ = 3.81 kJ/kg K) flowing at the rate of 6.93 kg/s through a heat exchanger made from a 25.4 mm outer diameter tube is cooled from 65.6°C to 39.4°C, using 6.30 kg/s of water available at 10°C. Assuming $U_0$ = 568 W/m$^2$ K, estimate the surface area required for the following arrangements:

(a) Parallel-flow tube and shell
(b) Counterflow tube and shell,
(c) Counterflow exchanger with 2 shell passes and 72 tube passes, the liquid flowing through the shell and the water flowing through the tubes,
(d) Cross-flow, with one tube pass and one shell pass, shell side fluid mixed.

[*Ans.* (a) 66.2 m$^2$, (b) 41.4 m$^2$, (c) 42.7 m$^2$, (d) 47.0 m$^2$]

8.17 Water is required to be preheated for a boiler using flue gases from the boiler stack. The flue gases are available at the rate of 0.25 kg/s at 150°C, with a specific heat of 1 kJ/kg K. The water entering the exchanger at

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15°C at the rate of 0.05 kg/s is to be heated to 90°C. The heat exchanger is to be of the reversed current type with one shell pass and four tube passes. The water flows inside the tubes, which are made of copper (25 mm inner diameter, 30 mm outer diameter). The heat transfer coefficient at the gas side is 115 W/m$^2$ K, while the heat transfer coefficient on the water side is 1150 W/m$^2$ K. A scale on the water side offers an additional thermal resistance of 0.002 m$^2$ K/W. (a) Determine the overall heat transfer coefficient based on the outer tube diameter. (b) Determine the appropriate mean temperature difference for the heat exchanger. (c) Estimate the required tube length. (d) What would be the outlet temperature and the effectiveness if the water flow rate is doubled, giving a heat transfer coefficient of 1820 W/m$^2$ K?

8.18 Water flowing at a rate of 10 kg/s through 50 double-pass tubes in a shell-and-tube heat exchanger heats air that flows through the shell side. The length of the brass tubes is 6.7 cm, and they have an outer diameter of 26 mm and an inner diameter of 23 mm. The heat transfer coefficients of the water and air are 470 W/m$^2$ K and 210 W/m$^2$ K, respectively. Air enters the shell at a temperature of 15°C and a flow rate of 1.6 kg/s. The temperature of the water as it enters the tubes is 75°C. Calculate (a) the heat exchanger effectiveness, (b) the heat transfer rate to the air and (c) the outlet temperatures of the air and water.

8.19 Water flowing at the rate of 12.6 kg/s is to be cooled from 90°C to 65°C by means of an equal flow rate of cold water entering at 40°C. The water velocity will be such that the overall coefficient of heat transfer $U$ is 2300 W/m$^2$ K. Calculate the heat exchanger surface area needed for each of the following arrangements: (a) parallel flow, (b) counterflow (c) a multipass heat exchanger with the hot water making one pass through a well-baffled shell and the cold water making two tube passes through the tubes and (d) a cross-flow heat exchanger with both sides unmixed.

8.20 Determine the maximum heat transport capability and the liquid flow rate of a water heat pipe operating at 100°C and atmospheric pressure. The heat pipe is 30 cm long and has an inner diameter of 1 cm. It is inclined at 30°C with the evaporator above the condenser. The wick consists of four layers of 250-mesh wire screen (wire diameter of 0.045 mm) on the inner surface of the pipe. The pore radius is 0.002 cm and the permeability is 0.3 × 10$^{-10}$ m$^2$. Water properties at 100°C are $\rho_1$ = 958 kg/m$^3$, $\mu_1$ = 279 × 10$^{-6}$ N s/m$^2$, $\sigma_1$ = 58.9 × 10$^{-3}$ N/m and $h_{fg}$ = 2260 kJ/kg. Assume perfect wetting.

(*Ans.* 20 W)

8.21 Air enters a gas-fired furnace at 20°C at a mass flow rate of 0.2 kg/s and is burned with an air-fuel ratio by volume of 12. The gases leave the furnace at 350°C. A run-around coil is installed to recover some of the energy of exhaust gases in order to preheat the air entering the furnace. Using the data given, neglecting thermal losses, calculate

(a) the required mass flow rate of secondary fluid,
(b) the effectiveness of the overall heat recovery process,
(c) the rate of energy recovery and
(d) the temperature of the gases at exit and the temperature of the air at entry to the burner.

Given: $\bar{c}_p$ of air = 1.01 kJ/kg K, $\bar{c}_p$ of gases = 1.15 kJ/kg K, $\rho$ of inlet air = 1.204 kg/m$^3$, $\rho$ of inlet gas = 0.715 kg/m$^3$, $\bar{c}_p$ of secondary fluid = 1.6 kJ/kg K, $U_0$ for each heat exchanger = 60 W/m$^2$ K and heat transfer area for each heat exchanger = 11 m$^2$.

[*Ans.* (a) 0.138 kg/s, (b) 0.652, (c) 43.45 kW, (d) 170°C, 235.1°C]

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# Some Special Heat Transfer Processes

# 9

In this chapter the following six topics would be discussed:

1. Heat transfer in high velocity flows
2. Heat transfer in rarefied gases
3. Transpiration and film cooling
4. Ablative cooling
5. Thermodynamic optimization of convective heat transfer
6. Heat transfer in a circulating fluidized bed (CFB) boiler

## 9.1 HEAT TRANSFER IN HIGH VELOCITY FLOWS

Problems related to heat transfer in high-speed flows have assumed considerable importance, since aircrafts, rockets, missiles, satellites and space probes move at huge velocities, often considerably exceeding the sound velocity. The convective heat transfer, so far we have studied, is valid for flows of low Mach number ($M < 0.3$). The equations require to be modified when the velocity is very high. An increase in the flow velocity results in a decrease in the thickness of the hydrodynamic boundary layer, and consequently, an increase in the velocity gradient in the boundary layer at the wall and a rise in viscous stresses or friction. Because of this steep velocity gradient, the kinetic energy of the fluid stream is dissipated to an increase in the internal energy of the fluid at the wall and hence the temperature. This process, known as *aerodynamic heating*, raises the surface temperatures of bodies placed in high-speed fluid stream or of bodies moving at high speed through a stagnant fluid. An example of the latter is the heating of the skins of high-speed aeroplanes and missiles. Aerodynamic heating becomes a serious problem at very high speeds because the rate of heat flow to the skin increases roughly in proportion to the flight velocity if the surface is maintained at a constant temperature.

Assumption of constant fluid properties may no longer be valid because of steep temperature gradients within the boundary layer. Since the speed is supersonic, there may be shock waves which may interact with the boundary layer. If the temperature gets high enough, the gas molecules in the boundary layer may partially dissociate, thereby changing the properties of the fluid. Also, the gas may get slightly ionised, thus electrically conducting, and its physical behaviour gets influenced by the presence of a magnetic field. At very high altitudes, the gas rarefaction must be considered, necessitating a critical reappraisal of the underlying hypothesis of continuum flow. Clearly, problems of heat transfer at high velocities can be extremely complicated. Here we shall merely attempt to outline the nature of the problems with a few simple examples and confine our attention to transfer processes occurring between a gas and a shock-free surface.

### 9.1.1 Adiabatic Wall Temperature and Recovery Factor

Let us consider a gas flowing at high velocity over an insulated plate surface in laminar motion (Fig. 9.1). The velocity distribution is qualitatively similar to that observed at low Mach number, but the temperature profile is quite different.

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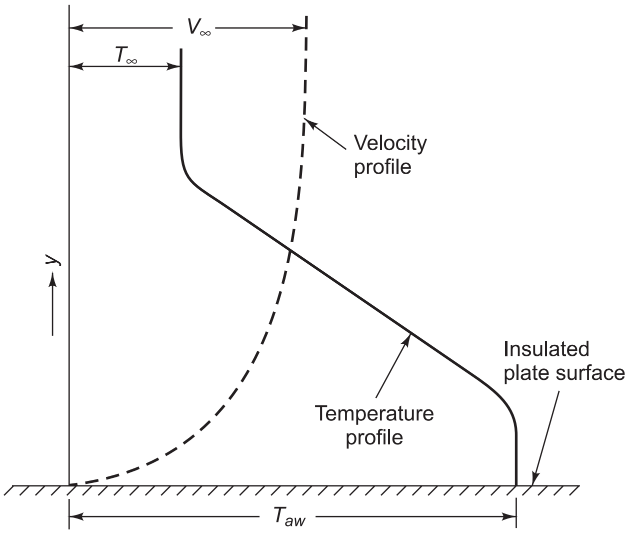


**Fig. 9.1** *Velocity and temperature distribution in high-speed flow over an insulated plate*

Since the velocity gradient is steep, there is high viscous stress which slows down the gas at the wall, and the kinetic energy of the gas gets dissipated into friction to increase the internal energy or temperature of the surface. If the surface is kept at a constant temperature and it is adiabatic, then the temperature profile will depend on the rate at which internal energy is produced by shear work at the boundary wall and the rate at which the energy gets conducted from the wall to the free stream. So the plate surface temperature will adjust to some equilibrium temperature known as *adiabatic wall temperature*, which is considerably higher than the free-stream static temperature, and slightly less than the stagnation temperature $T_0$. When the flowing gas is brought down to zero velocity adiabatically, then the temperature rises to stagnation temperature.

$$h_0 = h + \frac{V_\infty^2}{2}$$

$$T_0 = T + \frac{V_\infty^2(\gamma - 1)}{2\gamma R}$$

$$\mathbf{\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} M^2} \tag{9.1}$$

where $M_\infty = V_\infty/C$, $C = (\gamma R T_\infty)^{1/2}$, the gas being ideal. Dynamic temperature rise, $\theta_{dy} = T_0 - T_\infty = V_\infty^2/2c_p$.

When $T = 288$ K, sonic velocity in air $C = (1.4 \times 287\ T_\infty)^{1/2} \cong 20\ (T_\infty)^{1/2} = 340$ m/s. For a velocity around 85 m/s, $M = 85/340 = 0.25$. Then,

$$\frac{T_0}{T_\infty} = 1 + \frac{\gamma - 1}{2} M^2 = 1 + 0.2 \times (0.25)^2$$
$$= 1.0125$$

$$\mathbf{T_0 \cong T_\infty \text{ when } M < 0.25}$$

But at $M \geq 0.25$, such an assumption may result in large errors.

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Figure 9.2 illustrates the dependence of stagnation temperature on the velocity of air flow. At a Mach number $M = 1$, $T_0 = 1.2\ T_\infty$, but if $M = 3$, $T_0 = 2.8\ T_\infty$; at $M = 5$, $T_0 = 6T_\infty$. Thus, if the thermodynamic temperature $T_\infty$ is 288 K, the stagnation temperature $T_0$ will be 1728 K at $M = 5$; the increase in temperature is very large.

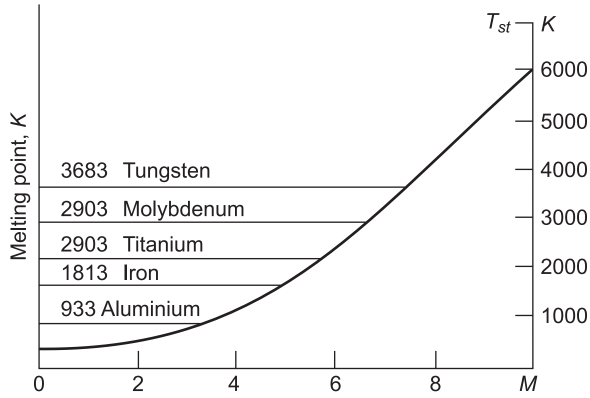


**Fig. 9.2** *Dependence of stagnation temperature on Mach number ($\gamma = 1.4$)*

Figure 9.2 also gives the melting points of some metals. A stagnation temperature equal to the melting point of aluminium is reached at $M = 3.35$, and at $M = 5.14$ a temperature equal to the melting point of pure iron is reached, and so on. The physical properties of a gas change essentially at very high temperatures. Molecular dissociation and even ionisation of the gas may take place at very high temperatures.

For a gas flowing past an insulated surface at a high velocity, the temperature at the surface will rise above the static temperature of the gas, but will not quite reach the stagnation temperature. The temperature reached at the surface is the adiabatic wall temperature $T_{aw}$, as mentioned earlier.

In practice it has been found convenient to relate $T_{aw}$ and $T_0$ by the *recovery factor r*, which is a measure of the fraction of the free stream dynamic temperature rise recovered at the wall. The recovery factor is defined as

$$\mathbf{r} = \frac{\mathbf{T_{aw} - T_\infty}}{\mathbf{T_0 - T_\infty}} = \frac{\mathbf{T_\infty [T_{aw}/T_\infty - 1]}}{\mathbf{T_\infty}\left[\mathbf{(\gamma - 1)/2}\,\dfrac{\boldsymbol{\gamma} \mathbf{- 1}}{\mathbf{2}}\mathbf{M_\infty^2}\right]}$$

$$= \frac{2}{(\gamma - 1)M_\infty^2}\left(\frac{T_{aw}}{T_\infty} - 1\right) \tag{9.2}$$

Also,

$$T_{aw} - \dot{T}_\infty = r(T_0 - T_\infty) = rT_\infty \frac{\gamma - 1}{2} M_\infty^2$$

$$T_{aw} = T_\infty\left(1 + r\frac{\gamma - 1}{2} M_\infty^2\right) \tag{9.3}$$

Also,

$$\mathbf{T_{aw} = T_\infty + rT_\infty \frac{\boldsymbol{\gamma} - 1}{2} \frac{V_\infty^2}{\boldsymbol{\gamma} RT_\infty} = T_\infty + r\frac{V_\infty^2}{2c_p}} \tag{9.4}$$

Experiments with air in laminar flow have shown that for practical purposes [1]

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$$\mathbf{r = (Pr)^{1/2}} \tag{9.5}$$

For turbulent flow, it is

$$\mathbf{r = (Pr)^{1/3}} \tag{9.6}$$

When a surface is not insulated, the rate of heat flow between the gas and the solid surface, $Q_w$, is governed by

$$\frac{Q_w}{A} = -k\left(\frac{\partial T}{\partial y}\right)_{y=0} \tag{9.7}$$

The influence of heat transfer to or from the plate on the temperature distribution is illustrated in Fig. 9.3.

We observe that, at high speeds, heat can flow to the surface even when the surface temperature is higher than the free-stream temperature. This rather unexpected phenomenon is a result of the aerodynamic heating of the boundary layer.

The heat transfer rate in high-speed flow over a flat surface can be predicted from the boundary-layer energy equation [1].

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} + \frac{\mu}{\rho c_p}\left(\frac{\partial u}{\partial x}\right)^2$$

where the last term accounts for the viscous dissipation.

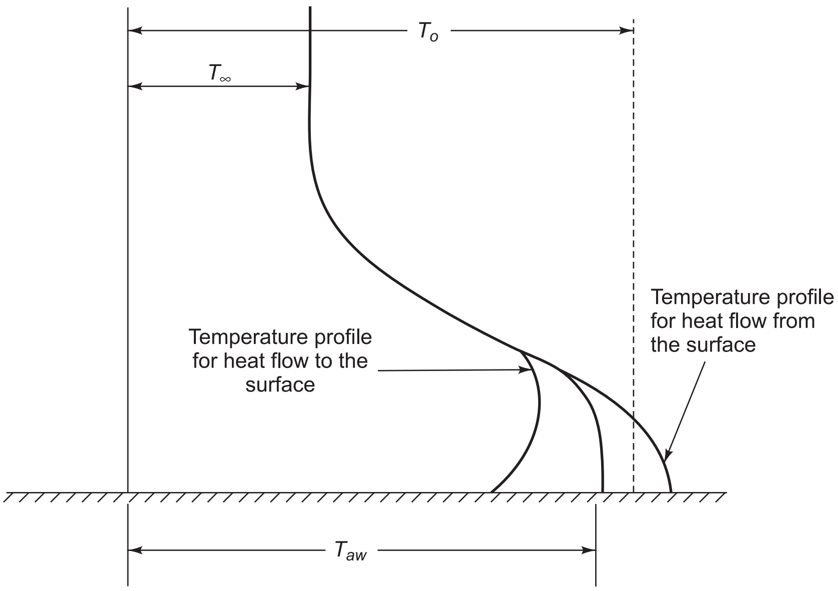


**Fig. 9.3** *Temperature profiles in high speed boundary layer for heating and cooling*

However, for most practical purposes the rate of heat transfer can be calculated with the same relations used for low-speed flow, if the average convection heat transfer coefficient is redefined by the relation [2]

$$Q = \bar{h}A(T_w - T_{aw}) \tag{9.8}$$

which will yield zero value when the wall temperature $T_w$ is equal to the adiabatic wall temperature $T_{aw}$.

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Since in high-speed flow the temperature gradients in a boundary layer are large, variations in the physical properties of the fluid will also be substantial. The constant-property heat transfer equations can still be used if the properties are evaluated at a reference temperature $T^*$ as recommended by Eckert [3].

$$T^* = T_\infty + 0.5(T_w - T_\infty) + 0.22\,(T_{aw} - T_\infty) \tag{9.9}$$

The analogy between heat transfer and fluid friction can also be used when the friction coefficient is known. Summarising the relations used for high-speed heat transfer calculations:

Laminar boundary layer: ($\mathrm{Re}_x < 5 \times 10^5$)

$$\mathrm{St}_x^* \, \mathrm{Pr}^{*2/3} = 0.332(\mathrm{Re}_x^*)^{-1/2} \tag{9.10}$$

Turbulent boundary layer: ($5 \times 10^5 < \mathrm{Re}_x < 10^7$)

$$\mathrm{St}_x^* \, \mathrm{Pr}^{*2/3} = 0.0296\,(\mathrm{Re}_x^*)^{-1/5} \tag{9.11}$$

Turbulent boundary layer: ($10^7 < \mathrm{Re}_x < 10^9$)

$$\mathrm{St}_x^* \, \mathrm{Pr}^{*2/3} = 0.185\,(\log \mathrm{Re}_x^*)^{-2.584} \tag{9.12}$$

The superscript * in the above equations indicates that the properties are evaluated at the reference temperature given in Eq. (9.9).

To obtain an average heat transfer coefficient, the above expressions must be integrated over the length of the plate.

When the speed of the gas is very high, the boundary layer may become so hot that the gas begins to dissociate. In such situations, Eckert [2] recommends that the heat transfer coefficient be based on the enthalpy difference between the wall and the adiabatic state ($i_w - i_{aw}$), and be defined by

$$Q = h_i A(i_w - i_{aw}) \tag{9.13}$$

The enthalpy recovery factor is then defined by

$$r_i = \frac{i_{aw} - i_\infty}{i_0 - i_\infty} \tag{9.14}$$

where $i_{aw}$ is the enthalpy at the adiabatic wall conditions. The same relations as before are used to calculate the recovery factor and heat transfer except that all properties are evaluated at a reference enthalpy $i^*$ given by

$$i^* = i_\infty + 0.5(i_w - i_\infty) + 0.22\,(i_{aw} - i_\infty) \tag{9.15}$$

The Stanton number is redefined as

$$\mathrm{St}_i^* = \frac{h_i}{\rho^* u} \tag{9.16}$$

This Stanton number is then used in Eqs (9.10) – (9.12) to calculate the heat transfer coefficient. When calculating the enthalpies for use in the above relations, the total enthalpy must be used, i.e., chemical energy of dissociation as well as internal thermal energy must be included. The reference-enthalpy method has proved *successful* for calculating high-speed heat transfer with an accuracy of better than 10% [2].

To summarise, we may state that when the velocity is low, the flow is defined by Reynolds number. When the velocity is comparable to the sonic velocity, the flow is defined by both Reynolds number and Mach number. For $M < 0.3$, compressibility effect may be neglected and the same equation may be used to evaluate $h$ for both compressible and incompressible fluids. But for $M > 0.3$, the equations have to be modified as given above.

For flow past cones, the following relations may be used [2]:

Laminar: $$\mathrm{Nu}_x = 0.575\,(\mathrm{Re}_x)^{1/2}\,\mathrm{Pr}^{1/3} \tag{9.17}$$

Turbulent: $$\mathrm{Nu}_x = 0.0292\,(\mathrm{Re}_x)^{0.8}\,\mathrm{Pr}^{1/3} \tag{9.18}$$

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## 9.2 HEAT TRANSFER IN RAREFIED GASES

In the usual macroscopic analysis of transfer processes, fluid media are treated as continua, and macroscopic properties such as density, velocity and temperature are assumed to vary continuously in time and space. The analysis of transfer processes is now extended to high-speed high-altitude flight where rarefaction effects are significant.

A rarefied gas is defined as a gas where the mean free path ($\lambda$) is considerably large compared with the characteristic dimension of the body. The average distance traversed by a molecule between two successive collisions is called the mean free path, denoted by $\lambda$. The dimensionless parameter formed by the ratio of $\lambda$ to a characteristic system dimension $L$ is known as the *Knudsen number:*

$$\text{Kn} = \frac{\lambda}{L}$$

When $\lambda << L$, continuum approach is valid and it postulates the existence of hydrodynamic and thermal boundary layers at the plate surface over which air is flowing (Fig. 9.4), for air at sea-level, $\lambda$ being $6 \times 10^{-6}$ cm. The rates of transfer of momentum and energy within the boundary layers are governed by a series of random molecular collisions. Velocity and temperature profiles within the boundary layers are given by curves labelled "*a*" in Fig. 9.4. When $\lambda \sim L$, intermolecular collisions become less frequent. When the gas pressure or density is low, the flow seems to "slip" along the surface and $V \neq 0$ at $y = 0$. This situation is appropriately called "*slip flow*". Molecules arriving at the solid surface cannot come to equilibrium with the surface. Consequently, significant velocity and temperature discontinuities may develop at the gas-solid boundary shown by curves labelled "*b*". Rates of transfer are no longer governed solely by intermolecular collisions within the boundary layers, but depend also upon the effectiveness of property exchange between the gas molecules and the wall.

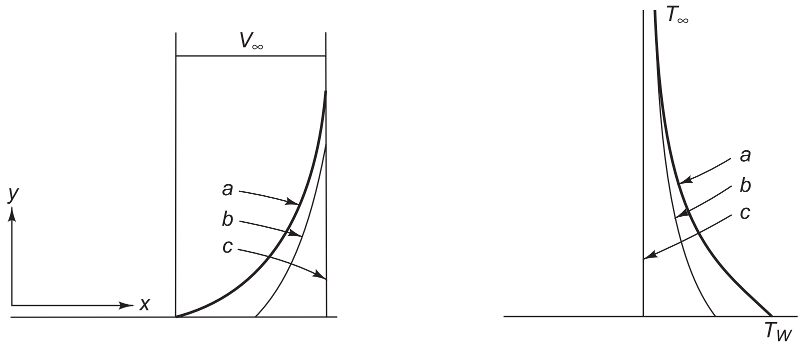


**Fig. 9.4** *Sketch of velocity and temperature distributions over a heated plate: a—continuum profiles, b—slip profiles, c—free-molecule profiles*

If the air is highly rarefied so that $\lambda >> L$, frequency of intermolecular collisions may be totally negligible. At the solid surface, the impinging and re-emitted streams of molecules do not interact, and the boundary layers disappear as shown by curves "*c*", (Fig. 9.4). It may be noted that even though the gas is highly rarefied, there are still a sufficient number of molecules per unit volume so that the gas properties such as density, velocity and temperature are meaningful. This regime is called "*free-molecule flow*".

The elementary kinetic theory [4] gives the viscosity of a gas as

$$\mu = \frac{1}{3} \rho \overline{V} \lambda \tag{9.19}$$

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where $\bar{V}$ is the mean molecular velocity, $(8KT/\pi m)^{1/2}$, $K$ is the Boltzmann constant and $m$ is the mass of a molecule.

The velocity of sound $\bar{V}_s$ in the gas is given by

$$\bar{V}_s = \left(\frac{\gamma KT}{m}\right)^{1/2} \tag{9.20}$$

By dividing,

$$\bar{V} = \left(\frac{8}{\pi\gamma}\right)^{1/2} \bar{V}_s \tag{9.21}$$

where $\gamma$ is the specific heat ratio. More rigorous assumptions regarding the behaviour of elastic, spherical molecules lead to modification of the constant in Eq. (9.19) from 1/3 to 1/2 [2].

$$\begin{aligned} \text{Kn} &= \frac{\lambda}{L} = \frac{2\mu}{\rho\bar{V}L}\left(\frac{\pi\gamma}{8}\right)^{1/2}\frac{1}{V_s} \\ &= \left(\frac{\pi\gamma}{2}\right)^{1/2}\frac{\mu}{\rho\bar{V}L}\frac{\bar{V}}{\bar{V}_s} = \left(\frac{\pi\gamma}{2}\right)^{1/2}\frac{M}{\text{Re}} \end{aligned} \tag{9.22}$$

or $$\mathbf{Kn = 1.48\,\frac{M}{Re}} \tag{9.23}$$

For flows at very low Reynolds numbers, the characteristic system dimension ($L$) is usually a body dimension $l$, and from Eq. (9.19), we can write,

$$\text{Kn}_1 = \frac{\lambda}{1} \sim \frac{M}{\text{Re}_1} \text{ for low Re}_1 \tag{9.24}$$

For flows at large Reynolds numbers, the significant characteristic dimension is of the order of the boundary layer thickness $\delta$. If the flow is laminar,

$$\frac{\delta}{1} \sim \frac{1}{(\text{Re}_1)^{1/2}}$$

$$\text{Kn}_\delta = \frac{\lambda}{\delta} = \frac{\lambda 1}{1\delta} \sim \frac{M}{(\text{Re}_1)^{1/2}} \text{ for high Re}_1 \tag{9.25}$$

The classification of flow regimes in terms of ranges of $M$ and $\text{Re}_1$ is shown in Fig. 9.5. The lines of demarcation between the various flow regimes are not known with precision for all Reynolds numbers. The following criteria are suggested:

Free-molecule and Transition $\dfrac{M}{\text{Re}_1} = 3 - 10$

Transition and Slip: $\begin{cases} \dfrac{M}{\text{Re}_1} = 0.1 & \text{Re}_1 < 1 \\ \dfrac{M}{(\text{Re}_1)^{1/2}} = 0.1 & \text{Re}_1 > 1 \end{cases}$

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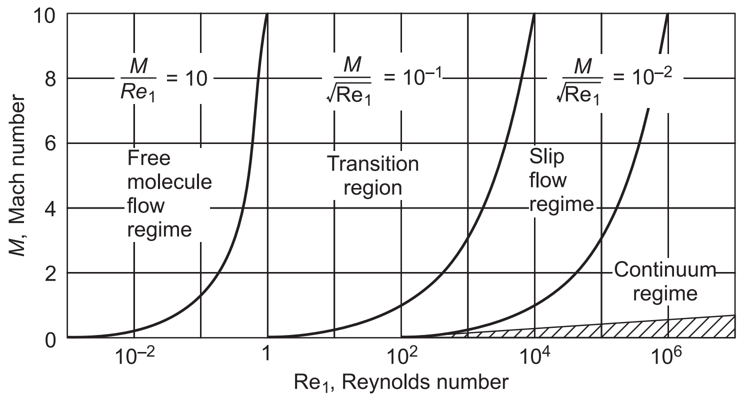


**Fig. 9.5** *Flow regimes in gas dynamics*

$$\text{Slip and Continuum:}\quad \begin{cases} \dfrac{M}{\mathrm{Re}_1} = 0.1 & \mathrm{Re}_1 < 1 \\[2ex] \dfrac{M}{(\mathrm{Re}_1)^{1/2}} = 0.01 & \mathrm{Re}_1 > 1 \end{cases}$$

The aerodynamic degree of gas rarefaction, considered as the degree of deviation from the state of continuum, is described by Knudsen number. If Kn < 0.001, the gas can be considered a continuum. When Kn > 10, the gas is considered a free molecular flux. Flow and heat transfer are then calculated by means of equations from the kinetic theory of gases. With Kn > 0.001, molecular collisions prevail over collisions with the wall. With free molecule flow on the contrary, collisions with the wall predominate.

A rarefied gas can neither be considered an absolutely continuous medium nor a free-molecule flux at 0.001 < Kn < 10. Two patterns are distinguished for this range: slip flow (0.001 < Kn < 1) and transition flow (1 < Kn < 10).

From kinetic theory of gases [4],

$$\lambda = \frac{0.707}{\sigma n} \tag{9.26}$$

where $\sigma$ is the molecular cross-section ($\pi d^2$) where $d = r_1 + r_2$ with $r_1$ and $r_2$ being the radii of the molecules. At 1 bar, 273 K, $\lambda = 10^{-7}$ m. In Eq. (9.26), $\lambda$ is inversely proportional to $n$, the number of molecules per unit volume ($p/KT$), i.e. the pressure. At low pressures $\lambda$ is very large. At $p = 10^{-3}$ mm Hg, $\lambda = 7$ cm and at $p = 10^{-8}$ mm Hg, $\lambda = 7$ km. An approximate relation for the mean free path of air molecules is given by

$$\lambda = 2.27 \times 10^{-5}\,\frac{T}{p}\ \text{metres} \tag{9.26a}$$

where $T$ is in degrees Kelvin and $p$ is in pascals.

Thus, the lower the pressure of the gas, the larger the $\lambda$ and smaller the probability of collision. According to Eq. (9.26), $\lambda \propto 1/\sigma$ and $\sigma$ diminishes with increasing $T$. Thus, the mean free path increases with diminishing pressure and rising temperature of the gas.

In slip flow at a wall (Fig. 9.4) an expression for the velocity discontinuity may be obtained as follows. An infinitesimal gas layer adjacent to the wall is acted upon on the gas side by a shear force proportional to the velocity gradient in the gas and on the solid side by a shear force which may be assumed to be proportional to the velocity difference between the gas and the wall. Equating these two forces, we can write

$$\mu\left(\frac{\partial v_x}{\partial y}\right) = \beta\,(V_0 - V_w) \tag{9.27}$$

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where $\mu$ is the coefficient of viscosity of the gas, $\beta$ is a proportionality constant, $V_0$ is the gas velocity at the wall and $V_w$ is the wall velocity. The ratio of $\mu/\beta$ measures the effectiveness of tangential momentum exchange at the wall.

By kinetic theory of gases, it can be shown [2],

$$\frac{\mu}{\beta} = \frac{2-F}{F}\lambda \tag{9.28}$$

where $F$ is the fraction of the tangential momentum of the molecules striking the surface. $F$ may be interpreted as the fraction of the incident molecules that are reflected diffusely from the surface, the rest being reflected specularly.

By a similar procedure, the temperature discontinuity at the wall may be expressed as

$$k\left(\frac{\partial T}{\partial y}\right) = K(T_0 - T_w) \tag{9.29}$$

where $k$ is the thermal conductivity of the gas and $K$ is a proportionality constant. The ratio $k/K$ can be expressed by the kinetic theory [2] in terms of a parameter $A$ called the *thermal accommodation coefficient*.

$$\frac{\mathbf{k}}{\mathbf{K}} = \frac{\mathbf{2}}{\mathbf{Pr}}\left(\frac{\mathbf{2-A}}{\mathbf{A}}\right)\left(\frac{\boldsymbol{\gamma}}{\boldsymbol{\gamma}+\mathbf{1}}\right) \tag{9.30}$$

The coefficient $A$ measures the effectiveness of energy exchange at the wall, given to a good approximation by

$$A = \frac{T_r - T_i}{T_w - T_i} \tag{9.31}$$

where $T_i$ is the temperature of the incident molecular stream, $T_r$ is the temperature of the reflected (or re-emitted) stream, and $T_w$ is the temperature of the wall.

Table 9.1 gives values of $A$ for some surfaces and the striking gas [2].

**Table 9.1** *Thermal accommodation coefficient, A*

| *Gas* | *Surface* | *A* |
|---|---|---|
| $H_2$ | Bright paint | 0.32 |
| $H_2$ | Black paint | 0.74 |
| $O_2$ | Bright paint | 0.81 |
| $O_2$ | Black paint | 0.93 |
| $CO_2$ | Bright paint | 0.84 |
| $CO_2$ | Black paint | 0.96 |
| Air | Flat laequer on bronze | 0.88 – 0.89 |
| Air | Polished bronze | 0.91 – 0.94 |
| Air | Machined bronze | 0.89 – 0.93 |
| Air | Polished cast iron | 0.87 – 0.93 |
| Air | Machined cast iron | 0.87 – 0.88 |
| Air | Polished aluminium | 0.87 – 0.95 |
| Air | Machined aluminium | 0.95 – 0.97 |
| Air | Etched aluminium | 0.8 – 0.97 |

The maximum probable molecular velocity as given by kinetic theory [4]

$$\bar{V}_{max} = (2RT)^{1/2}$$

The *speed ratio S* is defined by

$$S = \frac{\bar{V}}{\bar{V}_{max}} = \frac{(\gamma RT)^{1/2} M}{(2RT)^{1/2}} = M\left(\frac{\gamma}{2}\right)^{1/2} \tag{9.32}$$

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Also, from kinetic theory for free-molecule flow

$$\text{St} = \frac{\text{Nu}}{\text{Re Pr}} = \frac{h}{\rho c_p \bar{V}} = \frac{\gamma+1}{\gamma} A F_1(S) \tag{9.33}$$

The function $F_1(S) = \text{St}\ \gamma/(\gamma + 1)M$ is shown graphically in Fig. 9.6 against speed ratio, $S = M\ (\gamma/2)^{1/2}$, which gives heat transfer data from a flat plate, a sphere and a transverse circular cylinder in free-molecule flow [2].

The rate of heat flow for a rarefied gas is calculated from

$$q = h(T_r - T_w) \tag{9.34}$$

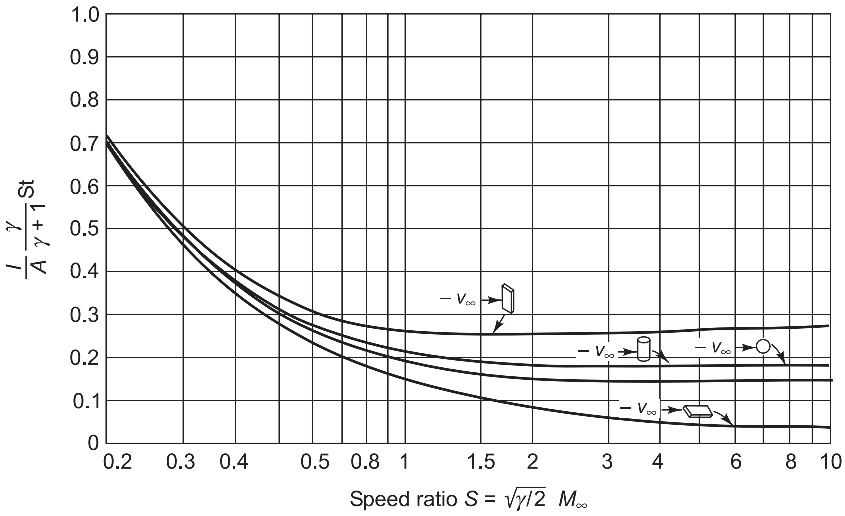


**Fig. 9.6** *Heat transfer from a flat plate, a sphere and a transverse circular cylinder in free-molecule flow*

According to the kinetic theory of gases, the recovery factor $r$ can be calculated from

$$r = \frac{\gamma}{\gamma+1} F_2(s) \tag{9.35}$$

The values of the function $F_2(s) = \dfrac{\gamma+1}{\gamma} r$ are given in Fig. 9.7 plotted against $s = (\gamma/2)^{1/2}\ M$. It is seen that $F_2\ (s)$ becomes practically constant and equal to 2, as soon as $s$ equals 5. At lower relative velocities, $s$, the value of $r$ depends materially upon the shape of the body exposed to flow.

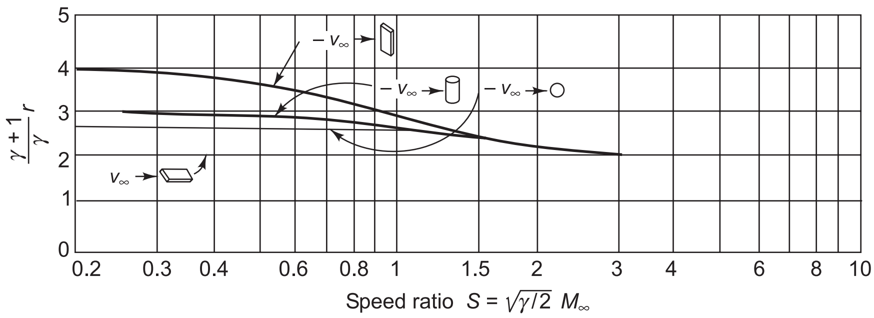


**Fig. 9.7** *Recovery factor for a flat plate, a sphere and a transverse circular cylinder in free-molecule flow*

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For air, $\gamma = 1.4$ and when $s > 5$, $\gamma + 1/\gamma = 2$, which gives $r = 1.17$. Thus, the recovery temperature of free-molecule flow exceeds the stagnation temperature.

Let us consider two infinite parallel plates maintained at two different temperatures and separated by a gaseous medium [Fig. 9.8(a)]. It is first assumed that the density or plate spacing is low enough that free convection effects are negligible, but with a gas density sufficiently high so that $\lambda \to 0$ and a linear temperature profile through the gas exists [$\lambda_1$ in Fig. 9.8 (b)]. As the gas density is lowered, the larger mean free paths require a greater distance from the heat transfer surfaces in order for the gas to accommodate to the surface temperatures. The anticipated temperature profiles are shown in Fig. 9.8(b). Extrapolating the straight-portion of the low density curves to the wall produces a temperature “jump” $T$. By making an energy balance,

$$\frac{Q}{A} = k\frac{T_1 - T_2}{g + L + g} = k\frac{\Delta T}{g} \tag{9.36}$$

assuming that the extrapolation $g$ is the same for both plate surfaces. However, the temperature jump will depend on the type of surface, and if the materials are different

$$\frac{Q}{A} = k\frac{T_1 - T_2}{g_1 + L + g_2} = k\frac{\Delta T_1}{g_1} = k\frac{\Delta T_2}{g_2} \tag{9.37}$$

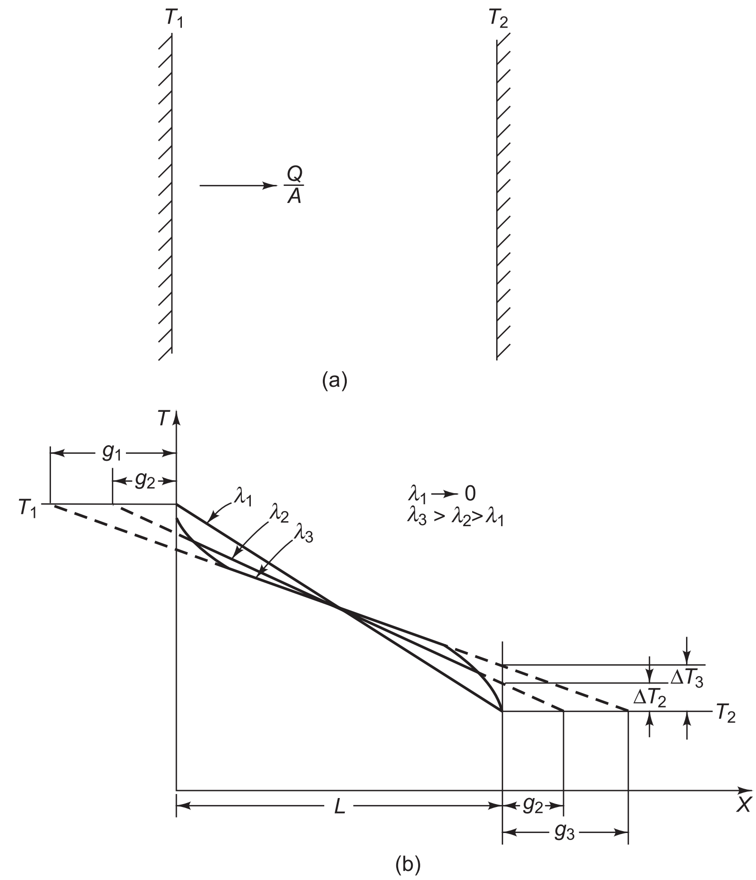


**Fig. 9.8** *Effect of mean free path on conduction heat transfer between parallel plates: (a) Physical model; (b) Anticipated temperature profiles*

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where $\Delta T_1$ and $\Delta T_2$ are the temperature jumps at the two heat transfer surfaces and $g_1$ and $g_2$ are the corresponding extrapolation distances. For identical surfaces the temperature jump would then be expressed as

$$T = \frac{g}{2g + L}(T_1 - T_2) \tag{9.38}$$

Similar expressions may be developed for low-density conduction between concentric cylinders [5].

The temperature jump effect arises as a result of the failure of the molecules to "accommodate" to the surface temperature when the mean free path becomes of the order of a characteristic body dimension. The accommodation coefficient ($A$) takes care of this behaviour of the particles.

It is possible to employ the kinetic theory of gases along with values of $A$ to determine the temperature jump at a surface as given below [5]:

$$T_{y=0} - T_w = \frac{2 - A}{A}\,\frac{2\gamma}{\gamma + 1}\,\frac{\lambda}{\Pr}\left|\frac{\partial T}{\partial y}\right|_{y=0} \tag{9.39}$$

The nomenclature for Eq. (9.39) is shown in Fig. 9.8.

The temperature gradient (Fig. 9.8) would be $(T_1 - T_2 - 2\,\Delta T)/L$.

For very low densities (high vacuum) the mean free path may become very large compared to the plate separation distance and the conduction–convection heat transfer will be zero. There can, however, be radiation heat transfer.

## 9.3 TRANSPIRATION AND FILM COOLING

Some special cooling methods have been developed in order to protect certain structural elements in turbojet and rocket engines from hot gases, like combustion chamber walls, exhaust nozzles, or gas turbine blades (Fig. 9.9). The upper left-hand sketch "a" indicates the standard convection cooling. The upper right-hand arrangement "b" shows the *film cooling* process in which a stream of coolant is blown through a series of slots in a direction tangential to the surface. In this way a layer is created which insulates the wall from the hot gases. The coolant film is gradually destroyed by mixing with the hot gases, so that its effectiveness decreases in the downstream direction. This disadvantage can be overcome by the process called "*transpiration cooling*" which is indicated in the lower left-hand sketch "c". In this method the wall is manufactured from a porous material and the coolant is blown through the pores. The coolant film on the hot gas side is, therefore, continuously renewed, and the cooling effectiveness remains uniform along the surface. It was assumed so far that the coolant as well as the hot medium are either both gases or both liquids. When the hot medium is a gas, the cooling effectiveness can be increased a lot by using a liquid as a coolant [Fig. 9.9(d)]. A liquid film is created on the hot gas side of the wall, the liquid is evaporated on its surface and the heat is absorbed by the evaporation process, thus substantially increasing the effectiveness of this cooling method. It may be called *evaporative film cooling* or *evaporative transpiration cooling* depending upon whether the coolant is discharged through slots or through a porous wall.

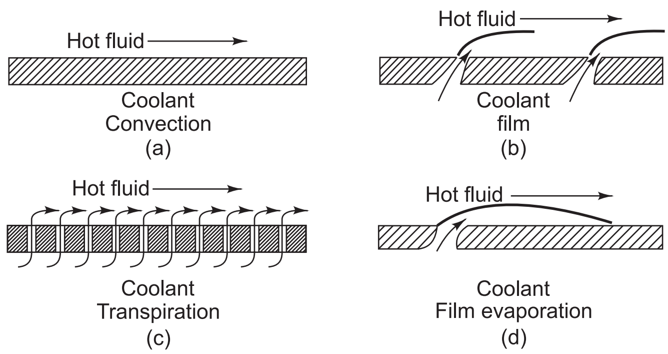


**Fig. 9.9** *Convection, transpiration and film cooling*

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In transpiration cooling a counterflow is created between the coolant ejected from the wall and the heat flux which moves from the hot gas towards the cooler surface (Fig. 9.10). This reduces the heat transfer coefficient as compared with conditions on a solid surface. This method is successfully used for cooling the combustion chamber walls as well as the blades of the gas turbine which are exposed to high velocity and high temperature gases.

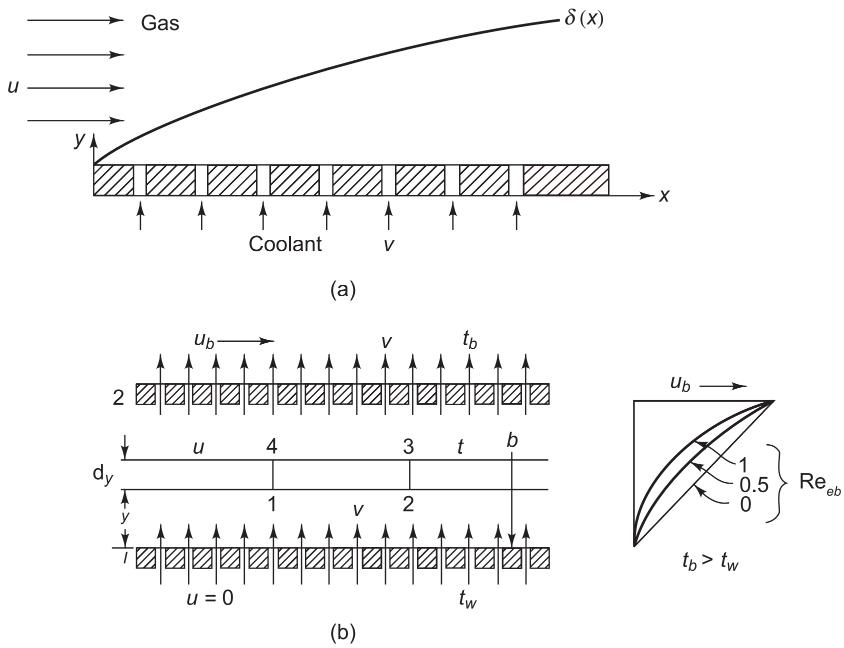


**Fig. 9.10** *(a) Transpiration cooling with a coolant injected through slots or porous plate; (b) Couette flow with injection and suction through the porous walls*

To solve the appropriate momentum and energy equations with a coolant so injected, Eckert and Drake [6] assumed a simplified model using a Couette flow as it exists between two parallel walls, one of which is stationary whereas the other one moves with a constant velocity within the plane. The pressure gradient in the boundary is zero, the velocity $u$ is changing along $y$-direction only, the velocity $v$ is constant along $y$-direction and the generation of heat due to viscous friction is neglected. With these assumptions, the following momentum and energy equations are formulated:

$$\mu \frac{d^2 u}{dy^2} - \rho v \frac{du}{dy} = 0$$

and
$$k \frac{d^2 T}{dy^2} - \rho v c_p \frac{dT}{dy} = 0$$

The boundary conditions are:

at $y = 0,\ u = 0,\ T = T_w$

at $y = \delta,\ u = U_\infty,\ T = T_\infty$

The solutions of the above equations give the velocity and temperature distribution as

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$$\frac{u}{U_\infty} = \frac{e^{(y/\delta)\,\mathrm{Re}_\delta} - 1}{e^{\mathrm{Re}_\delta} - 1} \tag{9.40}$$

and

$$\frac{T - T_w}{T_\infty - T_w} = \frac{e^{(y/\delta)\,\mathrm{Re}_\delta\,\mathrm{Pr}} - 1}{e^{\mathrm{Re}_\delta\,\mathrm{Pr}} - 1} \tag{9.41}$$

where $\mathrm{Re}_\delta = \frac{\bar{V}\delta}{\nu}$, based on injection velocity $v$ and boundary layer thickness $\delta$.

By using the above velocity and temperature distributions for the Couette flow, it can be shown that [6]

$$\frac{h_i}{h} = \frac{\mathrm{Re}\,\delta\,\mathrm{Pr}}{e^{\mathrm{Re}_\delta\,\mathrm{Pr}} - 1} \tag{9.42}$$

where $h_i$ and $h$ are the heat transfer coefficients with injection and without injection.

It has been found that the heat transfer coefficient decreases by 40% using air as injected fluid in the main stream of air with an injection factor

$$\frac{u_w}{U_\infty}(\mathrm{Re}_x)^{1/2} = 0.2$$

where $\mathrm{Re}_x = U_\infty x/\nu$

## 9.4 ABLATIVE COOLING

Ablating heat shields have been successfully used in satellite and missile re-entry to the earth's atmosphere and represent a major advance over the heat sink as a means of protecting surfaces from aerodynamic heating. In this application the high rate of heat transfer experienced at the surface first causes an initial temperature rise until the surface reaches the melting temperature $T_m$. Ablation, which means melting of the surface, starts after this heat-up period, and there follows a second short transient period after which a steady-state ablation velocity is reached. The melted material is assumed to run off immediately, e.g. around the sides of a nose cone.

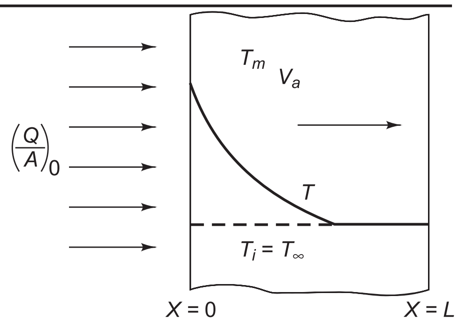


**Fig. 9.11** *Ablation at surface of flat plate*

We idealise the problem at the stagnation point of a missile nose as a one-dimensional heat conduction problem with the origin $x = 0$ at liquid–solid interface (Fig. 9.11). Then the wall is imagined to move to the left at the ablation velocity $V_a$. After steady-state ablation velocity has been reached, the temperature distribution being steady, with negative, $V_a$.

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) = -V_a \rho c \tag{9.43}$$

Materials such as glasses and plastics have proved the most successful. With the low thermal conductivity, the temperature gradient at the surface is very steep so that $x = L$ may be considered as $x = \infty$. The boundary conditions are

$$x = 0, \quad T = T_m$$

$$x = \infty, \quad T = T_\infty = T_i \text{ and } \frac{dT}{dx} = 0 \tag{9.44}$$

For constant properties $k$, $\rho$ and $c$, and for the case in which steady $(Q/A)_0$ net heat flux is transferred

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to the surface, Eq. (9.43) is solved by integrating twice and evaluating the integration constants with Eq. (9.44). The result is

$$\frac{T - T_\infty}{T_m - T_\infty} = \exp\left(-\frac{V_a x}{\alpha}\right) \tag{9.45}$$

where $\alpha = k/\rho c$

If $F$ is the heat of ablation (kJ/kg of material), an energy balance requires the following relations to hold:

$$\left(\frac{Q}{A}\right)_0 - \rho V_a F = -k\left(\frac{\partial T}{\partial x}\right)_{x=0} = \rho V_a c(T_m - T_\infty) \tag{9.46}$$

which states that the net heat flux into the solid material is equal to the net rate of change in enthalpy of the solid "flowing" through the coordinate system. From this the ablation velocity follows:

$$V_a = \frac{(Q/A)_0}{\rho F[1 + c(T_m - T_\infty)/F]} \tag{9.47}$$

The total heat conducted into the solid evaluated with the temperature distribution, Eq. (9.45), is

$$\left(\frac{Q}{A}\right)_0 = \rho c \int_{x=0}^{\infty} (T - T_\infty)\,dx = \frac{k(T_m - T_\infty)}{V_a} \tag{9.48}$$

The total heat transferred to the surface in time $t$ is $(Q/A)_0\, t$. Then for this period of time, the fraction of the total heat transferred which was conducted into the solid body is obtained by combining Eqs (9.47) and (9.48) to give

$$\frac{(Q/A)_c}{(Q/A)_0} = \frac{\rho k(T_m - T_\infty)[F + c(T_m - T_\infty)]}{(Q/A)_0^2 t} \tag{9.49}$$

## 9.5 THERMODYNAMIC OPTIMIZATION OF CONVECTIVE HEAT TRANSFER

In the past, the efficient use of energy sources has often been neglected in favour of the initial cost and the overall operating cost. However, as energy sources get steadily depleted, attention is being diverted to the more effective use of available energy. The location and degree of inefficient use of energy in energy systems should now be a primary factor in the design and performance analysis of these systems.

To be more specific, in any thermal system, e.g. a power plant or a refrigeration unit, the thermodynamic irreversibility in any of its components like a heat exchanger, quantified in the entropy generated in it, amounts to a penalty in otherwise available work. Hence, from an engineering standpoint, it makes good sense to first identify the irreversibility associated with each component and then design for minimum irreversibility. This is known as Second Law analysis of systems.

Loss of available energy due to fluid friction and that due to heat transfer across a finite temperature difference characterize most convective heat transfer processes. Second law analysis is made to minimize the loss of available energy by keeping the entropy generated to a minimum.

Bejan [15–18] has described the methodology of estimating entropy generation through heat and fluid flow in heat exchangers and has discussed irreversibility minimization subject to various constraints. Nag and Mukherjee [19] made a thermodynamic optimization of convective heat transfer in an isothermal duct and recommended the use of an optimum fluid velocity in a design. Mukherjee, Biswas and Nag [20] presented a second law analysis for convective heat transfer in swirling flow through a tube. Nag and

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Mukherjee [21] demonstrated this method by analysis of the entropy generated due to heat transfer for laminar flow in a tube with constant wall temperature.

An attempt has been made below to illustrate the method for the important case of heat transfer from a duct with a constant heat flux [22], which occurs, e.g. in a nuclear reactor, by energy balance of the control volume of length d$x$ of the duct with constant heat flux (Fig. 9.12).

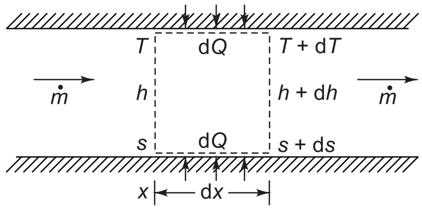


**Fig. 9.12** *Control volume*

$$\mathrm{d}Q = h_c\, 2\pi r\, \mathrm{d}x \cdot \phi = \pi r^2\, u\rho\, c_p\, \mathrm{d}T \tag{9.50}$$

where $u$ is the mean fluid velocity, $\phi$ is the temperature difference between the fluid ($T$) and wall ($T_1$), and $h_c$ is the average heat transfer coefficient. Since d$Q$/d$x$ = constant, the fluid temperature increases linearly with distance $x$, and so d$T$/d$x$ = constant (Eq. 4.177). The temperature profiles for both fluid and wall would be parallel (Fig. 4.40). Integrating Eq. (9.50).

$$T = T_0 + \frac{2h_c\phi}{ru\rho c_p} x$$

or
$$\theta = T - T_0 = \frac{A\phi x}{r} \tag{9.51}$$

where $A$ is equal to twice the Stanton number, $h_c/\rho\, c_p u$, and $T_0$ is the initial temperature of the fluid.

The temperature of the wall at any instant is $T' = T + \phi$. Considering an entropy balance in the same control volume, the main entropy generation is

$$\mathrm{d}\dot{S}_{\mathrm{gen}} = \dot{m}\, \mathrm{d}s - \frac{\mathrm{d}Q}{T^1} \tag{9.52}$$

Assuming the fluid to be an ideal gas or to be incompressible, the enthalpy change is d$h$ = $c_p$ d$T$, and using the thermodynamic relation.

$$T\mathrm{d}s = \mathrm{d}h - v\mathrm{d}p \text{ and } \mathrm{d}Q = \dot{m}\mathrm{d}h$$

Eq. (9.52) can be written as

$$\mathrm{d}\dot{S}_{\mathrm{gen}} = \dot{m}\left(\frac{c_p\mathrm{d}T}{T} - \frac{v\mathrm{d}p}{T}\right) - \frac{c_p\mathrm{d}T}{T+\phi} = \dot{m}\, c_p\mathrm{d}T\left(\frac{1}{T} - \frac{1}{T+\phi}\right) - \frac{\dot{m}v\mathrm{d}p}{T}$$

or,
$$\frac{\mathrm{d}\dot{S}_{\mathrm{gen}}}{\mathrm{d}x} = \frac{\dot{m}c_p\phi}{T(T+\phi)}\frac{\mathrm{d}T}{\mathrm{d}x} + \frac{\dot{m}}{\rho T}\left(-\frac{\mathrm{d}p}{\mathrm{d}x}\right) \tag{9.53}$$

The first and second terms on the RHS of Eq. (9.53) represent the entropy generations due to heat transfer across finite temperature difference and due to friction, respectively.

Substituting the value to $T$ from Eq. (9.51),

$$\frac{\mathrm{d}\dot{S}_{\mathrm{gen}}}{\mathrm{d}x} = \dot{m}c_p\left[\frac{1}{T_0 + \frac{A\phi x}{r}} - \frac{1}{T_0 + \phi + \frac{A\phi x}{r}}\right]\frac{\mathrm{d}T}{\mathrm{d}x} + \frac{\dot{m}}{\rho\left(T_0 + \frac{A\phi x}{r}\right)}\left(-\frac{\mathrm{d}p}{\mathrm{d}x}\right) \tag{9.54}$$

Integrating along the duct length,

$$\dot{S}_{\mathrm{gen}} = \frac{\dot{m}c_p r}{A\phi}\left(\frac{\mathrm{d}T}{\mathrm{d}x}\right)\ln\left[\frac{(T_0 + An\phi)(T_0+\phi)}{(T_0 + An\phi + \phi)T_0}\right] + \frac{\dot{m}r}{\rho\phi A}\left(-\frac{dp}{dx}\right)\ln\frac{T_0 + An\phi}{T_0} \tag{9.55}$$

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where $n$ is the length-to-radius ratio of the duct.

Defining the entropy generation unit $N_s$ as

$$N_s = \frac{\dot{S}_{\text{gen}}}{\dot{m}c_p}$$

and substituting

$$\frac{dT}{dx} = \frac{A\phi}{r}; \tau = \frac{\phi}{T_0}; -\frac{dp}{dx} = \frac{f\rho u^2}{r}$$

where $f$ is the friction factor, and $u = \dfrac{Q}{\pi r^2 \rho c_p A\phi n}$ (9.56)

Equation (9.69) can be written as

$$N_s = \ln\left[\frac{(1+An\tau)(1+\tau)}{(1+An\tau+\tau)}\right] + \frac{J^2}{n^2A^2\tau^3}\ln(1+An\tau) \tag{9.57}$$

where $J$ is the duty parameter defined as

$$J = \frac{f^{1/2}Q}{\rho a(c_pT_0)^{3/2}} \tag{9.58}$$

which accounts for the required heat transfer rate, fluid properties, initial fluid temperature $T_0$ and duct cross-section $a$.

Numerical solution of Eq. (9.57) is obtained with air as the fluid and a Stanton number of 0.005. Figure 9.13 shows the variation of $N_s$ with $\tau$ for $n$ = 60, $A$ = 0.01 and $J$ as a parameter, and Fig. 9.14 shows the variation of $N_s$ with $\tau$ for $J = 2 \times 10^{-4}$, $A$ = 0.01 and $n$ as a parameter. These two sets of curves show distinct minima in $N_s$ as $\tau$ is increased.

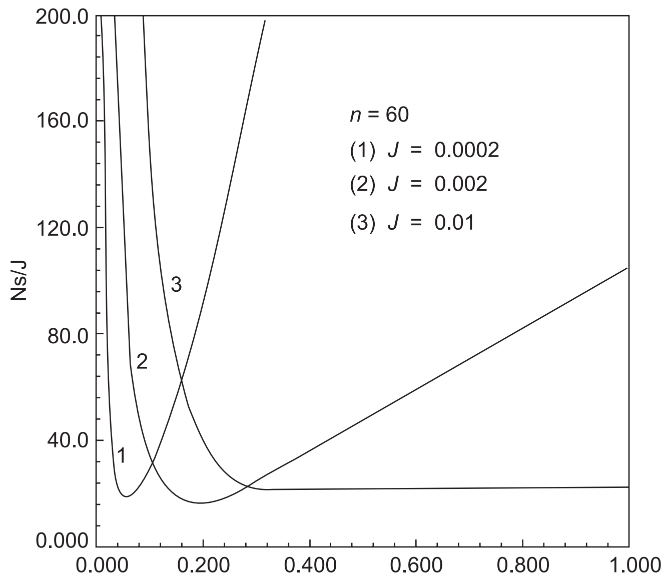


**Fig. 9.13** *Variation of $N_s$ with $\tau$ having $J$ as a parameter*

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Differentiating $N_s$ with respect to $\tau$ in Eq. (9.57) and equating it to zero, we get for $\tau << 1$,

$$\frac{J^2}{n^2 A^3} = \frac{(2 + \tau + AN\tau)\tau^4}{2 + 5An\tau + 4\tau + 8An\tau^2 + 3A^2n^2\tau^2 + 2\tau^2} \tag{9.59}$$

where $\ln(1 + An\tau) = An\tau$ has been used and higher powers of $\tau$ have been neglected.

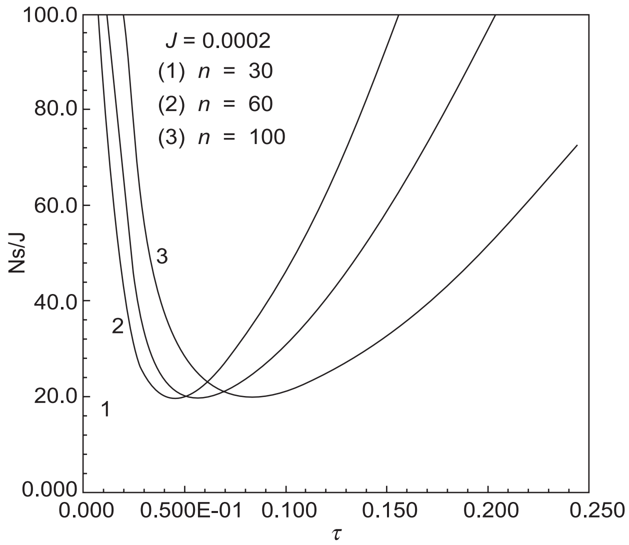


**Fig. 9.14** *Variation of $N_s$ with $\tau$ having n as a parameter*

Equation (9.57) has been solved numerically for $\tau_{opt}$, and the values of $\tau_{opt}$ are plotted against $J$ in Fig. 9.14 for air with $A = 0.01$ and $n$ as a parameter. Thus, for a particular heat load, there is an optimum temperature difference between the fluid and the wall from inlet $t_0$, and exit from, the duct, which corresponds to the minimum irreversibility.

Defining the ratio of heat transfer to pumping power $R$ as

$$R = \frac{Q}{a\Delta p u}$$

and substituting in Eq. (9.70),

$$R = \frac{n^2 A^3 \tau^3}{J^2} \tag{9.60}$$

Since $R$ is a function of $\tau$ and $N_s$ is also a function of $\tau$, hence $N_s$ is functionally dependent on $R$. The optimum value of $R$ corresponds to the condition when $N_s$ is a minimum, and this occurs when $\tau_{opt}$. occurs. By substituting Eq. (9.59) in Eq. (9.60),

$$R_{opt} = \frac{2 + 5An\tau_{opt} + 4\tau_{opt} + 8An\tau_{opt}^2 + 2\tau_{opt}^2 + 3A^2n^2\tau_{opt}^2}{\tau_{opt}(2 + \tau_{opt} + An\tau_{opt})} \tag{9.61}$$

For a particular value of heat load $J$ there is an optimum value of $R$. Using $\Delta p = f\rho u^2/r$ in Eq. (9.60),

$$u_{opt} = \left[\frac{Q}{\pi r f \rho L R_{opt}}\right]^{1/3} \tag{9.62}$$

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Thus, there is an optimum fluid velocity which corresponds to the minimum loss of available power, and this is recommended in the design of the heat exchangers.

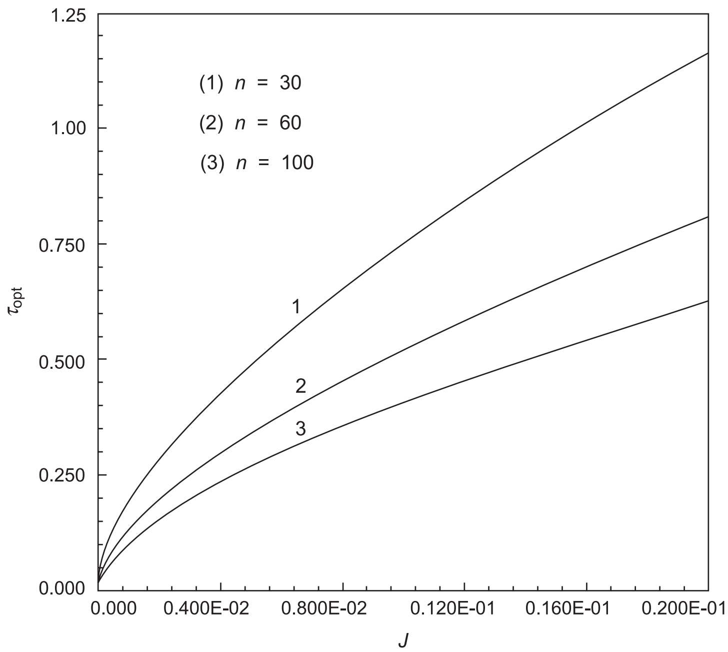


**Fig. 9.15** *Variation of $\tau_{opt}$ with J having n as a parameter*

In view of the discussion above, we recognize two dimensionless parameters in the entropy generation analysis of convective heat transfer problems. These parameters are the dimensionless temperature difference

$$\tau = \frac{\Delta T}{T}$$

and the irreversibility distribution ratio

$$\phi' = \frac{\dot{S}_{gen}\ \text{(fluid friction)}}{\dot{S}_{gen}\ \text{(heat transfer)}}$$

where the entropy generation is

$$\dot{S}_{gen} = \dot{S}_{gen}\ \text{(heat transfer)} + \dot{S}_{gen}\ \text{(fluid friction)}$$

Thus,

$$\frac{\dot{S}_{gen}\,\text{(heat transfer)}}{\dot{S}_{gen}} = \frac{1}{1+\phi'} = \text{Be} \tag{9.63}$$

which is the fraction of entropy generation due to heat transfer of the total entropy generation, called the *Bejan number,* Be (Ref. 15.) The value of Be = 1, sets the limit at which heat transfer irreversibility dominates and the opposite limit Be = 0 is the case when irreversibility is dominated by fluid friction. When Be = $\frac{1}{2}$, it refers to the situation in which the heat transfer and fluid friction entropy generation rates are equal.

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## 9.6 HEAT TRANSFER IN A CIRCULATING FLUIDIZED BED (CFB) BOILER

Dwindling reserves of high-quality coal and the imperative need for abatement of atmospheric pollution have spurred the development of circulating fluidized bed (CFB) technology for steam generation. The CFB boiler is outstanding in its fuel flexibility, low emission of pollutants and adaptability to load change, and therefore it has become a subject of worldwide attention as an improved new type of coal fired steam generator. Since the early 1980's, research on CFB systems continues at a brisk pace all around the world.

The heat transfer mechanisms of fluidized bed boilers are much different from those in conventional boilers. The heat generated due to combustion of fuel in the furnace is transferred to water in two sections: (i) the primary loop comprising the furnace operating in fast and turbulent fluidized bed regimes [23] and the solid return leg which includes cyclones and bubbling fludized bed, and (ii) the secondary loop or the back-pass which includes convective superheater/reheater, economiser etc. (Fig. 9.16).

The amount of heat transfer surfaces and their locations greatly influence the efficiency and output of a CFB boiler. A good understanding of heat transfer in such a boiler is thus essential for its proper design.

The heat released due to combustion of fuel in the furnace is transferred in two sections—the CFB loop and the backpass (Fig. 9.16). Five different situations prevail in the CFB loop:

(a) Gas to particle
(b) Bed to waterfall
(c) Bed to immersed surfaces in the furnace
(d) Bed to immersed surfaces in the external bubbling bed heat exchanger.
(e) Heat transfer to cyclone.

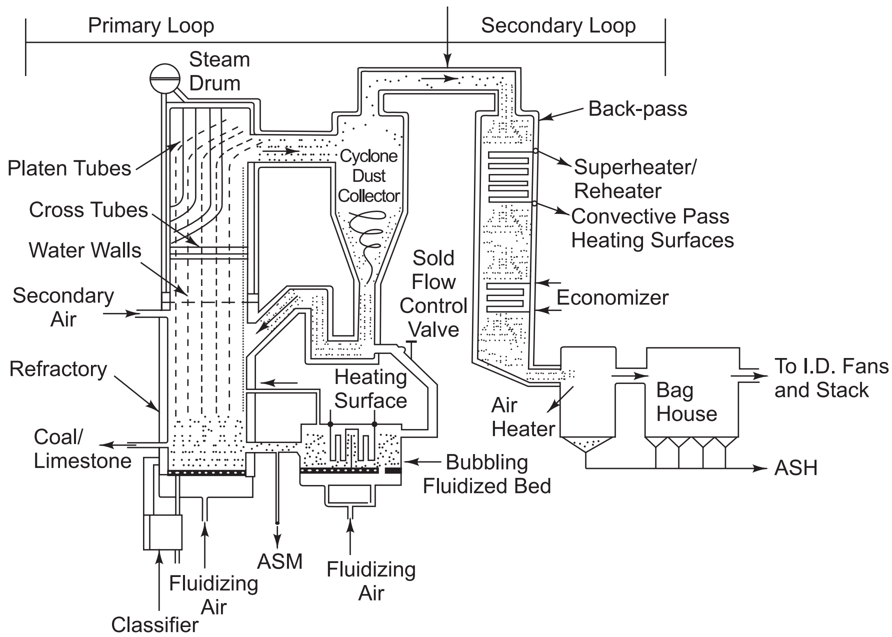


**Fig. 9.16** *Schematic of a circulating fludized bed boiler*

The orders of magnitude of heat transfer coefficients in the above situations are given in Table 9.7. The different modes of heat transfer prevailing in a CFB boiler are being explained below and some equations for estimating heat transfer coefficients are presented.

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**Table 9.2** *Heat transfer coefficients in different sections of a CFB boiler [Basu and Fraser (23)]*

| Locations | Types of section | Typical value of heat transfer coefficient (W/m² K) |
|---|---|---|
| 1. Water wall tubes above refractory walls in furnace (890–950°C) | Evaporator | 110–200 |
| 2. Wingwall or cross tubes inside the furnace (850–950°C) | Evaporator, Superheater and reheater | 50–150 |
| 3. Cyclone, horizontal tubes in backpass (800–200°C) | Evaporator, superheater and reheater | 340–510 |
| 4. Horizontal cross tubes in backpass (800–200°C) | Economiser, superheater and reheater | 50–150 |
| 5. Gas-to-bed material (420–177 µm, 50°C) | Furnace | 30–200 |

### 9.6.1 Gas-to-Particle Heat Transfer

Gas and solids in regions near the distributor, solid feed points and secondary air injection ports are at different temperatures from those of the bulk of the bed. The rate of heating of coal particles, the release of volatiles, attrition and fragmentation of particles are affected by heat transfer.

Particle Nusselt number ($h_{gp}\, d_p/k_g$) increases with the average Reynolds number $(U - U_p)\rho_p\, d_p/\mu_g$) where $(U - U_p)$ is the slip velocity.

Biot number ($h_g\, d_p$/kg) for a particle is very small so that the temperature variation within it can be neglected. An energy balance of a particle gives

$$\rho_p c_p \frac{\pi d_p^3}{6}\, dT_p = h_{gp}\, \pi d_p^2\, (T_g - T_p)\, dt \tag{9.64}$$

By assuming an average value of the gas-to-particle heat transfer coefficient, $h_{gp}$, the integration of Eq. (9.64) gives

$$t = \frac{\rho_p c_p d_p}{6 h_{gp}} \ln \frac{t_g - T_{pi}}{T_g - T_p} \tag{9.65}$$

where $T_{pi}$ is the initial temperature of the particle.

Thus, the time required for the gas-particle temperature difference to reduce to 1% of its original value or when $(T_g - T_p)/(T_g - T_{pi}) = 0.01$

$$T_{99\%} = \frac{0.765 \rho_p c_p d_p}{h_{gp}}$$

The distance travelled to reduce the difference $(T_g - T_p)$ by 99% can similarly be found by integrating over a distance d$x$ [23],

$$X_{99\%} = \frac{\rho_g U}{\rho_p S} \int_{T_{gi}}^{T_{99\%}} \frac{c_g\, dT_g}{h_{gp}(T_p - T_g)} \tag{9.66}$$

### 9.6.2 Bed-to-Wall Heat Transfer

Heat transfer from bed to the wall in a fast bed occurs through conduction from particle clusters, through convection from the dispersed phase, and through radiation from both phases.

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Clusters sliding down the wall (Fig. 9.17) experience unsteady heat conduction. Radiation is prominent in the dilute ($\rho_b$ < 30 kg/m$^3$) and high bed temperature (> 700°C), otherwise transient conduction predominates.

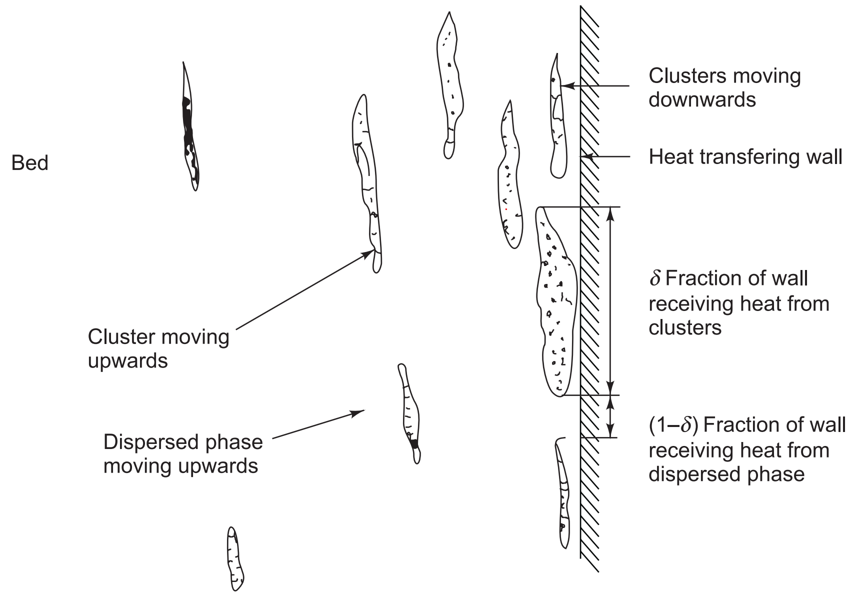


**Fig. 9.17** *Schematic representation of the heat transfer mechanism to the walls of a CFB*

Several models have been proposed to explain the mechanism of heat transfer in a CFB.

(a) Single particle model
(b) Cluster renewal model
(c) Continuous film model

The cluster renewal model is considered to be the closest to the observed behaviour of the bed [24, 25].

If $\delta_c$ is the average fraction of the wall area covered by clusters, the time-average heat transfer coefficient is given to be:

$$\bar{h} = \delta_c (h_c + h_{cr}) + (1 + \delta_c)(h_d + h_{dr}) \tag{9.67}$$

where $h_c$, $h_d$ represent convective and $h_{cr}$, $h_{dr}$ represent radiative contributions from the clusters and the dispersed phase respectively. Now [26]

$$d_c = 0.5\left[\frac{1 - e_w - Y}{1 - e_c^{0.5}}\right] \tag{9.68}$$

where $y$ is the volume fraction of solids in the dispersed phase. Again,

$$h_{\text{CONV.}} = \delta_c h_c + (1 - \delta_c) h_d \tag{9.69}$$

$$h_c = \frac{1}{\dfrac{d_p}{10k_g} + \left(\dfrac{\pi t_c}{4k_c C_c \rho_c}\right)^{0.5}} \tag{9.70}$$

and used Wen and Miller equation

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$$h_d = \frac{k_g}{d_p}\frac{C_p}{C_g}\left(\frac{\rho_{\text{dis}}}{\rho_p}\right)^{0.3}\left(\frac{U_t^2}{g d_p}\right)^{0.21} \text{Pr} \tag{9.71}$$

The total radiative heat transfer coefficient is given by

$$h_r = \delta_c h_{cr} + (1 - \delta_c) h_{dr} \tag{9.72}$$

where

$$h_{dr} = \frac{\sigma(T_b^4 - T_s^4)}{\left(\dfrac{1}{\varepsilon_d} + \dfrac{1}{\varepsilon_s} - 1\right)(T_b - T_s)} \tag{9.73}$$

assuming the two radiating surfaces to be infinitely parallel. Similarly, $h_{cr}$ can be obtained by substituting $\varepsilon_c$ for $\varepsilon_d$ in the above equation, where $\varepsilon_c = 0.5\,(1 + \varepsilon_p)$.

The bed-to-wall heat transfer depends on a number of design and operating parameters given, as follows.

### (i) Effect of Suspension Density

Suspension density ($\rho_{\text{sus}}$) is the most dominating parameter to influence the heat transfer coefficient (Fig. 9.18). Experimental data show that the heat transfer coefficient increases monotonically with suspension density (Fig. 9.19) where

$$\rho_{\text{sus}} = (1 - \varepsilon_c)\rho_p + \varepsilon_c\,\rho_g \tag{9.74}$$

where $\varepsilon$ is the voidage, subscript $c$, $d$ refer to clusters and dispersed phase.

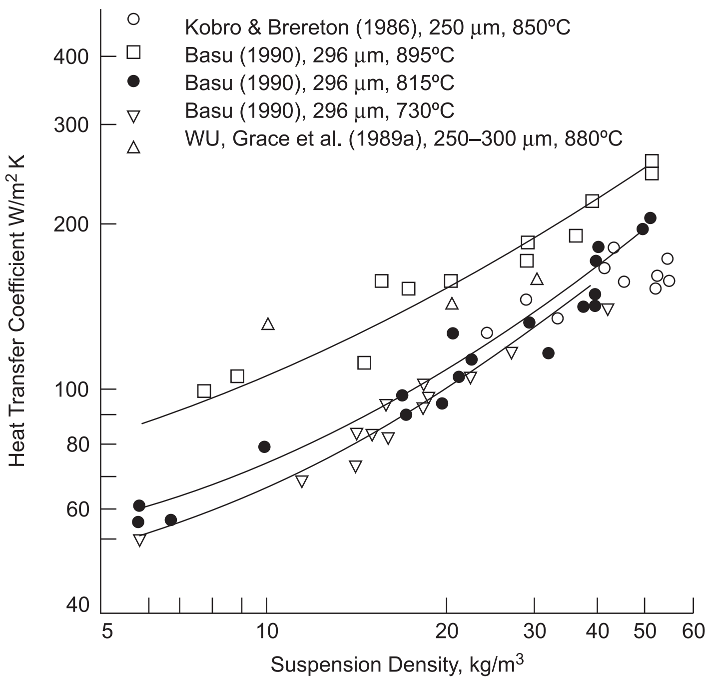


**Fig. 9.18** *Effect of suspension density on heat transfer coefficients measured on laboratory scale units*

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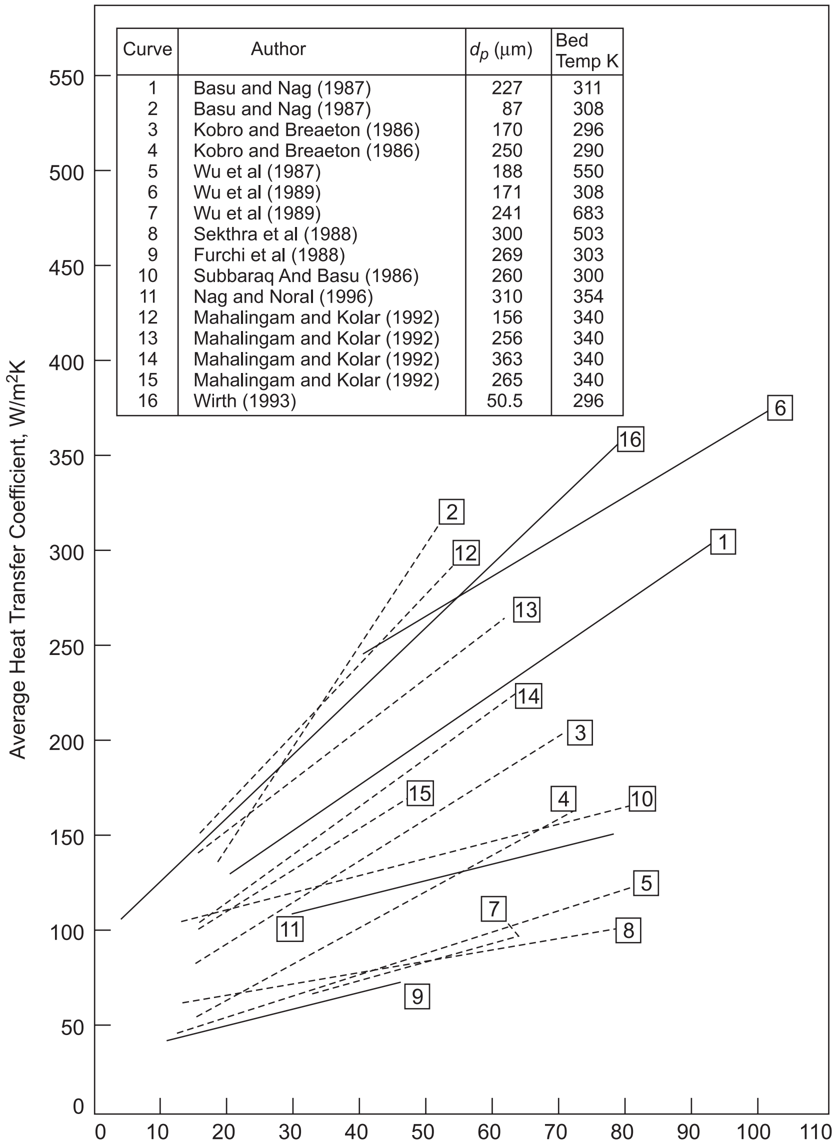


| Curve | Author | $d_p$ (μm) | Bed Temp K |
|---|---|---|---|
| 1 | Basu and Nag (1987) | 227 | 311 |
| 2 | Basu and Nag (1987) | 87 | 308 |
| 3 | Kobro and Breaeton (1986) | 170 | 296 |
| 4 | Kobro and Breaeton (1986) | 250 | 290 |
| 5 | Wu et al (1987) | 188 | 550 |
| 6 | Wu et al (1989) | 171 | 308 |
| 7 | Wu et al (1989) | 241 | 683 |
| 8 | Sekthra et al (1988) | 300 | 503 |
| 9 | Furchi et al (1988) | 269 | 303 |
| 10 | Subbaraq And Basu (1986) | 260 | 300 |
| 11 | Nag and Noral (1996) | 310 | 354 |
| 12 | Mahalingam and Kolar (1992) | 156 | 340 |
| 13 | Mahalingam and Kolar (1992) | 256 | 340 |
| 14 | Mahalingam and Kolar (1992) | 363 | 340 |
| 15 | Mahalingam and Kolar (1992) | 265 | 340 |
| 16 | Wirth (1993) | 50.5 | 296 |

**Fig. 9.19** *Effect of particle size and suspension density on average heat transfer coefficient measured on laboratory scale units*

This is due to the fact that the thermal capacity of solids is much higher than that of the gas. So, the heat transfer through particles contributes more than that across the fluid boundary layer. A dense bed yields higher heat transfer than a lean bed.

### *(ii) Effect of Superficial Velocity*

The superficial velocity of gas does not have any direct effect on heat transfer except through suspension density. When the velocity is increased at constant circulation rate, the heat transfer coefficient drops due to a decrease in suspension density.

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#### (iii) Effect of Bed Temperature

The heat transfer coefficient increases with bed temperature, almost linearly, due to higher values of $k_g$ and $k_P$ as well as due to radiation (Fig. 9.20).

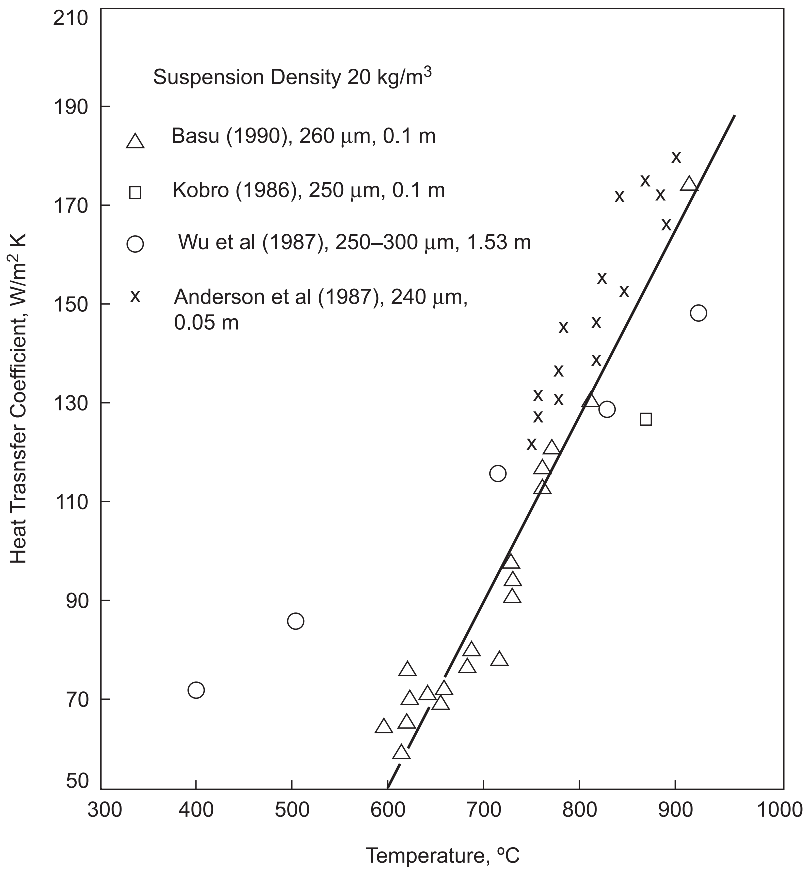


**Fig. 9.20** *Effect of bed temperature on heat transfer coefficient*

#### (iv) Effect of Particle Effect

Smaller diameter particles yield higher heat transfer coefficient. However, for long heat transfer surfaces as in commercial boilers, the particle size has no direct effect on heat transfer coefficient except through bed hydrodynamics [27].

#### (v) Effect of Vertical Length of Heat Transfer Surface

The heat transfer coefficient decreases with the vertical length [21] and becomes asymptotic beyond 1 – 1.5 m (Fig. 9.21). For commercial boilers, since the walls are long, vertical length has hardly any effect on heat transfer coefficient.

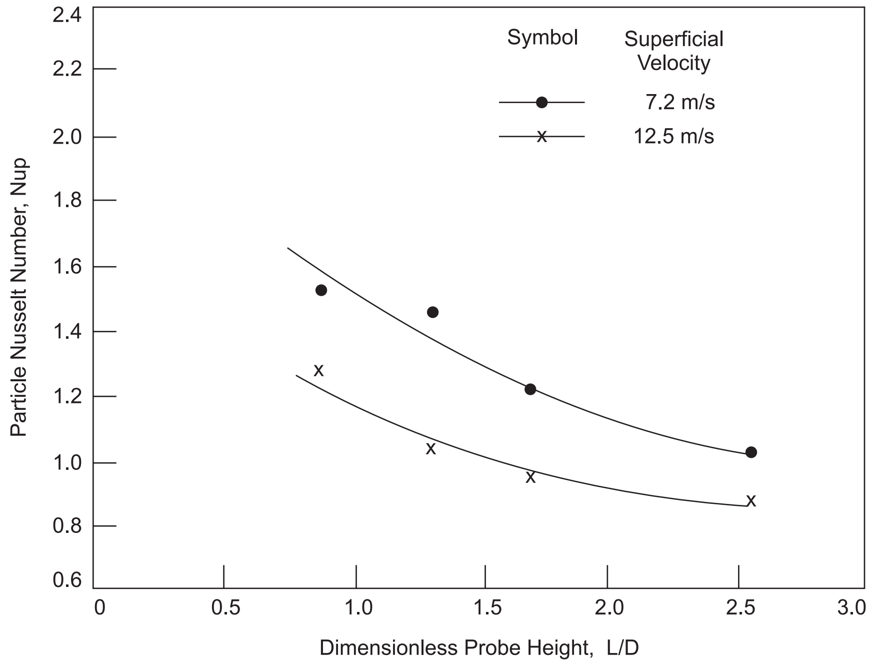


**Fig. 9.21** *Effect of vertical length of the heat transferring surface on heat transfer coefficient*

***(vi) Effect of Fins***

By using fins (on risers) projected into the furnace, surface area can be increased as a result of which the boiler becomes more compact and less tall. Extensive studies on the use of various types of fins in CFB boiler furnaces have been made [23, 24]. The external fluidized bed heat exchanger may be done away with. The heat transfer from the fins may be written as

$$Q_f = h\, A_f\, \eta_f\, (T_b - T_s),$$

where the efficiency $\eta_f$ is reported in the range of 80–90%.

***(vii) Effect of Pressure***

A mechanistic model similar to the cluster renewal model was proposed to predict the bed-to-wall heat transfer in the riser column of a CFB at elevated pressures by taking into account of the variation of gas density with pressure [24, 25] and it was validated by experiments [32].

### 9.6.3 Heat Transfer to Tubes Immersed in Fast Beds

High capacity boilers often need to have some evaporator or superheater tubes inside the furnace to absorb the required fraction of combustion heat released. These tubes are in the form of wing wall or cross tubes. The heat transfer coefficient is found to vary radially, increasing from the core to the wall, which is a direct consequence of varying suspension density in the core-annulus structure of the bed. The effect of riser exit geometry on bed hydrodynamics and heat transfer in a CFB riser column was discussed by Reddy and Nag [26].

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### 9.6.4 Heat Transfer in Cyclone Separator

The cyclone separator, an integral part of a CFB, is often used to extract heat from the gas-solid suspension to the evaporator or superheater tubes installed along its wall (Fig. 9.22). Unlike in the furnace the solid particles here swirl over the wall surface with high velocity. Experiments were conducted to study the heat transfer behaviour and the hydrodynamics in the cyclone separator of a CFB. The overall heat transfer coefficient in the cyclone has been found to be increasing with the increase in solid loading as well as gas superficial velocity [34]. The incorporation of fins to the tubes in the cyclone wall enhances the overall heat transfer, although decreasing the heat transfer coefficient [27, 28].

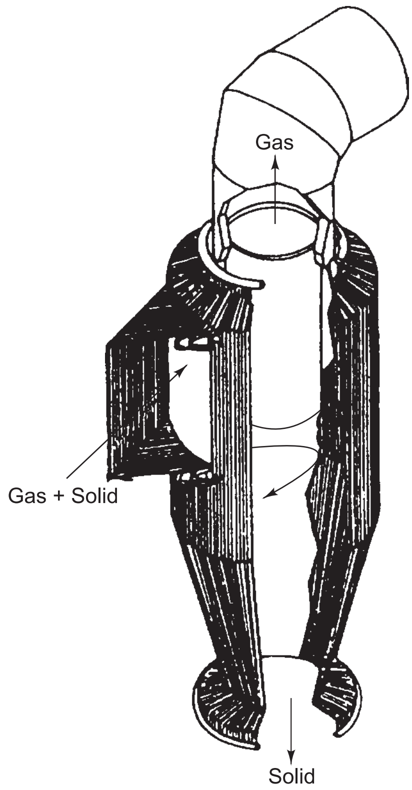


**Fig. 9.22** *A steam-cooled cyclone saves refractory and reduces radiation loss*

#### Conclusion

The understanding of the process of heat transfer in a CFB is still in a developing stage. However, a usable approximation of the processes involved is possible. Suspension density is the most significant factor influencing the heat transfer in a CFB furnace. It is followed by bed temperature and the vertical length of the heat exchanger. The direct effect of particle size is evident with short heat transfer surfaces, but is not significant with long surfaces similar to those used in commercial boilers. Although some mechanistic models are available, one depends on experimental data of the heat transfer coefficient which is generally in the range of 100–200 $W/m^2$ K and varies primarily with the suspension density.

## Solved Examples

**Example 9.1** Air at a pressure of 1/30 atm, temperature – 23°C and velocity 600 m/s flows over a flat plate 0.80 m long and 0.30 m wide. Calculate the amount of cooling needed to maintain the plate surface at a uniform temperature of 300 K.

**Solution** This is a high-speed flow problem. If it involves both laminar and turbulent regions, the heat transfer analysis for each region has to be considered separately because the recovery factors and hence the adiabatic wall temperatures, are different in the two regions.

***Laminar flow region*** At 300 K for air, Pr = 0.708 and $c_p$ = 1006 J/kg K. The recovery factor $r$ is found from Eq. (9.5),

$$r = \mathrm{Pr}^{1/2} = (0.708)^{1/2} = 0.841$$

and the adiabatic wall temperature $T_{aw}$ is found from Eq. (9.4),

$$T_{aw} = T_\infty + r\frac{V_\infty^2}{2c_p} = (-23 + 273) + 0.841\,\frac{600^2}{2 \times 1006}$$
$$= 400.5 \text{ K}$$

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The reference temperature is obtained from Eq. (9.9)

$$T^* = T_\infty + 0.5\,(T_w - T_\infty) + 0.22\,(T_{aw} - T_\infty)$$
$$= 250 + 0.5\,(300 - 250) + 0.22\,(400.5 - 250)$$
$$= 308.1 \text{ K}$$

Physical properties of air at 308 K, 1/30 atm are, $\rho$ = 0.0383 kg/m$^3$, $k$ = 0.0269 W/m K, $\mu$ = 1.998 × 10$^{-5}$ kgm s, Pr = 0.71 and, $c_p$ = 1007 J/kgK. Values of $c_p$ and Pr at $T^*$ are close to the earlier values.

Assuming that the transition takes place at a critical Reynolds number $\text{Re}_c = 5 \times 10^5$,

$$\text{Re}_c = 5 \times 10^5 = \frac{V_\infty x_c}{\gamma}$$

$$\therefore \qquad x_c = \frac{1.998 \times 10^{-5}}{0.0383 \times 600}\,(5 \times 10^5) = 0.43 \text{ m}$$

Average heat transfer coefficient for laminar boundary-layer flow over $0 \le x \le 0.43$ m is determined from Eq. (9.10) ($h_m = 2h_c$),

$$h_m = \frac{k}{x_c}\,0.664\,\text{Re}_c^{1/2}\,\text{Pr}^{1/3}$$
$$= \frac{0.0269}{0.43}\,(0.664)\,(5 \times 10^5)^{1/2}\,(0.71)^{1/3}$$
$$= 26.2 \text{ W/m}^2\text{ K}$$

Heat transfer rate from the laminar flow region

$$Q_1 = h_m A\,(T_w - T_{aw})$$
$$= (26.2)\,(0.3 \times 0.43)\,(300 - 400.5)$$
$$= -340 \text{ W}$$

***Turbulent flow region*** At 300 K, for air, $c_p$ = 1006 J/kg K and Pr = 0.708. Recovery factor $r$ is found from Eq. (9.6),

$$r = \text{Pr}^{1/3} = (0.708)^{1/3} = 0.891$$

$$T_{aw} = T_\infty + r\,\frac{V_\infty^2}{2c_p} = 250 + 0.891\,\frac{600^2}{2 \times 1006}$$
$$= 409.4 \text{ K}$$

$$T^* = T_\infty + 0.5\,(T_w - T_\infty) + 0.22\,(T_{aw} - T_\infty)$$
$$= 250 + 0.5\,(300 - 250)\,0.22\,(409.4 - 250)$$
$$= 310 \text{ K}$$

Physical properties of air at 310 K can be taken the same as those for the laminar flow region. From Eq. (9.11),

$$h_x = 0.0296\,\rho\,V_\infty\,c_p\,\text{Pr}^{-2/3}\,\text{Re}_x^{-0.2}$$
$$= (0.0296)\,(0.0383)\,(600)\,(1007)\,(0.71)^{-2/3}$$
$$\left(\frac{0.0383 \times 600}{1.998 \times 10^{-5}}\right)^{-0.2} x^{-0.2}$$
$$= 52.8\,x^{-0.2}$$

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The average value of $h_x$ over the region 0.43 m $\le x \le$ 0.8 m

$$h_m = \frac{1}{0.8 - 0.43} \int_{0.43}^{0.8} h_x \, dx = 58.4 \text{ W/m}^2\text{K}$$

Heat transfer rate from the turbulent flow region

$$\begin{aligned} Q_2 &= h_m A (T_w - T_{aw}) \\ &= 58.4 \times 0.3 \times (0.8 - 0.43)(300 - 409.4) \\ &= -709.2 \text{ W} \end{aligned}$$

Total amount of cooling required = 340 + 709 = 1049 W *Ans.*

**Example 9.2** An aluminium sphere 0.6096 m in diameter travels at $M_\infty$ = 10 at an altitude of 167,000 m. At that altitude the temperature is 232°C, the density is 48 × $10^9$ kg/m$^3$ and the mean free path is 3.048 m. Calculate (a) the adiabatic wall temperature and (b) the average heat transfer coefficient on the surface of the sphere.

**Solution**

(a)
$$\begin{aligned} V_\infty &= M_\infty (\gamma R T_\infty)^{1/2} \\ &= 10 (1.4 \times 0.287 \times 505 \times 10^3)^{1/2} \\ &= 4504.5 \text{ m/s} \end{aligned}$$

$$\text{Kn} = \frac{\lambda}{d} = \frac{3.048 \text{ m}}{0.6096 \text{ m}} = 5$$

It is free-molecule regime.

Speed ratio $S = M_\infty (\gamma/2)^{1/2} = 10 (1.4/2)^{1/2}$
$= 8.36$

From Fig. 9.6, $\frac{\gamma + 1}{\gamma} r = 2$,

$\therefore$ $r = 1.17$

Now,
$$r = \frac{T_{aw} - T_\infty}{V_\infty^2 / 2c_p}$$

$$1.17 = \frac{T_{aw} - 505}{(4504.5)^2 / (2 \times 1.005)}$$

$$T_{aw} = 12{,}316 \text{ K} = 12{,}043\text{°C} \quad \textit{Ans.}$$

From Fig. 9.6, at $S$ = 8.36

(b)
$$\frac{1}{A} \frac{\gamma}{\gamma + 1} \text{St} = 0.12$$

For an aluminium surface in air, from Table 9.2, $A$ = 0.95.

$$\text{St} = \frac{0.12 \times 0.95 \times 2.4}{1.4} = 0.195$$

$$\begin{aligned} h &= \text{St} \, \rho \, c_p \, V_\infty = 0.195 \times 48 \times 10^{-9} \times 1005 \times 4504.5 \\ &= 0.0424 \text{ W/m}^2 \text{ K} \quad \textit{Ans.} \end{aligned}$$

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**Example 9.3** Two polished aluminium plates of emissivity 0.06 are separated by a distance of 2.0 cm in air at a pressure of $2 \times 10^{-6}$ atm. The plates are maintained at 100°C and 30°C respectively. Calculate the conduction heat transfer through the air gap. Compare this with the radiation heat transfer and the conduction for air at normal atmospheric pressure.

**Solution**

$$T_{\text{mean}} = \frac{373+303}{2} = 338 \text{ K}$$

From Eq. 9.26(a), the mean free path is given by

$$\lambda = 2.27 \times 10^{-5} \frac{T}{p} \text{ metres}$$

$$= \frac{2.27 \times 10^{-5} \times 338}{1.01325 \times 10^5 \times 2 \times 10^{-6}}$$

$$= 378.6 \times 10^{-4} = 0.03786 \text{ m}$$

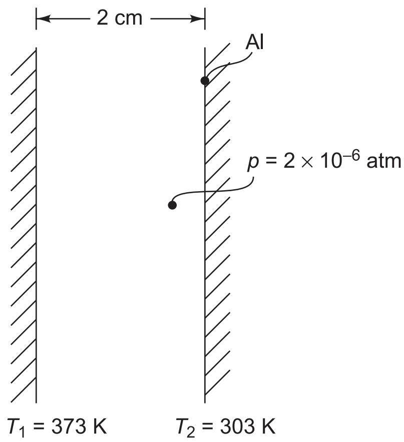


**Fig.** *Ex. 9.3*

Since the plate spacing is only 2 cm, low density effects are important. At the mean air temperature of 65°C, the properties are $k = 0.0291$ W/m K, $\gamma = 1.4$, Pr = 0.7 and $A = 0.9$.

From Eq. (9.39) using central temperature gradient relation,

$$\Delta T = \frac{2-A}{A} \frac{2\gamma}{\gamma+1} \frac{\bar{\lambda}}{\text{Pr}} \frac{T_1 - T_2 - 2\Delta T}{L}$$

$$= \frac{2-0.9}{0.9} \frac{2.8}{2.4} \frac{0.03786}{0.7} \frac{100-30-2\,\Delta T}{0.02}$$

$$= 3.856 \times 70 - 7.712\, T$$

$$\Delta T = 31°\text{C}$$

Conduction heat transfer is thus

$$\frac{Q}{A} = k \frac{T_1 - T_2 - 2\Delta T}{L} = 0.0291 \frac{70-62}{0.02}$$

$$= 11.64 \text{ W/m}^2 \quad \textit{Ans.}$$

At normal atmospheric pressure

$$\frac{Q}{A} = k \frac{T_1 - T_2}{L} = 0.0291 \frac{70}{0.02}$$

$$= 101.85 \text{ W/m}^2 \quad \textit{Ans.}$$

Radiation heat transfer is

$$\left(\frac{Q}{A}\right)_{\text{rad}} = \frac{(T_1^4 - T_2^4)}{(2/\varepsilon - 1)} = \frac{(5.67 \times 10^{-8})(393^4 - 303^4)}{(2/0.06) - 1}$$

$$= \frac{5.67 \times 154.25}{32.33} = 27.05 \text{ W/m}^2 \quad \textit{Ans.}$$

Thus, at the low density condition, the radiation heat transfer is about 2.5 times the conduction heat transfer.

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**Example 9.4** (a) For a heat flux $(Q/A)_0 = 3.1525 \times 10^6$ W/m$^2$ and the following material properties: $C$ = 1256.1 J/kg K; $k$ = 0.8655 W/m K; $\rho$ = 1601.8 kg/m$^3$; $F$ = 9304 J/kg; $T_m$ = 1649°C and $T_\infty$ = 15°C, what is the steady-state ablation velocity and the fraction $Q_c/Q_0$ in 20 s?

(b) At what distance $x$ is the temperature equal to 38°C?

(c) In 20 s how much material has ablated?

**Solution** From Eq. (9.47),

$$V_a = \frac{(Q/A)_0}{\rho F\left[1+\dfrac{C(T_m - T_\infty)}{F}\right]}$$

$$= \frac{3.1525\times 10^6 \text{ W/m}^2}{1601.8\dfrac{\text{kg}}{m^3}\times 9304\dfrac{\text{J}}{\text{kg}}\left[1+\dfrac{1256.1\text{ (J/kg K)}\times(1649-15)}{9304\text{ J/kg}}\right]}$$

$$= 0.211\times\frac{1}{221.6}\times 3600 = 3.43\text{ m/h} \qquad \textit{Ans. (a)}$$

From Eq. (9.49),

$$\frac{Q_C}{Q_0} = \frac{\rho K(T_m - T_\infty)[F + C(T_m - T_\infty)]}{(Q/A)_0^2\, t}$$

$$= \frac{1601.8\text{ kg/m}^3 \times 0.8655\text{ W/mK}\times 1634\text{ K }(9304\text{ J/kJ} + 1256.1\times 1634\text{ J/kg})}{(3.1525\times 10^6)^2\ (\text{W}^2/\text{m}^4)\times 20\text{ s}}$$

$$= 0.0235 \qquad \textit{Ans. (a)}$$

From Eq. (9.45),

$$\frac{T - T_\infty}{T_m - T_\infty} = \exp\left(\frac{V_a x}{\alpha}\right)$$

$$\frac{38-15}{1649-15} = \exp\left(-\frac{3.43}{3600}\frac{\text{m}}{\text{s}}\times x\text{ m}\times\frac{1601.8\text{ kg/m}^3\times 1256.1\text{ J/kg K}}{0.8656\text{ W/m K}}\right)$$

$$0.014076 = \exp(-2214.915\,x)$$

$$-4.2638 = -2214.915\,x$$

or, $$x = 1.925\text{ mm} \qquad \textit{Ans. (b)}$$

(c) Length of material ablated

$$x_{\text{abl}} = \frac{3.43}{3600}\frac{\text{m}}{\text{s}}\times 20\text{ s}\times\frac{1000\text{ mm}}{1\text{ m}}$$

$$= 19\text{ mm} \qquad \textit{Ans. (c)}$$

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## Summary

The chapter discusses some special topics of heat transfer processes. In high velocity flows adiabatic wall temperature and recovery factor are explained showing the dependence of stagnation temperature on Mach number. In rarefied gases Knudsen number is defined and different flow regimes are identified and accommodation coefficient is introduced. Some special cooling methods like transpiration and film cooling to protect certain high temperature structure elements as in turbojet and rocket engines are discussed along with ablative cooling for satellite and missile re-entry to the earth's atmosphere.

Most convective heat transfer processes are characterized by two losses of available energy: losses due to fluid friction and losses due to heat transfer across a finite temperature difference. These losses are quantified by entropy generation. An analysis of second law optimization for convective heat transfer with minimum irreversibility has been made to recommend an optimum fluid velocity.

Finally, a discussion on heat transfer in a circulating fluidized bed boiler is made to demonstrate that the effect of suspension density on heat transfer is the most important.

## Important Formulae and Equations

| *Equation number* | *Equation* | *Remarks* |
|---|---|---|
| (9.1) | $\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} M^2$ | Ratio of stagnation and static temperature |
| (9.2) | $r = \frac{2}{(\gamma - 1) M_\infty^2} \left( \frac{T_{aw}}{T_\infty} - 1 \right)$ | Recovery factor $r$ depends on $M_\infty$ and adiabatic wall temperature $T_{aw}$ |
| (9.3) | $T_{aw} = T_\infty \left( 1 + r \frac{\gamma - 1}{2} M_\infty^2 \right)$ | Adiabatic wall temperature |
| (9.5) | $r = (\text{Pr})^{1/2}$ | Recovery factor with air in laminar flow |
| (9.6) | $r = (\text{Pr})^{1/3}$ | Recovery factor with air in turbulent flow |
| (9.10) | $\text{St.Pr}^{2/3} = 0.332(\text{Re}_x)^{-1/2}$ | High speed heat transfer for turbulent flow for $\text{Re}_x < 5 \times 10^5$ |
| (9.11) | $\text{St. Pr}^{2/3} = 0.0296\ (\text{Re}_x)^{-1/5}$ | High speed heat transfer for turbulent flow for $5 \times 10^5 < \text{Re}_x < 10^7$ |
| (9.12) | $\text{St.Pr}^{2/3} = 0.185\ (\log \text{Re}_x)^{-2.584}$ | High speed turbulent flow for $\text{Re}_x$ in the range $10^7$ to $10^9$<br>All properties are evaluated at reference temperature |
| (9.20) | $\bar{V}_s = \sqrt{\frac{\gamma K T}{m}}$ | Velocity of sound |
| (9.21) | $\bar{V} = \sqrt{\frac{8KT}{\pi m}} = \left( \frac{8}{\pi_n} \right)^{1/2} \bar{V}_s$ | Mean molecular velocity |
| (9.23) | $\text{Kn} = 1.48 \frac{M}{\text{Re}}$ | Knudsen number |

(*Contd*)

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| Equation number | Equation | Remarks |
|---|---|---|
| (9.30) | $\frac{k}{K} = \frac{2}{\mathrm{Pr}}\left(\frac{2-A}{A}\right)\left(\frac{\gamma}{\gamma+1}\right)$ | Thermal accommodation coefficient $A$, where $k$ = thermal conductivity of the gas and $K$ is a proportionality constant |
| (9.43) | $\frac{\mathrm{d}}{\mathrm{d}x}\left(k\frac{\mathrm{d}T}{\mathrm{d}x}\right) = -V_a\rho_c$ | Temperature distribution at steady state ablation velocity $V_a$ |
| (9.45) | $\frac{T-T_\infty}{T_m-T_\infty} = \exp\left(-\frac{V_a x}{\alpha}\right)$ where $\alpha = k/\rho c$ and $T_m$ = melting temperature | Temperature distribution along $x$-coordinate, assuming constant properties of $k$, $\rho$ and $c$ |
| (9.46) | $\left(\frac{Q}{A}\right)_0 - \rho V_a F = -k\left(\frac{\partial T}{\partial x}\right)_{x=0} = \rho V_a c(T_m - T_\infty)$ where $F$ = heat of ablation (kJ/kg) | Net heat flux into the solid material |
| (9.47) | $V_a = \frac{(Q/A)_0}{\rho F[1+c(T_m-T_\infty]/F]}$ | Ablation velocity of molten material |
| (9.48) | $\left(\frac{Q}{A}\right)_0 = \frac{k(T_m-T_\infty)}{V_a}$ | Total heat conducted into the solid material |

## Review Questions

9.1 What is the limiting value of Mach number when the compressibility effect can be neglected?

9.2 What is aerodynamic heating? Why does it occur?

9.3 When does aerodynamic heating become a serious problem?

9.4 Explain the velocity and temperature distributions in high-speed flow over an insulated plate.

9.5 Define adiabatic wall temperature. Why is it less than the stagnation temperature and more than static temperature?

9.6 What do you mean by recovery factor? How does it relate $T_{aw}$ and $T_o$?

9.7 Explain the analogy of heat transfer and fluid friction in high-speed flows.

9.8 What is a rarefied gas? Define Knudsen number. When is the continuum approach valid?

9.9 Explain slip flow. Why do velocity and temperature discontinuities develop at the gas-solid boundary?

9.10 What do you mean by free-molecule flow? Are the properties like density and temperature meaningful in this regime?

9.11 Show that for a diatomic gas,

$$\mathrm{Kn} = \left(\frac{\pi\gamma}{2}\right)^{1/2}\frac{M}{\mathrm{Re}} = 1.48\frac{M}{\mathrm{Re}}$$

9.12 Give the classification of flow regimes in terms of ranges of $M$ and Re.

9.13 Define thermal accommodation coefficient. What does it physically mean?

9.14 Show that for air the recovery factor $r = 1.17$, which indicates that recovery temperature of free-molecule flow is greater than stagnation temperature.

9.15 Explain transpiration cooling. What is film cooling?

9.16 What do you understand by ablative cooling? What are its applications?

9.17 What is ablative velocity? What is heat of ablation?

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## Objective Type Questions

9.1 It becomes a serious problem in high speed flows:
(a) High flight velocity control
(b) Aerodynamic heating
(c) Maintaining the surface temperature constant
(d) Control of heat flow to the skin

9.2 Adiabatic wall temperature in high speed flows is
(a) equal to the stagnation temperature
(b) greater than the stagnation temperature
(c) less than the stagnation temperature
(d) the temperature of an adiabatic surface

9.3 The recovery factor in flow of high velocity gases is a function of
(a) Reynolds number
(b) Prandtl number
(c) Peclet number
(d) Rayleigh number

9.4 The stagnation temperature is almost equal to static temperature of a gas for
(a) $M \le 0.25$ (b) $M \ge 0.25$
(c) $M = 1$ (d) $M > 1$

9.5 When the velocity of a gas is comparable to the sonic velocity, the flow is defined by
(a) Reynolds number
(b) Reynolds number and Mach number
(c) Reynolds number and Prandtl number
(d) Prandtl number

9.6 Flow of rarefied gases is characterized by
(a) Mach number
(b) Knudsen number
(c) Reynolds number
(d) Kinetic theory of gases

9.7 A gas can be considered a continuum if
(a) Kn < 0.01 (b) Kn = 1
(c) Kn = 10 (d) Kn > 0.001

9.8 The wall made of a porous material allows a coolant to be blown through the pores for effective cooling of the structural element. This is called
(a) film cooling
(b) transpiration cooling
(c) convection cooling
(d) ablative cooling

9.9 To protect surfaces from aerodynamic heating, the surfaces are often made of material which melts taking off the heat of melting and producing cooling effect. It is called
(a) ablative cooling
(b) film cooling
(c) convection cooling
(d) transpiration cooling

## Answers

| | | | | |
|---|---|---|---|---|
| 9.1 (b) | 9.2 (c) | 9.3 (b) | 9.4 (a) | 9.5 (b) |
| 9.6 (b) | 9.7 (a) | 9.8 (b) | 9.9 (a) | |

## Problems for Practice

9.1 Calculate the local heat transfer coefficient that may be expected to occur at a point 30 cm behind the leading edge of the fin of a supersonic plane travelling at $M = 2$ at (a) sea level ($p$ = 1.01325 bar, $T$ = 15°C), (b) 7000 m altitude ($p$ = 0.46 bar, $T$ = –10°C). The surface to be maintained is at 95°C in both cases. What is the equilibrium surface temperature to be expected in each case if no internal cooling is provided? What is the amount of cooling necessary to keep the surface at 95°C in each case?

9.2 Air at 18°C flows over a plate 20 cm long at a velocity 218 m/s. If the plate is maintained at 35°C, calculate the adiabatic wall temperature. What do you conclude regarding heat transfer at the plate? If the plate is maintained at 150°C, find the rate of heat transfer to or from the plate?

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9.3 A thermocouple 0.063 cm in diameter is held normal to an air stream, and at $M = 0.7$, it reads 60°C. If the recovery factor is 0.9, find the static temperature of the air.

9.4 Air at 1/20 atm and 275 K flows with a free stream velocity of 700 m/s over a flat plate 1.2 m long. If the surface of the plate is to be maintained at 325 K, determine the amount of cooling needed per metre width of the plate. (*Ans.* 14.384 kW/m)

9.5 A scale model of an aeroplane wing section is tested in a wind tunnel at $M = 1.5$. The air pressure and temperature in the test section are 20 kPa and – 30°C, respectively. If the wing section is to be kept at an average temperature of 60°C, determine the rate of cooling required. The wing model can be approximated by a flat plate of 0.3 m length in the flow direction.

9.6 Mechanical energy is stored in an experimental car by means of a flywheel. The flywheel is shaped like a relatively thin disk of 0.6 m diameter and rotates at 18,000 rpm. The composite material of which the flywheel is constructed has an effective thermal conductivity so small that the flywheel is essentially insulated. Estimate the maximum temperature on the flywheel surface when spinning in 20°C air. Assume that the boundary layer flow is turbulent. (*Ans.* $T_{aw} = T_{\max} = 165$°C)

9.7 For a heat flux $(Q/A)_0 = 4 \times 10^6$ W/m$^2$ and the following material properties, estimate the steady-state ablation velocity and the fraction $Q_c/Q_0$ in 30s. At what distance $x$ is the temperature equal to 60°C? In 30 s how much material has ablated? Given: $\rho = 1602$ kg/m$^3$, $c = 1256$ J/kg K, $k = 0.86$ W/m K, $F = 9300$ J/kg, $T_m = 1650$°C and $T_\infty = 20$°C.

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# Mass Transfer 10

The bulk flow of fluid due to pressure gradient occurring at a macroscopic level is a kind of mass transfer usually treated in the subject of fluid mechanics. In this chapter we are concerned with mass transfer occurring at a microscopic or molecular level, which deals with the transport of one constituent of a fluid solution or gas mixture from a region of higher concentration to a region of lower concentration. Heat is transferred in a direction which reduces an existing temperature gradient, and mass is transferred in a direction which reduces an existing concentration gradient. Drying, evaporation, chemical reaction, absorption, adsorption, solution and so on are all instances of mass transfer.

## 10.1 MASS TRANSFER BY MOLECULAR DIFFUSION: FICK'S LAW OF DIFFUSION

Mass transfer by molecular diffusion is analogous to heat transfer by conduction or momentum transfer in laminar flow. Mass transfer by molecular diffusion may occur in a stagnant fluid or in a fluid in laminr flow. Like the Fourier's equation of heat conduction $\nabla^2 T = (1/\alpha)\ \partial T/\partial t$, the concentration field of the diffusing species $A$ is given by

$$\nabla^2 C_A = \frac{1}{D}\frac{\partial C_A}{\partial t} \tag{10.1}$$

where $C_A$ is the concentration of component $A$ in a mixture of $A$ and $B$ in kgmol/m$^3$, $t$ is the time in seconds and $D$ is the mass diffusivity in m$^2$/s.

For one-dimensional mass diffusion,

$$\frac{\partial^2 C_A}{\partial y^2} = \frac{1}{D}\frac{\partial C_A}{\partial t} \tag{10.2}$$

where $y$ is the distance in the direction of diffusion.

Like the conduction equation

$$\bar{q} = -k \text{ grad } T$$

the mass transfer equation is

$$\frac{N_A}{A} = -D \text{ grad } C_A \tag{10.3}$$

For one-dimensional mass transfer,

$$\frac{N_A}{A} = -D\frac{dC_A}{dy} \tag{10.4}$$

where $N_A/A$ is the mass flux in kgmol/m$^2$s. The negative sign appears because the concentration gradient is negative in the direction of mass transfer. Equation (10.4) is called *Fick's law of diffusion*, which states

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that the mass flux of a constituent per unit area is proportional to the concentration gradient.

In Fig. 10.1 a thin partition separates the two gases $A$ and $B$. When the partition is removed, the two gases diffuse through each other until equilibrium is established and the concentration of the gases is uniform throughout the box. The diffusion rate is given by Eq. (10.4). It can also be expressed in terms of mass flow:

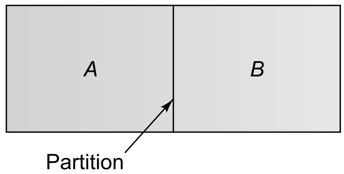


**Fig. 10.1** *Diffusion of component A into component B and vice versa*

$$\frac{\dot{m}_A}{A} = -D\frac{\partial C'_A}{\partial y} \tag{10.5}$$

where $\dot{m}_A$ is the mass flux per unit time, kg/s; $C'_A$ is the mass concentration of component $A$ per unit volume, kg/m$^3$; and $D$ is the proportionality constant called diffusion coefficient, m$^2$/s.

The concentration (moles per unit volume) of component $A$ is greater on the left side of the partition than on the right side. More molecules of $A$ cross the plane from left to right than in the opposite direction. This results in a net mass transfer from the region of higher concentration to the region of lower concentration till equilibrium is restored. Both Eqs (10.4) and (10.5) represent Fick's law of diffusion, the former in terms of moles/s and the latter in terms of kg/s.

We can observe the similarity of three transport processes: momentum, energy and mass. Momentum transfer occurs due to velocity gradient (Newton's law of viscosity), energy transfer occurs due to temperature gradient (Fourier heat conduction equation) and mass transfer occurs due to concentration gradient (Fick's law of diffusion).

Momentum transfer in laminar flow (Newton's law) is

$$\tau = -\mu\frac{\mathrm{d}u}{\mathrm{d}y}$$

or,

$$\tau = -\nu\frac{\mathrm{d}(u\rho)}{\mathrm{d}y} \tag{10.6}$$

Heat conduction per unit area (Fourier's law) is

$$q = -k\frac{\mathrm{d}T}{\mathrm{d}y} = -\alpha\frac{\mathrm{d}}{\mathrm{d}y}(\rho c_p T) \tag{10.7}$$

Rate of mass flux (Fick's law) is

$$\frac{\mathbf{N_A}}{\mathbf{A}} = -\mathbf{D_v}\frac{\mathbf{dC_A}}{\mathbf{dy}} \tag{10.4}$$

In each of these equations, Flux $\infty$ Force i.e., Potential (or concentration) gradient, and the constant of proportionality is a property of the medium through which the flux occurs. Force is the cause and flux is the effect. These are linear laws. Here, $(u\rho)$ is the momentum concentration, $(\rho c_p T)$ is the thermal concentration, and $C_A$ is the mass concentration, $\nu$ is the momentum (kinematic) diffusivity, $\alpha$ is the thermal diffusivity $(k/\rho c)$, and $D$ is the mass diffusivity. The units of $\nu$, $\alpha$ and $D$ are each m$^2$/s. The ratios of any two of these represent the interaction of the two relevant fields. Thus,

$\dfrac{\nu}{\alpha}$ = Prandtl number (Pr), which relates the velocity field with the temperature field, and is the ratio of the transport properties $\nu$ and $\alpha$, which govern the transport of momentum and energy respectively.

$\dfrac{\nu}{D}$ = Schmidt number (Sc), which relates the velocity field with the concentration field, and is the ratio of the transport properties $\nu$ and $D$, which govern the transport of momentum and mass respectively.

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$\dfrac{\alpha}{D}$ = Lewis number (Le), which relates the temperature field with the concentration field and is the ratio of $\alpha$ and $D$, which govern the transport of energy and mass respectively.

A high value of Lewis number indicates $\alpha > D$ i.e., the rate at which energy propagates is faster than the rate of mass transfer. Similar are the cases of Prandtl number ($\nu/\alpha$) and Schmidt number ($\nu/D$).

## 10.2 EQUIMOLAR COUNTER DIFFUSION

In the gas phase, the concentrations are usually expressed in terms of partial pressures. If ideal gas law is assumed to hold good.

$$p_A = \frac{n_A \bar{R} T}{V} = C_A \bar{R} T$$

where $\bar{R}$ is the universal gas constant.

Fick's law could then be written as

$$\frac{N_A}{A} = -D\frac{dC_A}{dy} = -\frac{D}{\bar{R}T}\frac{dp_A}{dy}$$

$$\int_{y_1}^{y_2} \frac{N_A}{A}\,dy = -\frac{D}{\bar{R}T}\int_{p_{A1}}^{p_{A2}} d\,p_A$$

or,

$$\mathbf{\frac{N_A}{A} = -\frac{D}{\bar{R}T}\frac{p_{A_2} - p_{A_1}}{y_2 - y_1}} \tag{10.8}$$

where $p_{A_1}$ is the partial pressure of $A$ at $y_1$ and $p_{A_2}$ is the partial pressure of $A$ at $y_2$.

Equation (10.8) is valid for *equimolar counter diffusion* in which gases $A$ and $B$ diffuse simultaneously in opposite directions through each other. The rates of diffusion are equal but in opposite direction (Fig. 10.2) i.e., $N_A = -N_B$.

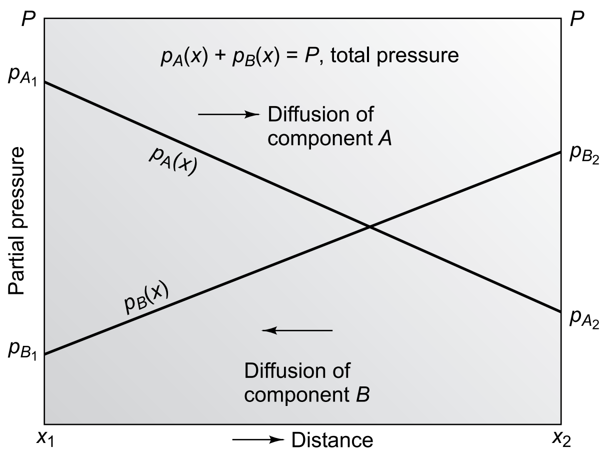


**Fig. 10.2** *Distribution of partial pressure $p_A$ and $p_B$ in equimolar counterdiffusion of a binary gas mixture*

$$P = p_{A_1} + p_{B_1} = p_{A_2} + p_{B_2}$$

$$\mathbf{\frac{dp_{A_1}}{dy} = -\frac{dp_{B_1}}{dy}} \tag{10.9}$$

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The situation has got no counterpart in heat transfer.

For equimolar counter diffusion the partial pressure gradients of the two diffusing species must be equal but of opposite sign.

## 10.3 MOLECULAR DIFFUSION THROUGH A STATIONARY GAS

Let us consider a gas $A$ diffusing through a stationary gas $B$ into a liquid–vapour interface where the gas $A$ is absorbed (Fig. 10.3). Since the gas $A$ is diffusing towards the interface, there must be a partial pressure gradient for $A$ in the direction of diffuson. The rate of mass transfer of $A$ is

$$\frac{\mathbf{N_A}}{\mathbf{A}} = -\frac{\mathbf{D}}{\mathbf{\bar{R}T}}\frac{\mathbf{dp_A}}{\mathbf{dy}}$$

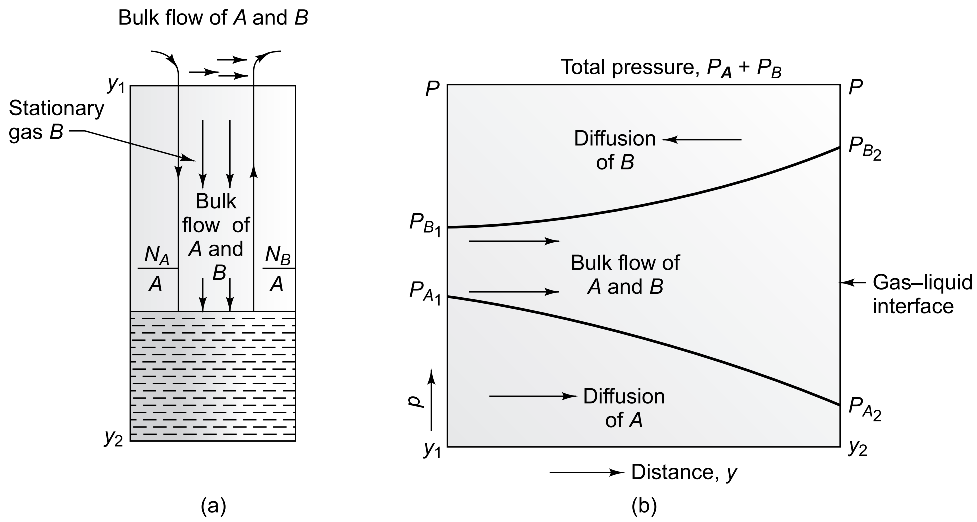


**Fig. 10.3** *Distribution of partial pressures $p_A$ and $p_B$ for unidirectional diffusion of gas A through gas B*

Now, the total pressure, $P = p_A + p_B$

$$\frac{dp_A}{dy} = -\frac{dp_B}{dy} \tag{10.10}$$

A gradient in $p_A$ will cause a gradient in $p_B$ in the opposite direction. This gradient will force diffusion of gas $B$ away from the interface.

$$\frac{N_A}{A} = -\frac{D}{\bar{R}T}\frac{dp_B}{dy} = \frac{D}{\bar{R}T}\frac{dp_A}{dy} \tag{10.11}$$

Since $B$ is not produced at the interface, even though it is diffusing away from the interface, some other mechanism must supply gas $B$ to maintain a constant concentration of gas $B$ at the interface. A bulk flow of gas towards the interface replenishes gas $B$ which is diffusing away. The bulk flow consists of a mixture of $A$ and $B$. The bulk flow of $B$ toward the interface must equal $-N_B/A$ to balance the diffusion of $B$ in the opposite direction. The presence of $A$ in the bulk flow will effectively enhance the rate of transfer of $A$ towards the interface. The bulk flow rate of $A$ ($N_A/A$) towards the interface equals

$$\frac{\text{Moles of } A \text{ in bulk flow}}{\text{Moles of } B \text{ in bulk flow}} \times \text{Bulk flow of } B$$

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or, $$\left(\frac{N_A}{A}\right)_{\text{bulk flow}} = \frac{p_A}{p_B}\left(-\frac{N_B}{A}\right) = \frac{p_A}{p - p_A}\left(-\frac{N_B}{A}\right) \tag{10.12}$$

The total bulk flow rate equals the sum of bulk flow rates of $A$ and $B$

$$= \left(\frac{N_A}{A}\right)_{\text{Bulk flow}} + \left(\frac{N_B}{A}\right)_{\text{Bulk flow}} = -\frac{N_B}{A}\frac{p_A}{p - p_A} - \frac{N_B}{A}$$

$$= -\frac{N_B}{A}\frac{p_A}{p - p_A} \tag{10.13}$$

Total flux of $A$ toward the interface

$$\frac{N_{A_t}}{A} = \text{Flux due to diffusion} + \text{Flux due to bulk flow}$$

$$= -\frac{D}{\bar{R}T}\frac{dp_A}{dy} + \left(-\frac{N_B}{A}\right)\frac{p_A}{P - p_A}$$

$$= -\frac{D}{\bar{R}T}\frac{dp_A}{dy} - \frac{p_A}{P - p_A}\frac{D}{\bar{R}T}\frac{dp_A}{dy}$$

$$= -\frac{D}{\bar{R}T}\frac{p_A}{P - p_A}\frac{dp_A}{dy} \tag{10.14}$$

or $$\int_{y_1}^{y_2}\frac{N_{A_t}}{A}dy = -\frac{D}{\bar{R}T}P\int_{p_{A_1}}^{p_{A_2}}\frac{dp_A}{P - p_A}$$

$$\frac{N_{A_t}}{A} = \frac{DP}{\bar{R}T}\frac{1}{y_2 - y_1}\ln\frac{P - p_{A_2}}{P - p_{A_1}}$$

or, $$\frac{N_{A_t}}{A} = \frac{DP}{\bar{R}T}\frac{1}{y_2 - y_1}\ln\frac{p_{B_2}}{p_{B_1}} \tag{10.15}$$

If we define log-mean partial pressure of $B$ as

$$p_{B_m} = \frac{p_{B_2} - p_{B_1}}{\ln(p_{B_2}/p_{B_1})} \tag{10.16}$$

Eq. (10.15) can be written as

$$\frac{N_{A_t}}{A} = \frac{DP}{\bar{R}T}\frac{1}{y_2 - y_1}\frac{p_{B_2} - p_{B_1}}{p_{B_m}}$$

Again, $p_{B_2} - p_{B_1} = p_{A_1} - p_{A_2}$

$$\mathbf{\frac{N_{A_t}}{A} = -\frac{D}{\bar{R}T}\frac{P}{p_{B_m}}\frac{p_{A_2} - p_{A_1}}{y_2 - y_1}} \tag{10.17}$$

If we compare the above equation with Eq. (10.8)

$$\frac{N_A}{A} = -\frac{DP}{\bar{R}T}\frac{p_{A_2} - p_{A_1}}{y_2 - y_1}$$

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we observe that the factor $p/p_{B_m}$ is introduced when diffusion through a stationary gas is considered. "Stationary" character of $B$ does not imply that $B$ is not moving, but refers to the *net* behaviour of $B$. Since $B$ is supplied by bulk flow at the rate it diffuses away, there is no net movement of $B$. The partial pressure gradients for diffusion are not linear through a stationary gas, contrary to the linear gradients of equimolar counter diffusion. Eq. (10.17) may be used to determine $D$ experimentally.

## 10.4 DIFFUSIVITY FOR GASES AND VAPOURS

For kinetic theory of gases it can be shown that the mass diffusivity $D$ for gases and vapours is proportional to $T^{3/2}/P$. Gilliland [1] proposed a semiempirical equation for the diffusion coefficient in gases.

$$\mathbf{D} = 435.7 \frac{\mathbf{T}^{3/2}}{\mathbf{p}(\mathbf{V}_\mathbf{A}^{1/3} + \mathbf{V}_\mathbf{B}^{1/3})^2} \left( \frac{1}{\mathbf{M}_\mathbf{A}} + \frac{1}{\mathbf{M}_\mathbf{B}} \right)^{1/2} \tag{10.18}$$

where $D$ is in cm$^2$/s, $T$ is in $K$, $p$ is the total system pressure in pascals and $V_A$ and $V_B$ are the molecular volumes of constituents $A$ and $B$ as calculated from the atomic volumes in Table 10.1; $M_A$ and $M_B$ are the molecular weights of constituents $A$ and $B$. An abbreviated table of diffusion coefficients is given in Table 10.2.

**Table 10.1** *Atomic volume*

| | |
|---|---|
| Air | 29.9 |
| Bromine | 27.0 |
| Carbon | 14.8 |
| Carbon dioxide | 34.0 |
| Chlorine | |
| Terminal as in R—Cl | 21.6 |
| Medial as in R—CHCl—R | 24.6 |
| Fluorine | 8.7 |
| Hydrogen, molecule ($H_2$) | 14.3 |
| In compounds | 3.7 |
| Iodine | 37.0 |
| Nitrogen, molecule ($N_2$) | 15.6 |
| In primary amines | 10.5 |
| In secondary amines | 1.20 |
| Oxygen, molecule ($O_2$) | 7.4 |
| Coupled to two other elements | |
| In aldehydes and ketones | 7.4 |
| In methyl esters | 9.1 |
| In ethyl esters | 9.9 |
| In higher esters and ethers | 11.0 |
| In acids | 12.0 |
| In union with S.P.N. | 8.3 |
| Phosphorus | 27.0 |
| Sulphur | 25.6 |
| Water | 18.8 |

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**Table 10.2** *Diffusion coefficients of gases and vapours in air at 25°C and 1 atm*

| *Substance* | *D* ($cm^2/s$) | $Sc = \frac{\nu}{D}$ |
|---|---|---|
| Ammonia | 0.28 | 0.78 |
| Carbon dioxide | 0.164 | 0.94 |
| Hydrogen | 0.410 | 0.22 |
| Oxygen | 0.206 | 0.75 |
| Water | 0.256 | 0.60 |
| Ethyl ether | 0.093 | 1.66 |
| Methanol | 0.159 | 0.97 |
| Ethyl alcohol | 0.119 | 1.30 |
| Formic Acid | 0.159 | 0.97 |
| Acetic acid | 0.133 | 1.16 |
| Aniline | 0.073 | 2.14 |
| Benzene | 0.088 | 1.76 |
| Toluene | 0.084 | 1.84 |
| Ethyl benzene | 0.077 | 2.01 |
| Propyl benzene | 0.059 | 2.62 |

## 10.5 CONCENTRATION BOUNDARY LAYER AND MASS TRANSFER COEFFICIENT

Just as the calculation of momentum and heat transfer requires the knowledge of velocity and temperature profiles within the boundary layer, the calculation of mass transfer requires that concentration profile within the boundary layer should be known.

Let us consider the flow of a fluid mixture on a surface (Fig. 10.4). Let the free stream velocity and concentration be $u_\infty$ and $C_{A_\infty}$. Let the plate surface be maintained at a concentration $C_{A_w} > C_{A_\infty}$. Then, species $A$ diffuses from the surface into the fluid. A *concentration boundary layer* develops (Fig. 10.4), thickness of which can be defined in the same way as that of the hydrodynamic or thermal boundary layer. The distance $\delta_c$ to which the boundary layer extends may be defined as the thickness at which the concentration is equal to 99% of the free-stream concentration, i.e.,

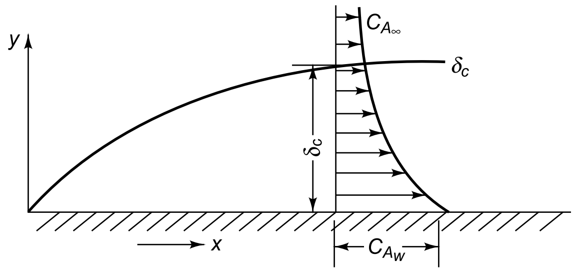


**Fig. 10.4** *Concentration boundary layer*

$$C_{A_w} - C_{A_\delta} = 0.99\,(C_{A_w} - C_{A_\infty}) \tag{10.19}$$

By analogy with heat transfer, a *mass transfer coefficient* $h_m$ may be defined by expressing mass flux as follows

$$\left(\frac{N_A}{A}\right)_{y=0} = -D\left(\frac{\partial c_A}{\partial y}\right)_{y=0}$$

$$= h_m\,(C_{A_w} - C_{A_\infty}) \tag{10.20}$$

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Accordingly,

$$h_m = \frac{\left(\frac{N_A}{A}\right)_{y=0}}{C_{A_\omega} - C_{A_\infty}}$$

$$= \frac{-D\left(\frac{\partial C_A}{\partial y}\right)_{y=0}}{C_{A_\omega} - C_{A_\infty}} \tag{10.21}$$

Thus, the unit of $h_m$ is given by $\frac{\text{m}^2}{\text{m·s}}$, or, m/s i.e., that of velocity.

The molecular diffusion equation is

$$\frac{N_A}{A} = -D\frac{\Delta C_A}{\Delta y} = h_m \Delta C_A$$

For a gas,

$$\mathbf{\frac{N_A}{A} = \frac{-D}{RT}\frac{dp_A}{dy} = \frac{h_m}{\bar{R}T}\Delta p_A} \tag{10.22}$$

## 10.6 ANALOGY BETWEEN MOMENTUM, HEAT AND MASS TRANSFER

The momentum and energy equations of a laminar boundary layer are

*Momentum:* $$u\frac{\partial u}{\partial x} + \bar{v}\frac{\partial u}{\partial y} = v\frac{\partial^2 u}{\partial y^2}$$

*Energy:* $$u\frac{\partial T}{\partial x} + \bar{v}\frac{\partial T}{\partial y} = v\frac{\partial^2 T}{\partial y^2}$$

The concentration boundary layer can similarly be simplified to

*Mass:* $$u\frac{\partial C_A}{\partial x} + \bar{v}\frac{\partial C_A}{\partial y} = D\frac{\partial^2 C_A}{\partial y^2}$$

The form of the three equations is exactly similar. To illustrate the analogy further, the following equations, as mentioned earlier in Eqs 10.6, 10.7 and 10.4, confirm similarity of boundary conditions which help in the solution of mass transfer problems.

Shear stress, $$\tau = -\nu\frac{\partial(\rho u)}{\partial y}$$

Heat flux, $$Q/A = -\alpha\frac{\partial(\rho c_p T)}{\partial y}$$

Mass flux, $$N_A/A = -D\frac{\partial C_A}{\partial y}$$

We know that the velocity and temperature profiles have the same shape if

$$\nu = \alpha$$

or, $$\Pr = \frac{\nu}{\alpha} = 1$$

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Similarly, the momentum and concentration profiles will have the same shape if

$$\nu = D$$

or,
$$\text{Sc} = \frac{\nu}{D} = 1$$

Thus, the Schmidt number plays the same role in mass transfer as does the Prandtl number in heat transfer.

It is also seen that the temperature and concentration profiles will be similar if

$$\alpha = D$$

or,
$$\text{Le} = \frac{\alpha}{D} = 1$$

Thus, Lewis number is important in the solution of simultaneous heat and mass transfer problems. When, Sc = Pr = Le = 1, all the three boundary layers coincide.

The dimensionless mass transfer number corresponding to the *Nusselt number* is the *Sherwood number* defined as

$$\mathbf{Sh} = \frac{\mathbf{h_m x}}{\mathbf{D}} \tag{10.23}$$

where $x$ is the characteristic length.

Similarly, corresponding to Stanton number, which is expressed as

$$\text{St} = \frac{\text{Nu}}{\text{Re.Pr}} = \frac{h}{\rho c_p u}$$

We have a dimensionless quantity

$$\frac{\text{Sh}}{\text{Re.Sc}} = \frac{h_m}{u} \tag{10.23a}$$

where $u$ is any characteristic velocity of the system.

We know that forced convection heat transfer correlations are of the form

$$\text{Nu} = f(\text{Re, Pr})$$

Likewise, in forced convection mass transfer problems we write the functional relation

$$\text{Sh} = f(\text{Re, Sc}).$$

As for free convection, a new Grashof number needs to be defined as it is a case of free convection due to density difference resulting from concentration difference, and not temperature difference. First, we define here a quantity $\beta_m$ analogous to $\beta$ which indicates the variation of density with composition, by

$$\beta_m = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial C_A^*}\right) \tag{10.24}$$

where $\rho$ is the local mixture density and $C_A^* = C_A/\rho\left(\frac{\text{kg mol}}{\text{m}^3} \times \frac{\text{m}^3}{\text{kg}} \text{ or M, M being the molecular weight}\right)$ and the unit of $\beta_m$ is $\text{M}^{-1}$.

The buoyancy force is then given by

$$\mathbf{F} = \frac{\mathbf{g(\rho - \rho_\infty)}}{\mathbf{\rho_\infty}} = \mathbf{g\beta_m (C_A^* - C_{A\infty}^*)} \tag{10.25}$$

where $\rho_\infty$ is the mixture density far away from the section. The mass Grashof number $\text{Gr}_m$ is then defined as

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$$\text{Gr}_m = \frac{F x^3}{\nu^2} = \frac{g\,\beta_m (C^*_{A_\infty} - C^*_{A0})\,x^3}{\nu^2} = \frac{g(\rho_\infty - \rho_0)}{\nu^2 \rho_\infty} \tag{10.26}$$

where subscript zero refers to the wall.

Accordingly, for natural convection mass transfer, the following correlation holds

$$\text{Sh} = f(\text{Gr}_m, \text{Sc}).$$

## 10.7 FORCED CONVECTION MASS TRANSFER IN LAMINAR FLOW IN A TUBE

In practice, such a case may arise by the evaporation of liquid from the wetted wall of a tube into gas flowing in laminar flow or by the diffusion of material from the tube surface.

The concentration boundary layer will develop as shown in Fig. 10.5. Let the concentration of the fluid at inlet be $C_{A_1}$ and that at the tube wall $C_{Aw}$. It is assumed that the concentration of $A$ is quite low so that the mixture density $\rho$ can be assumed constant, and there is no natural convection. It is intended to find the concentration profile at a distance $x$ from the inlet at any section 2, where the velocity and concentration profiles are fully developed.

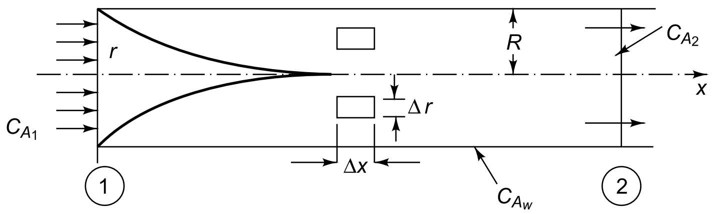


**Fig. 10.5** *Mass transfer in a tube*

Let us consider an annular control volume formed by $\Delta r$ and $\Delta x$ in the boundary layer. Under steady state, for constant $\rho$ and $D$, mass balance of component $A$ yields

$$u\frac{\partial C_A}{\partial x} + \bar{v}\frac{\partial C_A}{\partial r} = D\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C_A}{\partial r}\right) + \frac{\partial^2 C_A}{\partial x^2}\right] \tag{10.27}$$

where $u$ and $\bar{v}$ are the bulk velocities in axial and radial directions respectively. The left hand side represents convective flux, whereas the right hand side represents diffusion flux.

In the region of fully developed flow, the radial velocity $u$ is zero. Also, diffusion in the axial direction can be neglected so that

$$D\frac{\partial^2 C_A}{\partial x^2} = 0$$

Equation (10.27) then simplifies to

$$u\frac{\partial C_A}{\partial x} = D\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C_\text{A}}{\partial r}\right)\right] \tag{10.28}$$

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Equation (10.28) is of the same form as Eq. (4.176) for heat transfer in laminar tube flow. The solution is also valid for boundary condition

*Case I:* $C_A = C_{A_w}$ at $r = R$

*Case II:* $N_A/A$ = const. at $r = R$

We know that the velocity distribution for laminar flow in a tube is given by

$$\mathbf{u = 2u_m\left[1 - \left(\frac{r}{R}\right)^2\right]}$$

where $u_m$ is the average velocity given by Eq. (4.169).

$$\mathbf{u_m = \frac{R^2}{8\mu}\frac{dp}{dx}}$$

*Case I: Uniform mass flux*

Here, $\dfrac{\partial C_A}{\partial x}$ = constant

Also, $\dfrac{\partial C_A}{\partial r} = 0$ at $r = 0$

$C_A = C_{A_w}$ at $r = R$

Substituting for $u$ and $u_m$ into Eq. (10.28) and simplifying leads to the asymptotic value of Sherwood number for uniform heat flux

$$\mathbf{Sh = \frac{h_m\,(2R)}{D} = 4.364}$$

*Case II: Uniform wall concentration*

Here, $\dfrac{\partial C_A}{\partial x}$ is not constant.

Similarly for laminar flow constant tube wall temperature, for constant wall concentration

$$\mathbf{Sh = \frac{h_m\,(2R)}{D} = 3.66}$$

## 10.8 MASS TRANSFER BY CONVECTION IN TURBULENT FLOW

The mechanism of mass transfer in turbulent flow is similar to that of heat transfer in turbulent flow. Let us consider an air stream flowing over the surface of a pool of water (Fig. 10.6). The velocity distribution of air is similar to that over a flat plate. Near the surface there is a laminar sub-layer, followed by a buffer layer, and a turbulent stream. Water vapour is the diffusing component $A$. Water vapour moles are transported from the interface first by molecular diffusion through the laminar convection film and then get transported in the air by eddies. More the mixing motion, more the rate of transfer.

Rate of mass transfer of water vapour to air by molecular diffusion through the stagnant film is

$$\frac{N_A}{A} = -\frac{D}{\bar{R}T}\frac{P}{p_{B_m}}\frac{p_{A_G} - p_{A_i}}{y_{fg} - y_i} \tag{10.29}$$

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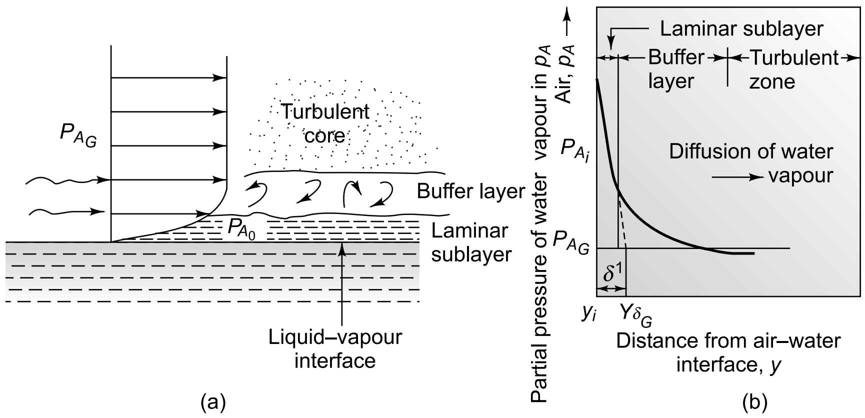


**Fig. 10.6** *Steady-state concentration gradient of water vapour in air flowing over a horizontal water surface*

where $p_{A_G}$ is the partial pressure of water vapour in the bulk air phase and $p_{A_i}$ that at the interface, and $y_{fg} - y_i$ is the effective boundary layer thickness for mass transfer $\delta_m$.

The effective film thickness for mass and heat transfer are approximately the same. Eq. (10.29) can be written in the following form:

$$\frac{N_A}{A} = k_G(p_{A_i} - p_{A_g}) \tag{10.30}$$

where $k_G$ is the gas-phase mass transfer coefficient (analogous to $h$) defined by

$$k_G = \frac{DP}{\bar{R}T\, p_{B_m}\,(y_{fg} - y_i)}\left(\frac{\text{kg moles}}{\text{m}^2\text{s atm}}\right) \tag{10.31}$$

For mass transfer in the liquid phase

$$\frac{N_A}{A} = \frac{-DC_t(C_{A_L} - C_{A_i})}{C_{B_m}(y_i - y_{fL})} \tag{10.32}$$

where, $C_{A_L}$ = concentration of the diffusing components in the bulk liquid phase,

$C_{A_i}$ = concentration of diffusing component at interface

$C_t = C_A + C_B$, total concentration

$y_{fL} - y_i$ = thickness of the effective liquid film

$C_{B_m}$ = log-mean concentration of component $B$

Equation (10.32) can be written as

$$\frac{N_A}{A} = k_L(C_{A_L} - C_{A_L}) \tag{10.33}$$

where $k_L$ is the liquid-phase mass transfer coefficient defined by

$$k_L = \frac{DC_t}{C_{B_m}(y_{fL} - y_i)} \tag{10.34}$$

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The mass transfer coefficients defined by Eqs (10.31) and (10.34) apply to diffusion of one component through a second stationary component. Coefficients for equimolar counter diffusion can be obtained in a similar way. In the humidification of air their is no resistance to diffusion of water in the liquid phase, since only water is present. Therefore, $k_L$ is infinite, and only gas-phase resistance need be considered.

## 10.9 EVALUATION OF MASS TRANSFER COEFFICIENTS BY DIMENSIONAL ANALYSIS

Mass transfer coefficients ($k_G$ and $k_L$) are evaluated experimentally. Where direct experimental data are not available, empirical equations are often used for predicting the coefficients. These equations are similar to the equations derived for predicting heat transfer coefficients.

The liquid-phase mass transfer coefficient can be assumed to be a function of velocity, density, viscosity and mass diffusivity of the fluid, and some characteristic dimension $L$, or

$$k_L = f(V, \rho, \mu, D, L) \tag{10.35}$$

Dimensional analysis yields the dimensionless equation

$$\frac{k_L L}{D} = \phi\left(\frac{LV\rho}{\mu}\right)\psi\left(\frac{\mu}{\rho D}\right) \tag{10.36}$$

or,

$$\mathbf{Sh} = \boldsymbol{\phi}(\mathbf{Re})^{\mathbf{a}}\,(\mathbf{Sc})^{\mathbf{b}} \tag{10.37}$$

where Sh = Sherwood number = $\dfrac{k_L L}{D}$,

Re = Reynolds number = $\dfrac{LV\rho}{\mu}$ and

Sc = Schmidt number = $\dfrac{\mu}{\rho D}$

Similarly, for liquid phase diffusion through a stationary fluid,

$$\frac{k_L L C_{B_m}}{D C_t} = \phi\left(\frac{LV\rho}{\mu}\right)^a\left(\frac{\mu}{\rho D}\right)^b \tag{10.38}$$

or,

$$\mathrm{Sh} = \phi(\mathrm{Re})^a\,(\mathrm{Sc})^b \tag{10.39}$$

where $\mathbf{Sh} = \dfrac{\mathbf{k_L L C_{B_m}}}{\mathbf{D C_t}}$

For gas-phase mass transfer (equimolar counter diffusion)

$$k_G = f(V, \rho, \mu, D, L, \bar{R}T) \tag{10.40}$$

By dimensional analysis,

$$\mathrm{Sh} = \frac{k_G \bar{R} T L}{D} = \phi(\mathrm{Re})^a\,(\mathrm{Sc})^b \tag{10.41}$$

For gas phase diffusion through a stagnant fluid

$$\mathbf{Sh} = \frac{\mathbf{k_G \bar{R} T L}}{\mathbf{D}}\,\frac{\mathbf{p_{B_m}}}{\mathbf{p}} = \boldsymbol{\phi}(\mathbf{Re})^{\mathbf{m}}\,(\mathbf{Sc})^{\mathbf{n}} = \mathbf{Sh} \tag{10.42}$$

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Sherwood number, as defined by the group, on the left side of Eqs (10.37), (10.39), (10.41) and (10.42), is analogous to Nusselt number in heat transfer. The mass transfer coefficient depends on the Reynolds number and the ratio of momentum diffusivity ($\nu$) and mass diffusivity ($D$) i.e., the Schmidt number. The Schmidt number characterises mass transfer in the same manner as the Prandtl number characterises heat transfer. The constants are evaluated experimentally.

## 10.10 ANALOGY OF HEAT AND MASS TRANSFER

As discussed earlier, the mechanism of mass, heat and momentum transfers are closely related. Therefore, one might expect data taken for one transfer operation to be useful in predicting the rate of transfer in other operations.

The heat and momentum transfers are interrelated by Reynolds analogy for fluids having unity Prandtl number (Chapter 4).

$$\frac{\mathbf{Nu}}{\mathbf{Re\,Pr}} = \mathbf{St} = \frac{\mathbf{C_{fx}}}{\mathbf{2}} \text{ or } \frac{\mathbf{f}}{\mathbf{2}} = \frac{\mathbf{h}}{\boldsymbol{\rho}\,\mathbf{c_p V}} \tag{10.43}$$

A similar analysis for mass and momentum transfer yields (for Sc = 1),

$$\frac{\text{Sh}}{\text{Re Sc}} = \frac{f}{2} \text{ or } \frac{C_{fx}}{2}$$

or,
$$= \frac{k_G \bar{R}\,TL}{D}\frac{p_{B_m}}{p}\frac{\mu}{\rho VL}\frac{\rho D}{\mu}$$

$$= \frac{\mathbf{k_G \bar{R} T\, p_{B_m}}}{\mathbf{VP}} \tag{10.44}$$

Similarly, for liquid phase

$$\frac{\text{Sh}}{\text{Re Sc}} = \frac{k_L C_{B_m}}{VC_t} \tag{10.45}$$

The interrelation of heat and mass transfer can similarly be developed. For Pr = 1 and Sc = 1 fluids,

$$\frac{k_G \bar{R}T\, p_{B_m}}{VP} = \text{St} = \frac{h}{\rho c_p V}$$

$$\mathbf{k_G} = \frac{\mathbf{hp}}{\mathbf{c_p \bar{R} T\, p_{B_m}}} \tag{10.46}$$

When Prandtl number and Schmidt number are not unity, Colburn's equation of fluids flowing turbulently inside tubes gives

$$\text{Nu} = 0.023\ \text{Re}^{0.8}\ \text{Pr}^{0.33} \tag{10.47}$$

The analogous relation for mass transfer in a wetted wall column is given by

$$\mathbf{Sh = 0.023\ Re^{0.8}\ Sc^{0.33}} \tag{10.48}$$

A wetted wall column consists of a vertical tube with liquid flowing in a thin film down the inside wall of the tube and a gas flowing upward in the tube. Mass transfer takes place from the liquid film to the gas, or vice versa. Equation (10.48) may be used to predict coefficients in wetted wall columns.

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In terms of Colburn's *j*-factors,

$$j_H = \frac{\text{Nu}}{\text{Re Pr}^{1/3}} = \text{St Pr}^{2/3} = \left(\frac{\bar{h}}{\rho c_p V}\right)\left(\frac{c_p \mu}{k}\right)^{0.67}$$

$$= \frac{f}{2} \tag{10.49}$$

and

$$\mathbf{j_M} = \frac{\mathbf{Sh}}{\mathbf{Re\ Sc^{1/3}}} = \left(\frac{\mathbf{k_G \bar{R} T p_{B_m}}}{\mathbf{VP}}\right)\left(\frac{\mu}{\rho \mathbf{D}}\right) = \frac{\mathbf{f}}{\mathbf{2}} \tag{10.50}$$

Combining Eq. (10.47) with Eq. (10.49),

$$j_H = 0.023\ \text{Re}^{-0.2} \tag{10.51}$$

Similarly, combining Eq. (10.48) with Eq. (10.50),

$$j_M = 0.023\ \text{Re}^{-0.2} \tag{10.52}$$

Experimental data for flow in tubes also confirm that

$$j_H = j_M = \frac{f}{2} \tag{10.53}$$

$$\left(\frac{h}{\rho c_p V}\right)\left(\frac{c_p \mu}{k}\right)^{2/3} = \left(\frac{k_G \bar{R} T}{V}\right)\left(\frac{p_{B_m}}{p}\right)\left(\frac{\mu}{\rho D}\right)^{2/3}$$

or,

$$\mathbf{k_G} = \left(\frac{\mathbf{h}}{\rho \mathbf{c_p}}\right)\left(\frac{\mathbf{p}}{\mathbf{\bar{R} T p_{B_m}}}\right)\left(\frac{\mathbf{c_p}\mu}{\mathbf{k}}\ \frac{\rho \mathbf{D}}{\mu}\right)^{2/3} \tag{10.54}$$

When $\text{Pr} = \text{Sc}$,

$$k_G = \frac{hp}{c_p \bar{R} T\, p_{B_m}}$$

which is the same as Eq. (10.46).

When direct mass transfer data for a system are not available, Eq. (10.54) may be used to predict mass transfer coefficients from heat transfer data taken in a system of identical geometry and flow characteristics.

## 10.11 MASS TRANSFER IN BOUNDARY LAYER FLOW OVER A FLAT PLATE

The heat transfer correlations for laminar and turbulent boundary-layer flows over a flat plate presented in Chapter 4 can be recast as mass transfer operations by changing the Nusselt number to the Sherwood number and the Prandtl number to the Schmidt number.

For example, for mass transfer from laminar boundary-layer flow over a flat plate,

$$\text{Sh}_x = 0.332\ \text{Re}_x^{1/2}\ \text{Sc}^{1/3} \text{ for } \text{Re}_x < 5 \times 10^5 \tag{10.55}$$

which can be rearranged as

$$j_M = \frac{k_m}{V}\ \text{Sc}^{2/3} = 0.332\ \text{Re}_x^{-1/2}$$

for

$$\text{Re}_x < 5 \times 10^5 \tag{10.56}$$

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where $k_m$ is the local mass transfer coefficient at $x$,

$$\mathrm{Sh}_x = \frac{k_x x}{D} = \text{local Sherwood number}$$

$$\mathrm{Sc} = \frac{\nu}{D} = \text{Schmidt number}$$

The average value of the mass transfer coefficient over the distance $0 \le x \le L$ is given by

$$k_m = 2(k_m)_{x=L} \tag{10.57}$$

In mass transfer from turbulent boundary layer flow over a flat plate,

$$j_M = \frac{k_x}{V}\mathrm{Sc}^{2/3} = 0.0296\,\mathrm{Re}_x^{0.2} \quad \text{for } 5 \times 10^5 < \mathrm{Re}_x < 10^7 \tag{10.58}$$

Since

$$\mathbf{St_x} = \frac{\mathbf{Nu_x}}{\mathbf{Re_x\ Pr}} \rightarrow \frac{\mathbf{Sh_x}}{\mathbf{Re_x\ Sc}} = \frac{\mathbf{k_x x}}{\mathbf{D}}\frac{\nu}{\mathbf{V_x}}\frac{\mathbf{D}}{\nu} = \frac{\mathbf{k_x}}{\mathbf{V}} \tag{10.59}$$

where $k_x$ is the local mass transfer coefficient.

For average values in the case of a flat plate, the $j$-factors are analogous to the skin friction coefficient as given below:

**Laminar:** $$\mathbf{j_H = j_M = \frac{C_f}{2} = \frac{0.664}{(Re_L)^{1/2}}}$$

**Turbulent:** $$\mathbf{j_H = J_M = \frac{C_f}{2} = \frac{0.037}{(Re_L)^{0.2}}} \tag{10.60}$$

When heat transfer and mass transfer take place simultaneously, the ratio of $h_x$ to $k_x$ can be of interest. Both for laminar and turbulent flows,

$$\mathbf{\frac{h_x}{k_x} = \rho c_p \left(\frac{Sc}{Pr}\right)^{2/3} = \rho c_p Le^{2/3}} \tag{10.61}$$

where the Lewis number is defined as

$$\mathbf{Le = \frac{Sc}{Pr} = \frac{\alpha}{D}} \tag{10.62}$$

The same relation is also applicable for the average values of the heat and mass transfer coefficients, namely,

$$\mathbf{\frac{h_m}{k_m} = \rho c_p Le^{2/3} = \rho c_p \left(\frac{\alpha}{D}\right)^{2/3}} \tag{10.63}$$

## SOLVED EXAMPLES

**Example 10.1** Gaseous hydrogen is stored at elevated pressure in a rectangular steel container of 10 mm wall thickness. The molar concentration of hydrogen in steel at the outer surface is 2 kg mol/m$^3$, while the concentration of hydrogen in steel at the outer surface is 0.5 kg mol/m$^3$. The binary diffusion coefficient for hydrogen in steel is $0.26 \times 10^{-12}$ m$^2$/s. What is the mass flux of hydrogen through the steel?

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**Solution** Assuming (a) steady-state one-dimensional diffusion of hydrogen, (b) the molar concentration of hydrogen much less than that of steel $C_A << C_B$, so that $C = C_A + C_B$ is uniform and (c) no chemical reaction (Fig. Ex. 10.1),

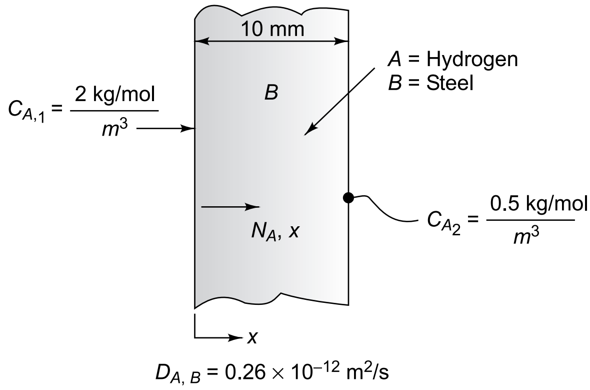


**Fig.** *Ex. 10.1*

$$\frac{N_A}{A} = -D_{AB}\frac{C_{A_2} - C_{A_1}}{L}$$

$$= 0.26 \times 10^{-12}\ \frac{\text{m}^2}{\text{s}}\ \frac{(2 - 0.5)\ (\text{kg mol})/\text{m}^3}{0.01\ \text{m}}$$

$$= 0.39 \times 10^{-10}\ \text{kg mol/m}^2\ \text{s}$$

Mass flux of hydrogen is $2 \times 0.39 \times 10^{-10}$

or $7.8 \times 10^{-11}$ kg/m$^2$ s *Ans.*

**Example 10.2** Hydrogen gas is maintained at 5 bar and 1 bar on opposite sides of a plastic membrane, which is 0.3 mm thick. The temperature is 25°C, and the binary diffusion coefficient of hydrogen in the plastic is $8.7 \times 10^{-8}$ m$^2$/s. The solubility of hydrogen in the membrane is $1.5 \times 10^{-3}$ kg mol/m$^3$ bar. What is the mass flux of hydrogen by diffusion through the membrane?

**Solution** The surface molar concentrations of hydrogen (Fig. Ex. 10.2) are

$$C_{A,1} = 1.5 \times 10^{-3} \times 5 = 7.5 \times 10^{-3}\ \text{kg mol/m}^3$$

$$C_{A,2} = 1.5 \times 10^{-3} \times 1 = 1.5 \times 10^{-3}\ \text{kg mol/m}^3$$

$$\frac{N_A}{A} = \frac{D_{AB}}{L}(C_{A,1} - C_{A,2})$$

$$= \frac{8.7 \times 10^{-8}\ \text{m}^2/\text{s}}{0.3 \times 10^{-3}\ \text{m}}\ (7.5 \times 10^{-3} - 1.5 \times 10^{-3})$$

$$= 17.4 \times 10^{-7}\ \text{kg mol/m}^2\ \text{s}$$

$$= 34.8 \times 10^{-7}\ \text{kg/m}^2\ \text{s} \quad \textit{Ans.}$$

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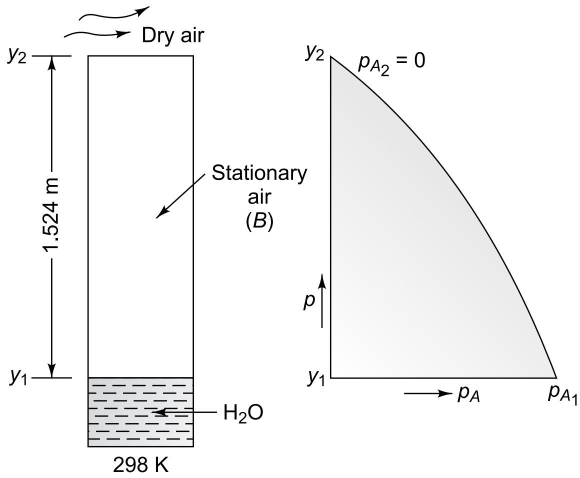


**Fig.** *Ex. 10.2*

**Example 10.3** Two large vessels contain uniform mixture of air (component *A*) and sulphur dioxide (component *B*) at 1 atm and 273 K, but at different concentrations. Vessel 1 contains 80% air and 20% $SO_2$ by volume or mole percentage whereas vessel 2 contains 30% air and 70% $SO_2$ by mole percentage. The vessels are connected by a 10 cm inner diameter 1.8 m long pipe. Determine the rate of transfer of air between these two vessels by assuming that a steady-state transfer takes place. The mass diffusivity for the air – $SO_2$ mixture at 1 atm and 273 K is $0.122 \times 10^{-4}$ m$^2$/s.

**Solution** Air diffuses from vessel 1 to 2 and $SO_2$ from vessel 2 to 1 due to concentration gradients (Fig. Ex. 10.3). Since the vessels are large, the partial pressure of air (*A*) in both vessels may be considered to remain constant, so that steady-state mass transfer takes place. The mass transfer process can be characterised as an *equimolar counter diffusion* and the mass flux of air through the connecting duct can be determined by Eq. (10.8),

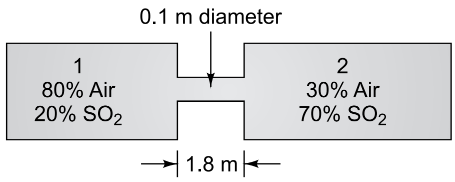


**Fig.** *Ex. 10.3*

$$\frac{N_A}{A} = -\frac{D}{\bar{R}T}\,\frac{p_{A_2} - p_{A_1}}{y_2 - y_1} \text{ kg mol/m}^2\text{ s}$$

The total mass transfer rate of air is given by

$$N_A = \frac{\pi}{4} d^2 \frac{D}{\bar{R}T}\,\frac{p_{A_2} - p_{A_1}}{y_2 - y_1}$$

where $d = 0.1$ m, $D = 0.122 \times 10^{-4}$ m$^2$/s, $T = 273$ K, $p_{A_1} = 0.8 \times 1$ atm $= 0.8$ atm, $p_{A_2} = 0.3 \times 1$ atm $= 0.3$ atm, $\bar{R} = 0.082$ m$^3$ atm/kg mol-K and $y_2 - y_1 = L = 1.8$ m.

$$N_A = \frac{\pi}{4} \times (0.1)^2 \,\frac{0.122 \times 10^{-4}}{0.082 \times 273} \times \frac{0.8 - 0.3}{1.8}$$

$$= 0.119 \times 10^{-6} \text{ kg mol/s} \quad \textit{Ans.}$$

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**Example 10.4** A deep narrow cylindrical vessel which is open at the top contains some water at the bottom. The air within the vessel is considered motionless, but there is sufficient air current at the top surface of the vessel so that any water vapour arriving at the top surface is immediately removed to ensure zero water vapour concentration. The entire system is at 1 atm, 298 K. The diffusivity of air-water vapour is $D = 0.26 \times 10^{-4}$ m$^2$/s and the saturated vapour pressure of water at the surface is 0.032 atm. Determine the rate of vaporation of water into the air per unit area if the distance between the water surface and the top of the vessel is 1.524 m.

**Solution** For steady-state unidirectional diffusion of water vapour (component $A$) through the stationary air (component $B$) of thickness 1.524 m from Eq. (10.17),

$$\frac{N_A}{A} = -\frac{D}{\bar{R}T}\frac{P}{p_{B_m}}\frac{p_{A_2} - p_{A_1}}{y_2 - y_1}$$

Here

$$p_{A_1} = (p_{\text{sat}})\ 25°\text{C} = 0.032 \text{ atm}$$

$$p_{A_2} = 0$$

$$p_{B_1} = p - p_{A_1} = 1 - 0.032 \text{ atm} = 0.968 \text{ atm}$$

$$p_{B_2} = 1 \text{ atm.}$$

$$p_{B_m} = \frac{p_{B_2} - p_{B_1}}{\ln \dfrac{p_{B_2}}{p_{B_1}}} = \frac{1 - 0.968}{\ln \dfrac{1}{0.968}} = 0.984 \text{ atm.}$$

$$\frac{N_A}{A} = -\frac{0.26 \times 10^{-4}\ \text{m}^2/\text{s}}{0.082\ \text{m}^2\ \text{atm/(kg mol-K)} \times 298\ \text{K}}\ \frac{1\ \text{atm}}{0.984\ \text{atm}}\ \frac{(0 - 0.032)\ \text{atm}}{1.524\ \text{m}}$$

$$= 6.868 \times 10^{-7} \text{ kg mol/m}^2 \text{ s,}$$

$$= 0.1236 \times 10^{-4} \text{ kg/m}^2 \text{ s} \quad \textit{Ans.}$$

**Example 10.5** In a Stefan tube experiment with carbon tetrachloride and oxygen, the following data are noted:

| | |
|---|---|
| Diameter of the tube | = 10 mm |
| Length of tube above liquid surface | = 150 mm |
| Temperature maintained | = 0°C |
| Pressure maintained | = 760 mm Hg |
| Vapour pressure of $CCl_4$ at 0°C | = 33 mm Hg |
| Evaporation of $CCl_4$ | = 0.03 g |
| Time of evaporation | = 10 hours |

Estimate the diffusion coefficient of carbon tetrachloride into air.

**Solution** Molecular weight of $CCl_4$ (A) and oxygen (B).

$$M_A = 12 + 4(35.5) = 154$$

$$M_B = 32$$

Partial pressures of $CCl_4$ and $O_2$ at the bottom and top is

$$p_{A_1} = 33 \text{ mm Hg,} \qquad p_{B_1} = 760 - 33 = 727 \text{ mm Hg}$$

$$p_{A_2} = 0 \qquad p_{B_2} = 760 \text{ mm Hg}$$

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$$N_A/A = \frac{0.03\times10^{-3}}{10\times3600\times\frac{\pi}{4}\times(0.01)^2}$$

$$= \frac{0.3\times4}{36000\times\pi} = \frac{1.2}{\pi\times0.36}\times10^{-5} = 1.061\times10^{-5}\ \text{kg/sm}^2$$

$$p = 1.01325\times10^5\ \text{N/m}^2,\ T = 273\ \text{K},$$

$$\Delta y = y_2 - y_1 = 0.15\ \text{m}$$

Using Eq. (10.15),

$$D = \frac{N_A}{A}\cdot\frac{\bar{R}T}{M_A p}\cdot(y_2-y_1)\cdot\frac{1}{\ln\frac{p_{B_2}}{p_{B_1}}}$$

$$= \frac{(1.061\times10^{-5})}{1.01325\times10^5}\times\frac{8.3143\times10^3\times273\times0.15}{154\ \ln\frac{760}{727}}$$

$$= 0.5206\times10^{-5}\ \text{m}^2/\text{s} \quad \textit{Ans.}$$

**Example 10.6** Calculate the rate of burning of a pulverized carbon particle, $d_0$ = 0.25 cm, in an atmosphere of pure oxygen at 1000 K and 1 atm pressure, assuming that a very large blanketing layer of $CO_2$ has formed around the particle. At the carbon surface, $p_{CO_2}$ = 1 atm, $p_{O_2}$ = 0, and at a very large radius, $p_{CO_2}$ = 0 and $p_{O_2}$ = 1 atm. Take $D$ = 0.4 m²/h.

**Solution** Since $C + O_2 \longrightarrow CO_2$, this is a case of equimolar counter diffusion. In spherical coordinates,

$$\frac{N_A}{A} = -\frac{D}{\bar{R}T}\frac{dp_A}{dy}$$

becomes

$$\frac{N_A}{4\pi r^2} = -\frac{D}{\bar{R}T}\frac{dp_A}{dr}$$

$$\int_{p_{A_0}}^{p_{A_\infty}} dp_A = \int_{r=r_0}^{r=\infty}\frac{\bar{R}T}{D}\frac{N_A}{4\pi}\frac{dr}{r^2}$$

$$p_{A_0} - p_{A_\infty} = \frac{N_A\bar{R}T}{4\pi D}\left(\frac{1}{r_0}-\frac{1}{\infty}\right)$$

Since $p_{A_\infty} = 0$, the burning rate of carbon is

$$N_A = \frac{4\pi D p_{A_0} r_0}{\bar{R}T}$$

$$= \frac{4\pi\times0.4\,(\text{m}^2/h)\times1\ \text{atm}\times0.25\times10^{-2}\ \text{m}}{0.082\ \text{m}^3\ \text{atm/kg mol K}\times1000\ \text{K}\times2}$$

$$= \frac{\pi\times0.2\times10^{-5}}{0.082}\ \text{kg mol/h} = 7.66\times10^{-5}\ \text{kg mol/h}$$

$$= 7.66\times10^{-5}\times12 = 9.192\times10^{-4}\ \text{kg/h} \quad \textit{Ans.}$$

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**Example 10.7** Determine the mass diffusivity for $CO_2$ in air at 1 atm, 298 K using Eq. (10.18) and compare this value with that in Table 10.2.

**Solution** From Table 10.1,

$$V_{CO_2} = 34.0,\ M_{CO_2} = 44$$
$$V_{air} = 29.9,\ M_{air} = 28.9$$

From Eq. (10.18),

$$D = 435.7 \frac{T^{3/2}}{p(V_A^{1/3} + V_B^{1/3})^2}\left(\frac{1}{M_A} + \frac{1}{M_B}\right)^{1/2}$$
$$= 435.7 \frac{(298)^{3/2}}{(1.01325 \times 10^5)[(34.0)^{1/3} + (29.9)^{1/3}]^2} \times \left[\frac{1}{44} + \frac{1}{28.9}\right]^{1/2}$$
$$= 0.132 \text{ cm}^2/\text{s} \quad \textit{Ans.}$$

From Table 10.2,

$$D = 0.164 \text{ cm}^2/\text{s}$$

Thus the two values are in good agreement.

**Example 10.8** Air at 1 atm, 25°C, containing small quantities of iodine flows with a velocity of 5.18 m/s inside a 3.048 cm diameter tube. Determine the mass transfer coefficient for iodine transfer from the gas stream to the wall surface. If $C_m$ is the mean concentrate of iodine in kg mol/m$^3$ in the air stream, determine the rate of deposition of iodine on the tube surface where the iodine concentration is zero.

**Solution** For air,

$$\nu = 1.58 \times 10^{-5} \text{ m}^2/\text{s},$$

$D$ for air–iodine system at 1 atm, 298 K = $0.826 \times 10^{-5}$ m$^2$/s

$$\text{Re}_d = \frac{u_m d}{\nu} = \frac{5.18 \times 3.048 \times 10^{-2}}{1.58 \times 10^{-5}} \cong 10^4$$

$$\text{Sc} = \frac{\nu}{D} = \frac{1.58 \times 10^{-5}}{0.826 \times 10^{-5}} = 1.91$$

$$\text{Sh} = \frac{k_m d}{D} = \frac{k_m \times 3.048 \times 10^{-2}}{0.826 \times 10^{-5}}$$

Now, $\text{Sh} = 0.023\,(\text{Re})^{0.8}\,(\text{Sc})^{1/3}$

$$\frac{k_m \times 3.048 \times 10^{-2}}{0.826 \times 10^{-5}} = 0.023\,(10^4)^{0.8}\,(1.91)^{1/3}$$

$$\therefore \quad k_m = 1.224 \times 10^2 \text{ m/s} \quad \textit{Ans.}$$

The rate of deposition of

$$\frac{N_A}{A} = k_m(C_m - C_w) = 1.224 \times 10^{-2} \times C_m \text{ kg mol/m}^2\text{s} \quad \textit{Ans.}$$

where $C_m$ is the mean iodine concentration in the air stream.

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**Example 10.9** Air at 100°C flows over a streamlined napthalene body. Naphthalene sublimes into air and its vapour pressure at 100°C is 20 mm Hg. The heat transfer coefficient for this system was previously found to be 17.45 W/m$^2$ K. The mass diffusivity of napthalene vapour in air at 100°C is 2.98 × 10$^{-2}$ m$^2$/h. The concentration of naphthalene in bulk air stream is negligibly small. Calculate the mass transfer coefficient and the mass flux for the system. For air at 100°C, take $\rho$ = 0.946 kg/m$^3$, $c_p$ = 1.005 kJ/kg K, $k$ = 0.032 W/m K and $\nu$ = 23.13 × 10$^{-6}$ m$^2$/s.

**Solution**

$$20 \text{ mm Hg} = \frac{20}{760} = 0.0263 \text{ atm}$$

Now,
$$k_G = \left(\frac{h}{\rho c_p u_\infty}\right)\left(\frac{p u_\infty}{\bar{R}T\, p_{B_m}}\right)\left[\left(\frac{c_p\mu}{k}\right)\left(\frac{\rho D}{\mu}\right)\right]^{0.67}$$

$$p = 1 \text{ atm}$$

$$p_{B_1} = 1 - 0.0263 = 0.9737 \text{ atm}$$

$$p_{B_2} = 1 - p_{A_2} = 1 \text{ atm}$$

$$p_{B_m} = \frac{1 - 0.9737}{\ln(1/0.9737)} = 0.9860 \text{ atm}$$

$$k_G = \frac{17.45 \times 1}{0.946 \times 1.005 \times 0.082 \times 373 \times 0.9868 \times 1000} \times \left(\frac{1.005 \times 0.946 \times 2.98 \times 10^{-2} \times 1000}{0.032 \times 3600}\right)^{0.67}$$

$$= 0.6081 \times 10^{-3} \times 0.3907$$

$$= 0.2376 \times 10^{-3} \text{ kg mol/m}^2 \text{ s atm}$$

$$= 0.855 \text{ kg mol/m}^2 \text{ h atm} \quad \textit{Ans.}$$

$$\frac{N_A}{A} = k_G (p_{A_2} - p_{A_1}) = 0.855 \times 0.0263$$

$$= 0.0225 \text{ kg mol/m}^2 \text{ h} \quad \textit{Ans.}$$

**Example 10.10** Calculate the mass transfer coefficient of water vapour in air in turbulent flow at 60 m/s at 1 atm, 300 K, over a flat plate 0.3 m long. Assume concentration of vapour in air is sufficiently dilute so that $p_{B_m}/p$ = 1.

**Solution** For air at 1 atm, 300 K, $\rho$ = 1.1774 kg/m$^3$ and $\nu$ = 15.68 × 10$^{-6}$ m$^2$/s. For air-water vapour system the mass diffusivity, $D$ = 0.260 × 10$^{-4}$ m$^2$/s.

$$\text{Sc} = \frac{\nu}{D} = \frac{15.68 \times 10^{-6}}{0.26 \times 10^{-4}} = 0.6$$

(a)
$$\text{Re}_L = \frac{60 \text{ m/s} \times 0.3 \text{ m}}{15.68 \times 10^{-6} \text{ m}^2/\text{s}} = 1.148 \times 10^6$$

Turbulent flow

$$j_M = \frac{0.037}{(\text{Re}_L)^{0.2}} = 2.27 \times 10^{-3}$$

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$$j_M = \frac{k_m}{V}\,\text{Sc}^{2/3} = 2.27 \times 10^{-3}$$

$$k_m = \frac{2.27 \times 10^{-3} \times 60 \text{ m/s}}{(0.6)^{0.67}} = 0.1918 \text{ m/s} \quad \textit{Ans.}$$

**Example 10.11** Estimate the value of mass transfer coefficient for the absorption of $NH_3$ by the wet surface of a cylinder placed in a turbulent air stream flowing across the cylinder at 5 m/s. No data on mass transfer exist for this process, but heat transfer tests with the same geometry and air velocity show $h$ = 56.8 W/m$^2$ K. For air, Pr = 0.74, $\rho$ = 1.2 kg/m$^3$ and $c_p$ = 1.005 kJ/g K. For dilute $NH_3$ – air mixture, $p_{B_m} = P$ and Sc = 0.61.

**Solution** Assuming $j_M = j_H$ and using Eq. (10.61)

$$\frac{h_x}{k_m} = \rho c_p \left(\frac{\text{Sc}}{\text{Pr}}\right)^{2/3}$$

$$\therefore \qquad k_m = \frac{h_x}{\rho c_p}\left(\frac{\text{Pr}}{\text{Sc}}\right)^{2/3} = \frac{56.8 \text{ W/m}^2\text{K} \times 10^{-3} \text{ kW/W}}{1.2 \text{ kg/m}^3 \times 1.005 \text{ kJ/kg K}}\left(\frac{0.74}{0.61}\right)^{0.67}$$

$$= 0.0536 \text{ m/s} \quad \textit{Ans.}$$

**Example 10.12** Dry air at 1 atm blows across a thermometer enclosed in a dampened cover. The wet-bulb temperature recorded is 18.3°C. Calculate the temperature of dry air.

Had the air stream been at 32.2°C while the wet-bulb temperature remains at 18.3°C, calculate the relative humidity of the air stream. Given: $D = 0.26 \times 10^{-4}$ m$^2$/s (for air–water vapour system), $\alpha = 0.221 \times 10^{-4}$ m$^2$/s (for air) and $c_p$ = 1.004 kJ/kg K.

**Solution** At steady state, the heat needed to evaporate water from the dampened cover must come from the air. So, by energy balance

$$hA(T_\infty - T_w) = \dot{m}_w h_{fg}$$

The mass of water evaporated

$$\dot{m}_w = k_m A\,(C_w - C_\infty)$$

$$hA(T_\infty - T_w) = k_m A(C_w - C_\infty)h_{fg}$$

From Eq. (10.61)

$$\rho c_p \left(\frac{\alpha}{D}\right)^{2/3} (T_\infty - T_w) = (C_w - C_\infty)h_{fg} \qquad (1)$$

From steam tables, at 18.3°C,

$$p_g = p_{\text{sat}} = 2.107 \text{ kPa}$$

and
$$\rho_w = C_w = \frac{P_w}{R_w T_w} = \frac{2.107 \times 18}{8.3143 \times 291.3}$$

$$= 0.01566 \text{ kg/m}^3$$

Now
$$\rho_\infty = C_\infty = 0 \text{ (since air is dry)}$$

$$\rho = \frac{p}{RT} = \frac{1.01325 \times 10^2}{0.287 \times 291.3} = 1.212 \text{ kg/m}^3$$

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$$\text{Le} = \frac{\alpha}{D} = \frac{0.221\times10^{-4}}{0.26\times10^{-4}} = 0.85$$

$$h_{fg} = 2.456 \text{ MJ/kg}$$

Substituting in Eq. (1) above,

$$T_\infty - T_w = \frac{(0.01566 - 0)(2.456\times10^6)}{(1.212)(1004)(0.85)^{2/3}} = 35.36°\text{C}$$

$$T = 53.69°\text{C} \quad \textit{Ans.}$$

Here, $T_w = 32.2°\text{C}$, $T_\infty = 18.3°\text{C}$, $\rho = 1.212 \text{ kg/m}^3$,

$h_{fg} = 2.456$ MJ/kg, $\rho_w = C_w$ $0.01566$ kg/m$^3$, Le = 0.85

Substituting in Eq. (1),

$$1.212 \times 1004\,(0.85)^{2/3}\,(32.2 - 18.3)$$
$$= (0.01566 - C_\infty)\,(2.456\times10^6)$$

$$C_\infty = \rho_\infty = 0.0095 \text{ kg/m}^3$$

From steam table, $C_g = \rho_g = 0.0342 \text{ kg/m}^3$

Relative humidity, $\text{RH} = \dfrac{0.0095}{0.0342} = 0.278$ or, 27.8% *Ans.*

## Summary

Mechanism of mass transfer by molecular diffusion and by convection are first explained. Fick's law of diffusion and the analogy of momentum transfer, heat transfer and mass transfer are discussed. The phenomenon of equimolar counter diffusion is introduced and it is followed by a discussion on convective mass transfer boundary layers for laminar and turbulent flows over a flat plate. The mass transfer coefficient is defined and its evaluation by dimensional analysis as well as by analogy with heat transfer is explained by illustrating the one-to-one correspondence of temperature with concentration of the diffusing species, the Sherwood number with Nusselt number and the Schmidt number with Prandtl number, and by using Colburn's *j*-factors.

## Important Formulae and Equations

| *Equation Number* | *Equation* | *Remarks* |
|---|---|---|
| (10.1) | $\nabla^2 C_A = \dfrac{1}{D}\dfrac{\partial C_A}{\partial t}$ | Concentration field of the diffusing species *A* |
| (10.2) | $\dfrac{\partial^2 C_A}{\partial y^2} = \dfrac{1}{D}\dfrac{\partial C_A}{\partial t}$ | One-dimensional mass diffusion of *A* along *y* direction |
| (10.4) | $\dfrac{N_A}{A} = -D\dfrac{\text{d}C_A}{\text{d}y}$ | Mass flux $\dfrac{N_A}{A}$ in kg mol/m$^2$ s. Fick's law of diffusion |

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| *Equation Number* | *Equation* | *Remarks* |
|---|---|---|
| (10.8) | $\frac{N_A}{A} = -\frac{D}{\bar{R}T}\frac{p_{A2}-p_{A1}}{y_2-y_1}$ | Equimolar counter diffusion in which gases $A$ and $B$ diffuse simultaneously in opposite direction |
| (10.17) | $\frac{N_A}{A} = -\frac{D}{RT}\cdot\frac{P}{p_{BM}}\cdot\frac{p_{A2}-p_{A1}}{y_2-y_1}$ | Molecular diffusion of gas $A$ through a stationary gas $B$ |
| (10.18) | $D = 435.7\frac{T^{3/2}}{p(V_A^{1/3}+V_B^{1/3})^2}\left(\frac{1}{M_A}+\frac{1}{M_B}\right)^{1/2}$ | Gulliland's equation for diffusion coefficient |
| (10.21) | $h_m = \frac{-D\left(\frac{\partial C_A}{\partial y}\right)_{y=0}}{C_{A_w}-C_{A_\infty}}$ | Mass transfer coefficient (m/s) |
| (10.22) | $\frac{N_A}{A} = -\frac{D}{RT}\frac{\mathrm{d}p_A}{\mathrm{d}y} = \frac{h_m}{RT}\Delta p_A$ | Molecular diffusion for a gas |
| (10.23) | $\mathrm{Sh} = \frac{h_m x}{D}$ | Sherwood number |
| (10.23a) | $\frac{\mathrm{Sh}}{\mathrm{Re.Sc}} = \frac{h_m}{u}$ | Analogous to Stanton number where $u$ is the characteristic velocity of the system |
| (10.26) | $\mathrm{Gr}_m = \frac{g(\rho_\infty-\rho_o)}{v^2\rho_\infty}$ | Mass Grashof number where subscript zero refers to the wall |
| (10.30) | $\frac{N_A}{A} = k_G(p_{Ai}-p_{AG})$ | Molecular diffusion of water vapour through air, where $p_{AG}$ is the partial pressure of water vapour in the bulk air phase and $p_{Ai}$ that at the interface |
| (10.31) | $k_G = \frac{DP}{\bar{R}T\,p_{Bm}(y_{fg}-y_i)}\left(\frac{\text{kg mol}}{\text{m}^2\text{s atm}}\right)$ | Gas-phase mass transfer coefficient where $(y_{fg} - y_i)$ is the effective film thickness |
| (10.33) | $\frac{N_A}{A} = k_L(C_{A_i}-C_{A_L})$ | Mass transfer in liquid phase where $C_L$ is the concentration of $A$ in the bulk liquid phase and $C_{Ai}$ that at the interface |
| (10.34) | $k_L = \frac{DC_t}{C_{Bm}(y_{fL}-y_i)}$ | Liquid phase mass transfer coefficient where $C_t = C_A + C_B$ |

(*Contd*)

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| *Equation Number* | *Equation* | *Remarks* |
|---|---|---|
| (10.44) | $\dfrac{\text{Sh}}{\text{Re}\,\text{Sc}} = \dfrac{k_G \bar{R}T}{VP}\dfrac{p_{Bm}}{} = \dfrac{C_{fx}}{2}$ | Analogy of mass and momentum transfer in gas phase |
| (10.46) | $k_G = \dfrac{hp}{C_p \bar{R}T p_{Bm}}$ | Inter-relating heat and mass transfer for Pr = 1 and Sc = 1 |
| (10.48) | $\text{Sh} = 0.023\ \text{Re}^{0.8}\ \text{Sc}^{0.33}$ | Mass transfer in a wetted wall column |
| (10.61) | $\dfrac{h_m}{k_m} = \rho c_p \text{Le}^{2/3} = \rho c_p \left(\dfrac{\alpha}{D}\right)^{2/3}$ | Ratio of average values of heat and mass transfer coefficients |

## Review Questions

10.1 What kind of mass transfer this chapter is concerned with? How is it different from mass flow in fluid mechanics?

10.2 Explain Fick's law of diffusion. What is mass diffusivity? What is its dimension?

10.3 Give a comparison of Newton's law of viscosity, Fourier's law of heat conduction and Fick's law of diffusion. How can you generalise them in terms of force and flux?

10.4 Explain the physical significance of Schmidt number, Lewis number and Prandtl number.

10.5 Explain equimolar counter diffusion. Does it have any counterpart in heat transfer?

10.6 Explain molecular diffusion through a stationary gas. What does the stationary character of the stagnant gas imply?

10.7 How does mass diffusivity of a gas depend on pressure and temperature?

10.8 Define gas-phase mass transfer coefficient. How is it different from the liquid-phase mass transfer coefficient?

10.9 How is mass transfer coefficient evaluated by dimensional analysis?

10.10 What is Sherwood number? What is its counterpart in heat transfer?

10.11 Bring out the analogy of heat transfer and mass transfer.

10.12 Explain how $j_H = j_M = f/2$.

10.13 When direct mass transfer data for a system are not available, show how the mass transfer coefficient can be predicted from the relevant heat transfer data.

10.14 For boundary-layer flow over a flat plate, how is the mass transfer coefficient related to heat transfer coefficient and skin friction coefficient?

10.15 For simultaneous flow of heat and mass in a flat plate show that

$$\frac{h_m}{k_m} = \rho c_p\, \text{Le}^{0.67}$$

## Objective Type Questions

10.1 It is not a property of the fluid:
(a) Prandtl number
(b) Reynolds number
(c) Schmidt number
(d) Lewis number

10.2 The unit of the following property is not $m^2/s$
(a) Thermal diffusivity
(b) Kinematic viscosity
(c) Dynamic viscosity
(d) Mass diffusivity

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10.3 Fick's law of diffusion in terms of mass fraction can be written as

(a) Mass flux = $D\dfrac{\partial C_A}{\partial x}$

(b) Mass flux = $-\rho D\dfrac{\partial C_A}{\partial x}$

(c) Mass flux = $\rho D\dfrac{\partial C_A}{\partial x}$

(d) Mass flux = $-\dfrac{D}{\rho}\dfrac{\partial C_A}{\partial x}$

where $C_A$ is the mass fraction of diffusing species $A$ and ρ is the mass density.

10.4 Match List I and List II and select the correct answers using the codes given below:

| List I | List II |
|---|---|
| A. Stanton number | 1. Surface tension |
| B. Mach number | 2. Mass transfer |
| C. Weber number | 3. Forced convection |
| D. Schmidt number | 4. Compressibility |

**Codes:**

| | A | B | C | D |
|---|---|---|---|---|
| (a) | 4 | 3 | 2 | 1 |
| (b) | 3 | 4 | 1 | 2 |
| (c) | 1 | 2 | 3 | 4 |
| (d) | 2 | 3 | 4 | 1 |

10.5 When there is forced convection with mass transfer over a flat plate, there is a similarity in momentum and concentration boundary layer if

(a) Pr = 1 (b) Sc = 1
(c) Sh = 1 (d) Sc × Re = 1

10.6 For complete similarity in momentum, heat and mass transfer, we should have

(a) Le = Re = 1 (b) Pr = Sc = 1
(c) Sh = Sc = 1 (d) Pr = Sh = 1

10.7 Choose the wrong statement:
During isothermal evaporation of water vapour into still air, it is assumed that

(a) total pressure remains constant
(b) air and water vapour behave like ideal gases
(c) the air movement creates a little turbulence
(d) the system is in steady state

10.8 The diffusion coefficient is a property of the system. Its value depends upon the system's

(a) pressure
(b) temperature
(c) composition
(d) all of the above

10.9 When both heat transfer and mass transfer take place simultaneously, the ratio of the average values of the heat and mass transfer coefficients both for laminar and turbulent flows is given by

(a) $\dfrac{h_m}{k_m} = \rho c_p\left(\dfrac{\nu}{D}\right)^{2/3}$

(b) $\dfrac{h_m}{k_m} = \rho\left(\dfrac{\alpha}{D}\right)^{2/3}$

(c) $\dfrac{h_m}{k_m} = \rho c_p\left(\dfrac{\alpha}{D}\right)^{2/3}$

(d) $\dfrac{h_m}{k_m} = \rho c_p\left(\dfrac{D}{\alpha}\right)^{2/3}$

10.10 The unit of mass transfer coefficient is

(a) m/s
(b) $m^2/s$
(c) kg mol/$m^2$ s
(d) kg mol/$m^2$ h atm

10.11 The counterpart of Sherwood number in heat transfer is

(a) Schmidt number
(b) Prandtl number
(c) Reynolds number
(d) Nusselt number

10.12 The counterpart of equimolar counter diffusion in heat transfer is

(a) heat conduction
(b) natural convection
(c) forced convection
(d) none

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## Answers

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 10.1 | (b) | 10.2 | (c) | 10.3 | (b) | 10.4 | (b) | 10.5 | (b) |
| 10.6 | (b) | 10.7 | (c) | 10.8 | (d) | 10.9 | (c) | 10.10 | (a) |
| 10.11 | (d) | 10.12 | (d) | | | | | | |

## Problems for Practice

10.1 Two large vessels contain uniform mixtures of nitrogen (component A) and carbon dioxide (component B) at 1 atm, $T$ = 289 K, but at different concentrations. Vessel 1 contains 90% $N_2$ and 10% $CO_2$ by moles, whereas vessel 2 contains 20% $N_2$ and 80% $CO_2$ by moles. The two vessels are connected by a duct of 0.1524 m ID, and 1.22 m long. Determine the rate of transfer of nitrogen between the two vessels by assuming that steady-state transfer takes place in view of the large capacity of the two reservoirs. Take the mass diffusivity for $N_2 - CO_2$ mixture as $0.16 \times 10^{-4}$ m²/s.

(*Ans.* $0.71 \times 10^{-8}$ kg mol/s)

10.2 An open tank contains benzene at 1 atm, 298 K. The distance from the surface of the benzene layer to the top of the tank is 3 m. It is assumed that the air in the vessel is motionless while there is sufficient air motion outside to remove the benzene vapour arriving at the top surface. Determine the rate of evaporation of benzene per sq. m. of the benzene surface ($p_{A_1}$ = 0.01 atm). Take $D = 0.0962 \times 10^{-4}$ m²/s.

(*Ans.* $0.13 \times 10^{-6}$ g mol/m² s)

10.3 A deep narrow tube open at the top contains toluene at the bottom. Air inside the tube is motionless while at the top toluene concentration is zero. The entire system is at 1 atm, 18.7°C, when $D = 0.826 \times 10^{-4}$ m²/s. The saturated vapour pressure of toluene at liquid surface is 0.026 atm. Determine the rate of evaporation of toluene per unit area if the distance from the liquid surface to the top is 1.524 m.

(*Ans.* $0.597 \times 10^{-8}$ kg mol/m² s)

10.4 One method of measuring diffusion coefficients of vapours is to measure the rate of evaporation of a liquid in narrow tubes. In one such experiment, a glass tube 1 cm in diameter was filled with water at 20°C to within 4 cm of the top. Dry air at 20°C and 1.013 bar was blown across the top of the tube. At the end of 24 hours of steady-state operation, the level of the water dropped 0.1 cm. Calculate the diffusivity of the air-vapour system at 20°C.

10.5 A pulverized coal particle burns in pure oxygen at 1200°C. The process is limited by diffusion of the oxygen counterflow to the $CO_2$ formed at the particle surface. Assume that the coal is pure carbon and has an initial diameter of 0.01 cm. If $D = 1.032 \times 10^{-4}$ m²/s, calculate the time required for 90% of the carbon to burn away. What is the final diameter of the coal particle?

10.6 Nitrogen diffuses steadily through a stagnant layer of air which is 0.5 cm thick. The concentration of $N_2$ is 0.08 kg/m³ at one face and zero at the other. The total pressure is 1 atm and the temperature 20°C. Calculate the time required for 1 kg of $N_2$ to diffuse across 1 sq. m. of this air film.

10.7 Calculate the mass diffusivity of the binary gas mixture of air-water vapour at 273 K and 1 atm, and compare the result with that given in Table 10.2.

10.8 Calculate the diffusion coefficient for benzene in atmospheric air at 25°C.

10.9 Estimate the diffusion rate of water from the bottom of a test tube 10 mm in diameter and 15 cm long into dry atmospheric air at 25°C. Given: $D = 0.256 \times 10^{-4}$ m²/s.

(*Ans.* $1.131 \times 10^{-10}$ kg/s)

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10.10 Predict the mass transfer coefficient for liquid ammonia vaporising into air at 1 atm, 25°C, knowing that the heat transfer coefficient in the same equipment, at the same gas and liquid flow rates, is 4.544 kW/$m^2$ K.

(*Ans.* 537.35 kg mol/$m^2$ h atm)

10.11 Air at 10°C and 1 atm flows over a plane surface covered with naphthalene. The flow velocity is such that the Reynolds number at a distance of 0.6 m from the leading edge of the plate is $9 \times 10^4$. Determine the average mass transfer coefficient for the transfer of napthalene over the 0.6 m length of the surface by making use of the analogy between heat and mass transfer in laminar flow along a flat plate.

10.12 Air at 25°C and atmospheric pressure flows with a velocity of 7.6 m/s inside a 2.5 cm inner diameter pipe. The inside surface of the tube contains a deposit of naphthalene. Determine the mass transfer coefficient for the transfer of naphthalene from the pipe surface into the air in regions away from the inlet.

10.13 Dry air at atmospheric pressure and 10°C flows over a flat plate with a velocity of 1 m/s. The plate is covered with a film of water which evaporates into the air stream. Determine the average mass transfer coefficient for the transfer of water vapour over a distance of 0.6 m from the leading edge of the plate.

10.14 Atmospheric air at 30°C flows over a wet-bulb thermometer which reads 20°C. Calculate the concentration of water vapour in the air stream and the relative humidity of the air.

10.15 The temperature of an air stream is to be measured, but the thermometer available does not have a sufficiently high range. Accordingly, a dampened cover is placed around the thermometer before it is placed in the air stream. The thermometer reads 32°C. Estimate the true air temperature, assuming that it is dry at atmospheric pressure.

10.16 Dry air at 25°C and atmospheric pressure blows over a 30 $cm^2$ surface of ice at a velocity of 1.5 m/s. Estimate the amount of moisture evaporated per hour, assuming that the block of ice is perfectly insulated except for the surface exposed to the air stream.

10.17 Dry air at atmospheric pressure blows over an insulated flat plate covered with a thin wicking material which has been soaked in ethyl alcohol. The temperature of the plate is 25°C. Calculate the temperature of the air stream assuming that the concentration of alcohol is negligible in the free stream. Also calculate the mass transfer rate of alcohol for a 30 $cm^2$ plate if the free stream velocity is 7 m/s.

10.18 A thin plastic membrane is used to separate helium from a gas stream. Under steady-state conditions the concentration of helium in the membrane is known to be 0.02 and 0.005 kg mol/$m^3$ at the inner and outer surfaces, respectively. If the membrane is 1 mm thick and the binary diffusion coefficient of helium with respect to the plastic is $10^{-9}$ $m^2$/s, what is the mass flux by diffusion?

10.19 Helium gas at 25°C and 4 bar is contained in a glass cylinder of 100 mm inner diameter and 5 mm thickness. What is the rate of mass loss per unit length of the cylinder?

10.20 Oxygen gas is maintained at pressure of 2 bar and 1 bar on opposite sides of a rubber membrane that is 0.5 mm thick, and the entire system is at 25°C. What is the molar diffusive flux of $O_2$ through the membrane? What are the molar concentrations of $O_2$ on both sides of the membrane (outside the rubber)?

10.21 Helium gas is stored at 20°C in a spherical container of fused silica, which has an inner diameter of 0.20 m and a wall thickness of 2 mm. If the container is charged to an initial pressure of 4 bar, what is the rate at which this pressure decreases with time?

(*Ans.* $2.63 \times 10^{-8}$ bar/s)

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10.22 To maintain a pressure close to atmosphere, an industrial pipeline containing ammonia gas is vented to ambient air. Venting is achieved by tapping the pipe and inserting a 3 mm diameter tube, which extends for 20 m into the atmosphere. With the entire system operating at 25°C, determine the mass rate at which ammonia is lost to the atmosphere and the mass rate of contamination of the pipe with air. What are the mole and mass fractions of air in the pipe when the ammonia flow rate is 5 kg/h. Given: For ammonia-air system at 298 K, $D = 0.28 \times 10^{-4}$ m$^2$/s.

(*Ans.* $2.48 \times 10^{-8}$ kg/h, $4.23 \times 10^{-8}$ kg/h, $x_B = 4.96 \times 10^{-9}$ and $c_B = 0.85 \times 10^{-8}$)

## REFERENCE

1. E.R. Gilliland, "Diffusion Coefficients in Gaseous Systems", *Ind. Eng. Chem.*, Vol. 26, p. 681, 1934.

# Appendix A: Thermophysical Properties of Matter

**Table A.1** *Thermophysical properties of selected metallic solids*

| Composition | Melting Point (K) | 300 K $\rho$ (kg/m$^3$) | 300 K $c_p$ (J/kg K) | 300 K $k$ (W/m K) | 300 K $\alpha \cdot 10^6$ (m$^2$/s) | k (W/m K)/$c_p$ (J/kg K) at various temperature (K) 100 | 200 | 400 | 600 | 800 | 1000 | 1200 | 1500 | 2000 | 2500 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Aluminium | | | | | | | | | | | | | | | |
| Pure | 993 | 2702 | 903 | 237 | 97.1 | 302 | 237 | 240 | 231 | 218 | | | | | |
| | | | | | | 482 | 798 | 949 | 1033 | 1146 | | | | | |
| Alloy 2024-T6 | 775 | 2770 | 875 | 177 | 73.0 | 65 | 163 | 186 | 186 | | | | | | |
| (4.5% Cu, 1.5% Mg, 0.6% Mn) | | | | | | 473 | 787 | 925 | 1042 | | | | | | |
| Alloy 195, Cast (4.5% Cu) | | 2790 | 883 | 168 | 68.2 | | | 174 | 185 | | | | | | |
| | | | | | | | | — | — | | | | | | |
| Beryllium | 1550 | 1850 | 1825 | 200 | 59.2 | 990 | 301 | 161 | 126 | 106 | 90.8 | 78.7 | | | |
| | | | | | | 203 | 1114 | 2191 | 2604 | 2823 | 3018 | 3227 | 3519 | | |
| Bismuth | 545 | 9780 | 122 | 7.86 | 6.59 | 16.5 | 9.69 | 7.04 | | | | | | | |
| | | | | | | 112 | 120 | 127 | | | | | | | |
| Boron | 2573 | 2500 | 1107 | 27.0 | 9.76 | 190 | 55.5 | 16.8 | 10.6 | 9.60 | 9.85 | | | | |
| | | | | | | 128 | 600 | 1463 | 1892 | 2160 | 2338 | | | | |
| Cadmium | 594 | 8650 | 231 | 96.8 | 48.4 | 203 | 99.3 | 94.7 | | | | | | | |
| | | | | | | 198 | 222 | 242 | | | | | | | |
| Chromium | 2118 | 7160 | 449 | 93.7 | 29.1 | 159 | 111 | 90.9 | 80.7 | 71.3 | 65.4 | 61.9 | 57.2 | 49.4 | |
| | | | | | | 192 | 384 | 484 | 542 | 581 | 616 | 682 | 779 | 937 | |
| Cobalt | 1769 | 8862 | 421 | 99.2 | 26.6 | 167 | 122 | 85.4 | 67.4 | 58.2 | 52.1 | 49.3 | 42.5 | | |
| | | | | | | 236 | 379 | 450 | 503 | 550 | 628 | 733 | 674 | | |
| Copper | | | | | | | | | | | | | | | |
| Pure | 1358 | 8933 | 385 | 401 | 117 | 482 | 413 | 393 | 379 | 366 | 352 | 399 | | | |
| | | | | | | 252 | 336 | 397 | 417 | 433 | 451 | 480 | | | |
| Commercial bronze (90% Cu, 10% A1) | 1293 | 8800 | 420 | 52 | 14 | | 42 | 52 | 59 | | | | | | |
| | | | | | | | 785 | 460 | 545 | | | | | | |
| Phosphor gear bronze (89% Cu, 11% Sn) | 1104 | 8780 | 355 | 54 | 17 | | 41 | 65 | 74 | | | | | | |
| | | | | | | | — | — | — | | | | | | |

(*Contd*)

**Table A.1** (*Contd*)

| Composition | Melting Point (K) | 300 K $\rho$ ($kg/m^3$) | $c_p$ (J/kg K) | k (W/m K) | $\alpha \cdot 10^6$ ($m^2/s$) | k (W/m K)/$c_p$ (J/kg K) at various temperature (K) 100 | 200 | 400 | 600 | 800 | 1000 | 1200 | 1500 | 2000 | 2500 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Cartridge brass (70% Cu. 30% Zn) | 1188 | 8530 | 380 | 110 | 33.9 | 75 | 95 | 137 | 149 | | | | | | |
| | | | | | | | 360 | 395 | 425 | | | | | | |
| Constantan (55% Cu, 45% Ni) | 1493 | 8920 | 384 | 23 | 6.71 | 17 | 19 | | | | | | | | |
| | | | | | | 237 | 362 | | | | | | | | |
| Germanium | 1211 | 5360 | 322 | 59.9 | 34.7 | 232 | 96.8 | 43.2 | 27.3 | 19.8 | 17.4 | 17.4 | | | |
| | | | | | | 190 | 290 | 337 | 348 | 357 | 375 | 395 | | | |
| Gold | 1336 | 19300 | 129 | 317 | 127 | 327 | 323 | 311 | 298 | 284 | 270 | 255 | | | |
| | | | | | | 109 | 124 | 131 | 135 | 140 | 145 | 155 | | | |
| Iridium | 2720 | 22500 | 130 | 147 | 50.3 | 172 | 153 | 144 | 138 | 132 | 126 | 120 | 111 | | |
| | | | | | | 90 | 122 | 133 | 138 | 144 | 153 | 161 | 172 | | |
| Iron | | | | | | | | | | | | | | | |
| Pure | 1810 | 7870 | 447 | 80.2 | 23.1 | 134 | 94.0 | 69.5 | 54.7 | 43.3 | 32.7 | 28.3 | 32.1 | | |
| | | | | | | 216 | 384 | 490 | 574 | 680 | 975 | 609 | 654 | | |
| Armco (99.75% pure) | | 7870 | 447 | 72.7 | 20.7 | 95.6 | 80.6 | 65.7 | 53.1 | 42.2 | 32.3 | 28.7 | 31.4 | | |
| | | | | | | 215 | 384 | 490 | 574 | 680 | 975 | 609 | 654 | | |
| Carbon steels | | | | | | | | | | | | | | | |
| Plain carbon (Mn ≤ 1%, Si ≤ 0.1%) | | 7854 | 434 | 60.5 | 17.7 | | | 56.7 | 48.0 | 39.2 | 30.0 | | | | |
| | | | | | | | | 487 | 559 | 685 | 1169 | | | | |
| AlS1 1010 | | 7832 | 434 | 63.9 | 18.8 | | | 58.7 | 48.8 | 39.2 | 31.3 | | | | |
| | | | | | | | | 487 | 559 | 685 | 1168 | | | | |
| Carbon-silicon (Mn ≤ 1%, 0.1% < Si ≤ 0.6%) | | 7817 | 446 | 51.9 | 14.9 | | | 49.8 | 44.0 | 37.4 | 29.3 | | | | |
| | | | | | | | | 501 | 582 | 699 | 971 | | | | |
| Carbon–manganese–silicon | | 8131 | 434 | 41.0 | 11.6 | | | 42.2 | 39.7 | 35.0 | 27.6 | | | | |
| | | | | | | | | 487 | 559 | 685 | 1090 | | | | |

(*Contd*)

**Table A.1** (*Contd*)

| Composition | Melting Point (K) | 300 K $\rho$ (kg/m$^3$) | $c_p$ (J/kg K) | $k$ (W/m K) | $\alpha\cdot10^{\circ}$ (m$^2$/s) | $k$ (W/m K)/$c_p$ (J/kg K) at various temperature (K) 100 | 200 | 400 | 600 | 800 | 1000 | 1200 | 1500 | 2000 | 2500 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| (1% < Mn ≤ 1.65%, 0.1% < Si ≤ 0.6%) | | | | | | | | | | | | | | | |
| Chromium (low) steels | | | | | | | | | | | | | | | |
| $\frac{1}{2}$ Cr – $\frac{1}{4}$ Mo–Si | | 7822 | 444 | 37.7 | 10.9 | | | 38.2 | 36.7 | 33.3 | 26.9 | | | | |
| (0.18% C, 0.65% Cr, 0.23% Mo, 0.6% Si) | | | | | | | | 492 | 575 | 688 | 969 | | | | |
| 1 Cr – $\frac{1}{2}$ Mo | | 7858 | 442 | 42.3 | 12.2 | | | 42.0 | 39.1 | 34.5 | 27.4 | | | | |
| (0.16% C, 1% Cr, 0.54% Mo, 0.39% Si) | | | | | | | | 492 | 575 | 688 | 969 | | | | |
| 1 Cr–V, | | 78.36 | 443 | 48.9 | 14.1 | | | 46.8 | 42.1 | 36.3 | 28.2 | | | | |
| (0.2% C, 1.02% Cr, 0.15% V) | | | | | | | | 492 | 575 | 688 | 969 | | | | |
| Stainless steels | | | | | | | | | | | | | | | |
| AISI302 | | 8055 | 480 | 15.1 | 3.91 | | | 17.3 | 20.0 | 22.8 | 25.4 | | | | |
| | | | | | | | | 512 | 559 | 585 | 606 | | | | |
| AISI 304 | 1670 | 7900 | 477 | 14.9 | 3.95 | 9.2 | 12.6 | 16.6 | 19.8 | 22.6 | 25.4 | 218.0 | 31.7 | | |
| | | | | | | 272 | 402 | 515 | 557 | 582 | 611 | 640 | 682 | | |
| AISI 316 | | 8238 | 468 | 13.4 | 3.48 | | | 15.2 | 18.3 | 21.3 | 24.2 | | | | |
| | | | | | | | | 504 | 550 | 576 | 602 | | | | |
| AISI 347 | | 7978 | 480 | 14.2 | 3.71 | | | 15.8 | 18.9 | 21.9 | 24.7 | | | | |
| | | | | | | | | 513 | 559 | 585 | 606 | | | | |

(*Contd*)

**Table A.1** (*Contd*)

| Composition | Melting Point (K) | 300 K $\rho$ ($kg/m^3$) | $c_p$ (J/kg K) | k (W/m K) | $\alpha \cdot 10^6$ ($m^2/s$) | k (W/m K)/$c_p$ (J/kg K) at various temperature (K) 100 | 200 | 400 | 600 | 800 | 1000 | 1200 | 1500 | 2000 | 2500 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Lead | 601 | 11340 | 129 | 35.3 | 24.1 | 39.7 | 36.7 | 34.0 | 31.4 | | | | | | |
| | | | | | | 118 | 125 | 132 | 142 | | | | | | |
| Magnesium | 923 | 1740 | 1024 | 159 | 87.6 | 169 | 159 | 153 | 149 | 146 | | | | | |
| | | | | | | 649 | 934 | 1074 | 1170 | 1267 | | | | | |
| Molybdenum | 2894 | 10240 | 251 | 138 | 53.7 | 179 | 143 | 134 | 126 | 118 | 112 | 105 | 98 | 90 | 86 |
| | | | | | | 141 | 224 | 261 | 275 | 285 | 295 | 308 | 330 | 380 | 459 |
| Nickel | | | | | | | | | | | | | | | |
| Pure | 1728 | 8900 | 444 | 90.7 | 23.0 | 164 | 107 | 80.2 | 65.6 | 67.6 | 71.8 | 76.2 | 82.6 | | |
| | | | | | | 232 | 383 | 485 | 592 | 530 | 562 | 594 | 616 | | |
| Nichrome | 1672 | 8400 | 420 | 12 | 3.4 | | | 14 | 16 | 21 | | | | | |
| (80% Ni, 20% Cr) | | | | | | | | 480 | 525 | 545 | | | | | |
| Inconel X-750 | 1665 | 8510 | 439 | 11.7 | 3.1 | 8.7 | 10.3 | 13.5 | 17.0 | 20.5 | 24.0 | 27.6 | 33.0 | | |
| (73% Ni, 15% Cr, 6.7% Fe) | | | | | | — | 372 | 473 | 510 | 546 | 626 | — | — | | |
| Niobium | 2741 | 8570 | 265 | 53.7 | 23.6 | 55.2 | 52.6 | 55.2 | 58.2 | 61.3 | 64.4 | 67.5 | 72.1 | 79.1 | |
| | | | | | | 188 | 249 | 274 | 283 | 292 | 301 | 310 | 324 | 347 | |
| Palladium | 1827 | 12020 | 244 | 71.8 | 24.5 | 76.5 | 71.6 | 73.6 | 79.7 | 86.9 | 94.2 | 102 | 110 | | |
| | | | | | | 168 | 227 | 251 | 261 | 271 | 281 | 291 | 307 | | |
| Platinum | | | | | | | | | | | | | | | |
| Pure | 2045 | 21450 | 133 | 71.6 | 25.1 | 77.5 | 72.6 | 71.8 | 73.2 | 75.6 | 78.7 | 82.6 | 89.5 | 99.4 | |
| | | | | | | 100 | 125 | 136 | 141 | 146 | 152 | 157 | 165 | 179 | |
| Alloy (60Pt – 40Rh | 1800 | 16630 | 162 | 47 | 17.4 | | | 52 | 59 | 65 | 69 | 73 | 76 | | |
| (60% Pt, 40% Rh) | | | | | | | | — | — | — | — | — | — | | |
| Rhenium | 3453 | 21100 | 136 | 47.9 | 16.7 | 58.9 | 51.0 | 46.1 | 44.2 | 44.1 | 44.6 | 45.7 | 47.8 | 51.9 | |
| | | | | | | 97 | 127 | 139 | 145 | 151 | 156 | 162 | 171 | 186 | |

(*Contd*)

**Table A.1** (*Contd*)

| Composition | Melting Point (K) | 300 K $\rho$ ($kg/m^3$) | $c_p$ (J/kg K) | $k$ (W/m K) | $\alpha \cdot 10^6$ ($m^2/s$) | $k$ (W/m K)/$c_p$ (J/kg K) at various temperature (K) 100 | 200 | 400 | 600 | 800 | 1000 | 1200 | 1500 | 2000 | 2500 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Rhodium | 2236 | 12450 | 243 | 150 | 49.6 | 186 | 154 | 146 | 136 | 127 | 121 | 116 | 110 | 112 | |
| | | | | | | 147 | 220 | 253 | 274 | 293 | 311 | 327 | 349 | 376 | |
| Silicon | 1685 | 2330 | 712 | 148 | 89.2 | 884 | 264 | 98.9 | 61.9 | 42.2 | 31.2 | 25.7 | 22.7 | | |
| | | | | | | 259 | 556 | 790 | 867 | 913 | 946 | 967 | 992 | | |
| Silver | 1235 | 10500 | 235 | 429 | 174 | 444 | 430 | 425 | 412 | 396 | 379 | 361 | | | |
| | | | | | | 187 | 225 | 239 | 250 | 262 | 277 | 292 | | | |
| Tantalum | 3269 | 16600 | 140 | 57.5 | 24.7 | 59.2 | 57.5 | 57.8 | 58.6 | 59.4 | 60.2 | 61.0 | 62.2 | 64.1 | 65.6 |
| | | | | | | 110 | 133 | 144 | 146 | 149 | 152 | 155 | 160 | 172 | 189 |
| Thorium | 2023 | 11700 | 118 | 54.0 | 39.1 | 59.8 | 54.6 | 54.5 | 55.8 | 56.9 | 56.9 | 58.7 | | | |
| | | | | | | 99 | 112 | 124 | 134 | 145 | 156 | 167 | | | |
| Tin | 505 | 7310 | 227 | 66.6 | 40.1 | 85.2 | 73.3 | 62.2 | | | | | | | |
| | | | | | | 188 | 215 | 243 | | | | | | | |
| Titanium | 1953 | 4500 | 522 | 21.9 | 9.32 | 30.5 | 24.5 | 20.4 | 19.4 | 19.7 | 20.7 | 22.0 | 24.5 | | |
| | | | | | | 300 | 465 | 551 | 591 | 633 | 675 | 620 | 686 | | |
| Tungsten | 3660 | 19300 | 132 | 174 | 68.3 | 208 | 186 | 159 | 137 | 125 | 118 | 113 | 107 | 110 | 95 |
| | | | | | | 87 | 122 | 137 | 142 | 145 | 148 | 152 | 157 | 167 | 176 |
| Uranium | 1406 | 19070 | 116 | 27.6 | 12.5 | 21.7 | 25.1 | 29.6 | 34.0 | 38.8 | 43.9 | 49.0 | | | |
| | | | | | | 94 | 108 | 125 | 146 | 176 | 180 | 161 | | | |
| Vanadium | 2192 | 6100 | 489 | 30.7 | 10.3 | 35.8 | 31.3 | 31.3 | 33.3 | 35.7 | 38.2 | 40.8 | 44.6 | 50.9 | |
| | | | | | | 258 | 430 | 515 | 540 | 563 | 597 | 645 | 714 | 867 | |
| Zinc | 693 | 7140 | 389 | 116 | 41.8 | 117 | 118 | 111 | 103 | | | | | | |
| | | | | | | 297 | 267 | 402 | 436 | | | | | | |
| Zirconium | 2125 | 6570 | 278 | 22.7 | 12.4 | 33.2 | 25.2 | 21.6 | 20.7 | 21.6 | 23.7 | 26.0 | 28.8 | 33.0 | |
| | | | | | | 205 | 264 | 300 | 322 | 342 | 362 | 344 | 344 | 344 | |

**Table A.2** *Thermophysical properties of selected nonmetallic solids*

| Composition | Melting Point (K) | 300 K: $\rho$ (kg/m$^3$) | 300 K: $c_p$ (J/kg K) | 300 K: $k$ (W/m K) | 300 K: $\alpha \cdot 10^6$ (m$^2$/s) | $k$ (W/m K)/$c_p$ (J/kg K) at various temperature (K): 100 | 200 | 400 | 600 | 800 | 1000 | 1200 | 1500 | 2000 | 2500 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Aluminium oxide | 2323 | 3970 | 765 | 46 | 15.1 | 450 | 82 | 32.4 | 18.9 | 13.0 | 10.5 | | | | |
| sapphire | | | | | | — | — | 940 | 1110 | 1180 | 1225 | | | | |
| Aluminium oxide, | 2323 | 3970 | 765 | 36.0 | 11.9 | 133 | 55 | 26.4 | 15.8 | 10.4 | 7.85 | 6.55 | 5.66 | 6.00 | |
| polycrystalline | | | | | | — | — | 940 | 1110 | 1180 | 1225 | — | — | — | |
| Beryllium oxide | 2725 | 3000 | 1030 | 272 | 88.0 | | | 196 | 111 | 70 | 47 | 33 | 21.5 | 15 | |
| | | | | | | | | 1350 | 1690 | 1865 | 1975 | 2055 | 2145 | 2750 | |
| Boron | 2573 | 2500 | 1105 | 27.6 | 9.99 | 190 | 52.5 | 18.7 | 11.3 | 8.1 | 6.3 | 5.2 | | | |
| | | | | | | — | — | 1490 | 1880 | 2135 | 2350 | 2555 | | | |
| Boron fiber epoxy (30% vol) composite | 590 | 2080 | | | | | | | | | | | | | |
| $k$, ‖ to fibers | | | | 2.29 | | 2.10 | 2.23 | 2.28 | | | | | | | |
| $k$, ⊥ to fibers | | | | 0.59 | | 0.37 | 0.49 | 0.60 | | | | | | | |
| $c_p$ | | | 1122 | | | 364 | 757 | 1431 | | | | | | | |
| Carbon Amorphous | 1500 | 1950 | — | 1.60 | — | 0.67 | 1.18 | 1.89 | 2.19 | 2.37 | 2.53 | 2.84 | 3.48 | | |
| | | | | | | — | — | — | — | — | — | — | — | | |
| Diamond, type 11a | — | 3500 | 509 | 2300 | — | 1000 | 4000 | 1540 | | | | | | | |
| insulator | | | | | | 21 | 194 | 853 | | | | | | | |
| Graphite, pyrolytic | 2273 | 2210 | | | | | | | | | | | | | |
| $k$, ‖ to layers | | | | 1950 | | 4970 | 3230 | 1390 | 892 | 667 | 534 | 448 | 357 | 262 | |
| $k$, ⊥ to layers | | | | 5.70 | | 16.8 | 9.23 | 4.09 | 2.68 | 2.01 | 1.60 | 1.34 | 1.08 | 0.81 | |
| $c_p$ | | | 709 | | | 136 | 411 | 992 | 1406 | 1650 | 1793 | 1890 | 1974 | 2043 | |
| Graphite fiber | 450 | 1400 | | | | | | | | | | | | | |
| epoxy (25% vol) composite k, heat flow | | | | | | | | | | | | | | | |
| ‖ to fibers | | | | 11.1 | | 5.7 | 8.7 | 13.0 | | | | | | | |

(*Contd*)

**Table A.2** *(Contd)*

| Composition | Melting Point (K) | $\rho$ (kg/m$^3$) | 300 K $c_p$ (J/kg K) | 300 K $k$ (W/m K) | 300 K $\alpha \cdot 10^6$ (m$^2$/s) | $k$ (W/m K)/$c_p$ (J/kg K) at various temperature (K): 100 | 200 | 400 | 600 | 800 | 1000 | 1200 | 1500 | 2000 | 2500 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $k$, heat flow ⊥ to fibers | | | | 0.87 | | 0.46 | 0.68 | 1.1 | | | | | | | |
| $c_p$ | | | 935 | | | 337 | 642 | 1216 | | | | | | | |
| Pyrocram, | 1623 | 2600 | 808 | 3.98 | 1.89 | 5.25 | 4.78 | 3.64 | 3.28 | 3.08 | 2.96 | 2.87 | 2.79 | | |
| Corning 9606 | | | | | | — | — | 908 | 1038 | 1122 | 1197 | 1264 | 1498 | | |
| Silicon carbide | 3100 | 3160 | 675 | 490 | 230 | | | — | — | — | 87 | 58 | 30 | | |
| | | | | | | | | 880 | 1050 | 1135 | 1195 | 1243 | 1310 | | |
| Silicon dioxide | 1883 | 2650 | | | | | | | | | | | | | |
| crystalline (quartz) | | | | | | | | | | | | | | | |
| $k$, ∥ to $c$ axis | | | | 10.4 | | 39 | 16.4 | 7.6 | 5.0 | 4.2 | | | | | |
| $k$, ⊥ to $c$ axis | | | | 6.21 | | 20.8 | 9.5 | 4.70 | 3.4 | 3.1 | | | | | |
| $c_p$ | | | 745 | | | — | — | 885 | 1075 | 1250 | | | | | |
| Silicon dioxide, | 1883 | 2220 | 745 | 1.38 | 0.834 | 0.69 | 1.14 | 1.51 | 1.75 | 2.17 | 2.87 | 4.00 | | | |
| polycrystalline (fused silica) | | | | | | — | — | 905 | 1040 | 1105 | 1155 | 1195 | | | |
| Silicon nitride | 2173 | 2400 | 691 | 16.0 | 9.65 | — | — | 13.9 | 11.3 | 9.88 | 8.76 | 8.00 | 7.16 | 6.20 | |
| | | | | | | — | 578 | 778 | 937 | 1063 | 1155 | 1226 | 1306 | 1377 | |
| Sulfur | 392 | 2070 | 708 | 0.206 | 0.141 | 0.165 | 0.185 | | | | | | | | |
| | | | | | | 403 | 606 | | | | | | | | |
| Thorium dioxide | 3573 | 110 | 235 | 13 | 6.1 | | | 10.2 | 6.6 | 4.7 | 3.68 | 3.12 | 2.73 | 2.5 | |
| | | | | | | | | 255 | 274 | 285 | 295 | 303 | 315 | 330 | |
| Titanium dioxide | 2133 | 4157 | 710 | 8.4 | 2.8 | | | 7.01 | 5.02 | 3.94 | 3.46 | 3.28 | | | |
| polycrystalline | | | | | | | | 805 | 880 | 910 | 930 | 945 | | | |

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**Table A.3** *Thermophysical properties of common materials*

| Description/composition | 300 K Density, $\rho$ $(kg/m^3)$ | 300 K Thermal conductivity, $k$ $(W/m\cdot K)$ | 300 K Specific Heat, $c_p$ $(J/kg\cdot K)$ |
|---|---|---|---|
| **Structural building materials** | | | |
| Building boards | | | |
| Asbestos–cement board | 1920 | 0.58 | — |
| Gypsum or plaster board | 800 | 0.17 | — |
| Plywood | 545 | 0.12 | 1215 |
| Sheathing, regular density | 290 | 0.055 | 1300 |
| Acoustic tile | 290 | 0.058 | 1340 |
| Hardboard, siding | 640 | 0.094 | 1170 |
| Hardboard, high density | 1010 | 0.15 | 1380 |
| Particle board, low density | 590 | 0.078 | 1300 |
| Particle board, high density | 1000 | 0.170 | 1300 |
| Woods | | | |
| Hardwoods (oak, maple) | 720 | 0.16 | 1255 |
| Softwoods (fir, pine) | 510 | 0.12 | 1380 |
| Masonry materials | | | |
| Cement mortar | 1860 | 0.72 | 780 |
| Brick, common | 1920 | 0.72 | 835 |
| Brick, face | 2083 | 1.3 | — |
| Clay tile, hollow | | | |
| 1 cell deep, 10 cm thick | — | 0.52 | — |
| 3 cells deep, 30 cm thick | — | 0.69 | — |
| Concrete block, 3 oval cores | | | |
| Sand/gravel, 20 cm thick | — | 1.0 | — |
| Cinder aggregate 20 cm thick | — | 0.67 | — |
| Concrete block, rectangular core | | | |
| 2 cores, 20 cm thick, 16 kg | — | 1.1 | — |
| Same with filled cores | — | 0.60 | — |
| Plastering materials | | | |
| Cement plaster, sand aggregate | 1860 | 0.72 | — |
| Gypsum plaster, sand aggregate | 1680 | 0.22 | 1085 |
| Gypsum plaster, vermiculite aggregate | 720 | 0.25 | — |
| Blanket and batt | | | |
| Glass fiber, paper faced | 16 | 0.046 | — |
| | 28 | 0.038 | — |
| | 40 | 0.035 | — |
| Glass fiber, coated; duct liner | 32 | 0.038 | 835 |

(*Contd*)

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**Table A.3** (*Contd*)

| *Description/composition* | *Density, ρ (kg/m³)* | *300 K* Thermal conductivity, k (W/m·K) | *Specific Heat, $c_p$ (J/kg·K)* |
|---|---|---|---|
| Board and slab | | | |
| Cellular glass | 145 | 0.058 | 1000 |
| Glass fiber, organic bonded | 105 | 0.036 | 795 |
| Polystyrene, expanded | | | |
| Extruded (R-12) | 55 | 0.027 | 1210 |
| Molded beads | 16 | 0.040 | 1210 |
| Mineral fiberboard; roofing material | 265 | 0.049 | — |
| Wood, shredded/cemented | 350 | 0.087 | 1590 |
| Cork | 120 | 0.039 | 1800 |
| Loose Fill | | | |
| Cork, granulated | 160 | 0.045 | — |
| Diatomaceous silica, coarse | 350 | 0.069 | — |
| Powder | 400 | 0.091 | — |
| Diatomaceous silica, fine powder | 200 | 0.052 | — |
| | 275 | 0.061 | — |
| Glass fiber, poured or blown | 16 | 0.043 | 835 |
| Vermiculite, flakes | 80 | 0.068 | 835 |
| | 160 | 0.063 | 1000 |
| Formed/foamed-in-place | | | |
| Mineral wool granules with Asbestos/ inorganic binders sprayed | 190 | 0.046 | — |
| Polyvinyl acetate cork mastic; sprayed or troweled | — | 0.100 | — |
| Urethane, two-part mixture; rigid foam | 70 | 0.026 | 1045 |
| Reflective | | | |
| Aluminium foil separating fluffy glass mats; 10–12 layers, evacuated; for cryogenic applications (150 K) | 40 | 0.00016 | — |
| Aluminium foil and glass paper laminate; 75–150 layers, evacuated; for cryogenic application (150 K) | 120 | 0.000017 | — |
| Typical silica powder, evacuated | 160 | 0.0017 | — |

**Table A.3** (*Contd*)

| *Description/ Composition* | *Maximum service tempera-ture (K)* | *Typical density (kg/m³)* | *Typical thermal conductivity, k (W/m K), at various temperatures (K)* | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | *200* | *215* | *230* | *240* | *255* | *270* | *285* | *300* | *310* | *365* | *420* | *530* | *645* | *750* |
| **Industrial insu-lation** | | | | | | | | | | | | | | | | |
| Blankets | | | | | | | | | | | | | | | | |
| Blanket, mineral fiber, metal | 920 | 96–192 | | | | | | | | | 0.038 | 0.046 | 0.056 | 0.078 | | |
| reinforced | 815 | 40–96 | | | | | | | | | 0.035 | 0.045 | 0.058 | 0.088 | | |
| Blanket, mineral fiber, glass; fine fiber, organic | 450 | 10 | | | | 0.036 | 0.038 | 0.040 | 0.043 | 0.048 | 0.052 | 0.076 | | | | |
| bonded | | 12 | | | | 0.035 | 0.036 | 0.039 | 0.042 | 0.046 | 0.049 | 0.069 | | | | |
| | | 16 | | | | 0.033 | 0.035 | 0.036 | 0.039 | 0.042 | 0.046 | 0.062 | | | | |
| | | 24 | | | | 0.030 | 0.032 | 0.033 | 0.036 | 0.039 | 0.040 | 0.053 | | | | |
| | | 32 | | | | 0.029 | 0.030 | 0.032 | 0.033 | 0.036 | 0.038 | 0.048 | | | | |
| | | 48 | | | | 0.027 | 0.029 | 0.030 | 0.032 | .033 | 0.035 | 0.045 | | | | |
| Blanket, alu-mina-silica fiber | 1530 | 48 | | | | | | | | | | | | 0.071 | 0.105 | 0.150 |
| | | 64 | | | | | | | | | | | | 0.059 | 0.087 | 0.125 |
| | | 96 | | | | | | | | | | | | 0.052 | 0.076 | 0.100 |
| | | 128 | | | | | | | | | | | | 0.049 | 0.068 | 0.091 |
| Felt, semirigid; | 480 | 50–125 | | | | | | 0.035 | 0.036 | 0.038 | 0.039 | 0.051 | 0.063 | | | |
| organic bonded | 730 | 50 | 0.023 | 0.025 | 0.026 | 0.027 | 0.029 | 0.030 | 0.032 | 0.033 | 0.035 | 0.051 | 0.079 | | | |
| Felt, laminated; no binder | 920 | 120 | | | | | | | | | | | 0.051 | 0.065 | 0.087 | |
| Blocks, boards and, pipe insula-tions | | | | | | | | | | | | | | | | |
| Asbestos paper, laminated and corrugated | | | | | | | | | | | | | | | | |
| 4-ply | 420 | 190 | | | | | | | 0.078 | 0.082 | 0.098 | | | | | |

(*Contd*)

**Table A.3** (*Contd*)

| *Description/ Composition* | *Maximum service tempera-ture (K)* | *Typical density (kg/m$^3$)* | *Typical thermal conductivity, k (W/m K), at various temperatures (K)* | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | *200* | *215* | *230* | *240* | *255* | *270* | *285* | *300* | *310* | *365* | *420* | *530* | *645* | *750* |
| 6-ply | 420 | 255 | | | | | | | | 0.071 | 0.074 | 0.085 | | | | |
| 8-ply | 420 | 300 | | | | | | | | 0.068 | 0.071 | 0.082 | | | | |
| Magnesia, 85% | 590 | 185 | | | | | | | | | 0.051 | 0.055 | 0.061 | | | |
| Calcium silicate | 920 | 190 | | | | | | | | | 0.055 | 0.059 | 0.063 | 0.075 | 0.089 | 0.104 |
| Cellular glass | 700 | 145 | | | 0.046 | 0.048 | 0.051 | 0.052 | 0.055 | 0.058 | 0.062 | 0.069 | 0.079 | | | |
| Diatomaceous | 1145 | 345 | | | | | | | | | | | | 0.092 | 0.098 | 0.104 |
| silica | 1310 | 385 | | | | | | | | | | | | 0.101 | 0.100 | 0.115 |
| Polystyrene, rigid | | | | | | | | | | | | | | | | |
| Extruded (R-12) | 350 | 56 | 0.023 | 0.023 | 0.022 | 0.023 | 0.023 | 0.025 | 0.026 | 0.027 | 0.029 | | | | | |
| Extruded (R-12) | 350 | 35 | 0.023 | 0.023 | 0.023 | 0.025 | 0.025 | 0.026 | 0.027 | 0.029 | | | | | | |
| Molded beads | 350 | 16 | 0.026 | 0.029 | 0.030 | 0.033 | 0.035 | 0.036 | 0.038 | 0.040 | | | | | | |
| Rubber, rigid, foamed | 340 | 70 | | | | | | 0.029 | 0.030 | 0.032 | 0.033 | | | | | |
| Insulating cement | | | | | | | | | | | | | | | | |
| Mineral fiber (rock, slag or glass) | | | | | | | | | | | | | | | | |
| With clay binder | 1255 | 430 | | | | | | | | | 0.071 | 0.079 | 0.088 | 0.105 | 0.123 | |
| With hydraulic setting binder | 922 | 560 | | | | | | | | | 0.108 | 0.115 | 0.123 | 0.137 | | |
| Loose Fill | | | | | | | | | | | | | | | | |
| Cellulose, wood or paper pulp | — | 45 | | | | | | | 0.038 | 0.039 | 0.042 | | | | | |
| Perlite, ex-panded | — | 105 | 0.036 | 0.039 | 0.042 | 0.043 | 0.046 | 0.049 | 0.051 | 0.053 | 0.056 | | | | | |
| Vermiculite, expanded | — | 122 | | | 0.056 | 0.058 | 0.0061 | 0.063 | 0.065 | 0.068 | 0.071 | | | | | |
| | | 80 | | | 0.049 | 0.051 | 0.055 | 0.058 | 0.061 | 0.063 | 0.066 | | | | | |

(*Contd*)

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**Table A.3** *(Contd)*

| *Description/ composition* | *Temperature (k)* | *Density, ρ (kg/m³)* | *Thermal conductivity, k (W/m K)* | *Specific heat $c_p$, (J/kg K)* |
|---|---|---|---|---|
| **Other materials** | | | | |
| Asphalt | 300 | 2115 | 0.062 | 920 |
| Bakelite | 300 | 1300 | 1.4 | 1465 |
| Brick, refractory | | | | |
| Carborundum | 872 | — | 18.5 | — |
| | 1675 | — | 11.0 | — |
| Chrome brick | 473 | 3010 | 2.3 | 835 |
| | 823 | | 2.5 | |
| | 1173 | | 2.0 | |
| Diatomaceous | 478 | — | 0.25 | — |
| silica, fired | 1145 | — | 0.30 | |
| Fire clay, burnt 1600 K | 773 | 2050 | 1.0 | 960 |
| | 1073 | — | 1.1 | |
| | 1373 | — | 1.1 | |
| Fire clay, burnt 1725 K | 773 | 2325 | 1.3 | 960 |
| | 1073 | | 1.4 | |
| | 1373 | | 1.4 | |
| Fire clay brick | 478 | 2645 | 1.0 | 960 |
| | 922 | | 1.5 | |
| | 1478 | | 1.8 | |
| Magnesite | 478 | — | 3.8 | 1130 |
| | 922 | — | 2.8 | |
| | 1478 | | 1.9 | |
| Clay | 300 | 1460 | 1.3 | 880 |
| Coal, anthracite | 300 | 1350 | 0.26 | 1260 |
| Concrete (stone mix) | 300 | 2300 | 1.4 | 880 |
| Cotton | 300 | 80 | 0.06 | 1300 |
| Foodstuffs | | | | |
| Banana (75.7% water content) | 300 | 980 | 0.481 | 3350 |
| Apple, red (75% water content) | 300 | 840 | 0.513 | 3600 |
| Cake, batter | 300 | 720 | 0.223 | — |
| Cake, fully baked | 300 | 280 | 0.121 | — |
| Chicken meat, white | 198 | — | 1.60 | |
| (74.4% water content) | 233 | — | 1.49 | |
| | 253 | | 1.35 | |
| | 263 | | 1.20 | |
| | 273 | | 0.476 | |
| | 283 | | 0.480 | |
| | 293 | | 0.489 | |

(*Contd* )

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**Table A.3** (*Contd*)

| *Description/ composition* | *Temperature (k)* | *Density, ρ (kg/m$^3$)* | *Thermal conductivity, k (W/m K)* | *Specific heat $c_p$, (J/kg K)* |
|---|---|---|---|---|
| Glass | | | | |
| Plate (soda lime) | 300 | 2500 | 1.4 | 750 |
| Pyrex | 300 | 2225 | 1.4 | 835 |
| Ice | 273 | 920 | 1.88 | 2040 |
| | 253 | — | 2.03 | 1945 |
| Leather (sole) | 300 | 998 | 0.159 | — |
| Paper | 300 | 930 | 0.180 | 1340 |
| Paraffin | 300 | 900 | 0.240 | 2890 |
| Rock | | | | |
| Granite, Barre | 300 | 2630 | 2.79 | 775 |
| Limestone, Salem | 300 | 2320 | 2.15 | 810 |
| Marble, Halston | 300 | 2680 | 2.80 | 830 |
| Quartzite, | | | | |
| Sioux | 300 | 2640 | 5.38 | 1105 |
| Sandstone, Berea | 300 | 2150 | 2.90 | 745 |
| Rubber, vulcanized | | | | |
| Soft | 300 | 1100 | 0.13 | 2010 |
| Hard | 300 | 1190 | 0.16 | — |
| Sand | 300 | 1515 | 0.27 | 800 |
| Soil | 300 | 2050 | 0.52 | 1840 |
| Snow | 273 | 110 | 0.049 | — |
| | | 500 | 0.190 | — |
| Teflon | 300 | 2200 | 0.35 | — |
| | 400 | | 0.45 | — |
| Tissue, human | | | | |
| Skin | 300 | — | 0.37 | — |
| Fat layer (adipose) | 300 | — | 0.2 | — |
| Muscle | 300 | — | 0.41 | — |
| Wood, cross gain | | | | |
| Balsa | 300 | 140 | 0.055 | — |
| Cypress | 300 | 465 | 0.097 | — |
| Fir | 300 | 415 | 0.11 | 2720 |
| Oak | 300 | 545 | 0.17 | 2385 |
| Yellow pine | 300 | 640 | 0.15 | 2805 |
| White pine | 300 | 435 | 0.11 | — |
| Wood, radial | | | | |
| Oak | 300 | 5.45 | 0.19 | 2385 |
| Fir | 300 | 420 | 0.14 | 2720 |